Manipulating Images with Fractal Julia Sets

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Abstract

The pictures associated with the fractal Julia sets are familiar to mathematicians and non-mathematicians. The main purpose of this paper is to look at ways by which an image can be manipulated using the same mathematical process. This paper looks at several ways that an arbitrary pixel image can be iteratively transformed using fractal Julia set algorithms. The standard approach of letting the pixels evolve towards attractors was interesting, but a second approach, one that inverted the process so that attractors became expellers, proved more fruitful for creating interesting images.

1. Introduction

Upon reading Pietgen and Richter's The Beauty of Fractals [1] for the first time, I used my programming skills and a long-forgotten set of notes on complex numbers to reproduce most of its pictures based on Julia set fractals, Yang Lee singularities and Newton's method for complex polynomials. The book also contains an interesting discussion concerning the nature of art and science (page 20), particularly with regard to the lack of choice and human concern. This discussion intrigued me into experimenting with something more than just conventional use of colour to represent the speed of convergence/convergence. This paper is limited to examples from Julia sets.

2. The Basic Process

The processes associated with Julia sets relate to a series of points each being formed from the previous point using a rule:
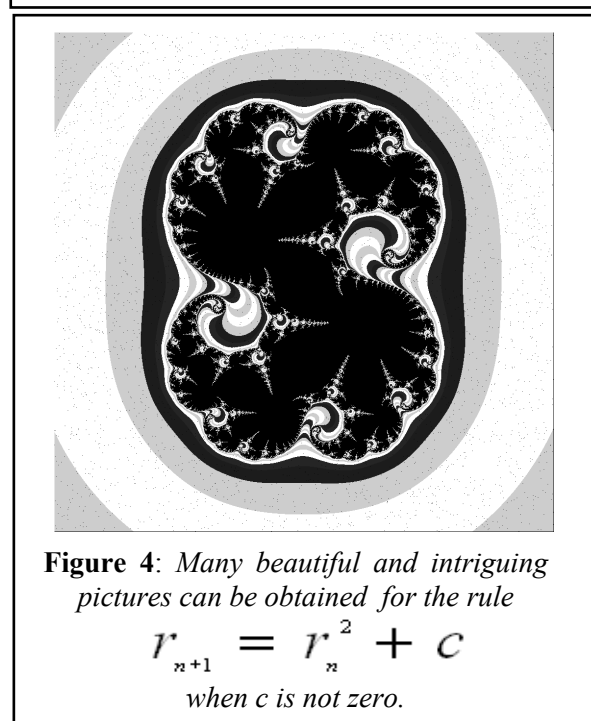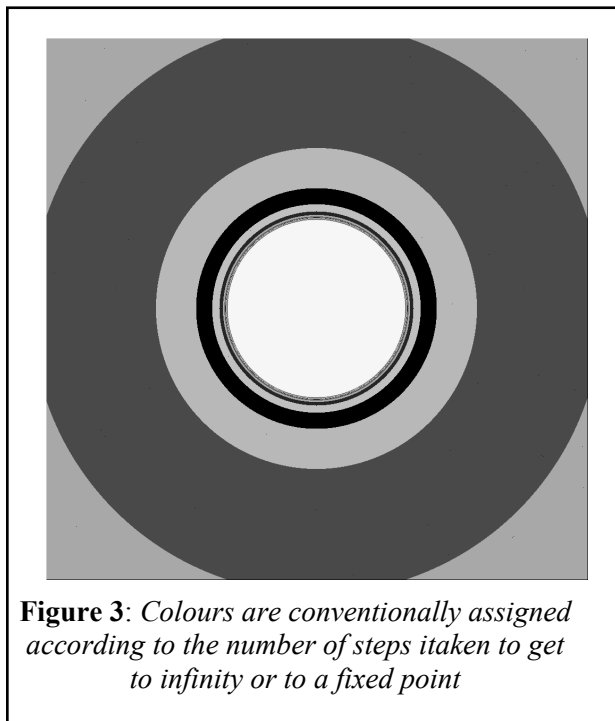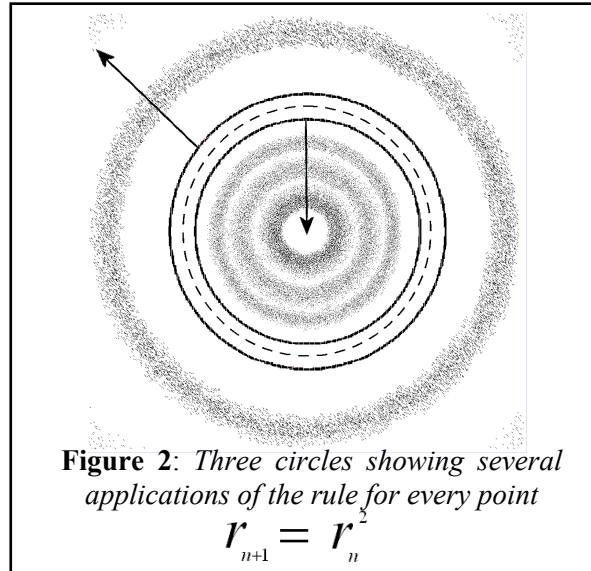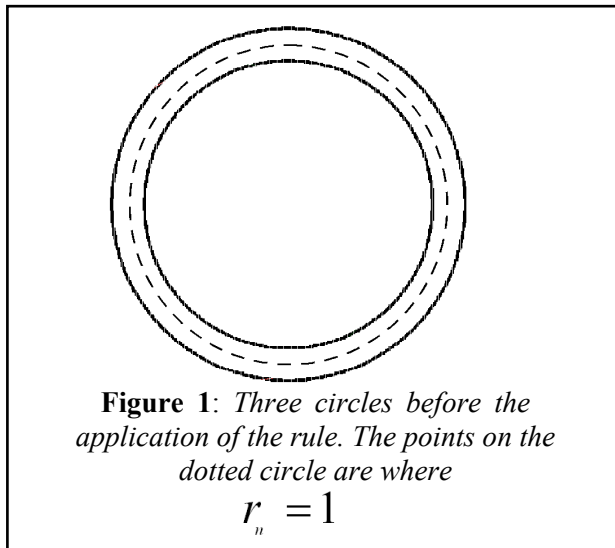
$$r_{n+1} = r_n^2 + c$$

The process produces a sequence of points formed by squaring the last point and adding a constant. If we consider the very simple case where $c=0$ then there are three scenarios for r depending upon the initial value of r.

a) The numbers become smaller and smaller. In this case we say that zero is an attractor for the process. All points less than 1 from this attractor are drawn into it.
b) The numbers become larger and larger, tending towards infinity. We say that infinity is an attractor for the process. All points further than a distance 1 from zero are drawn into it.
c) The points are at a distance 1 from zero and stay there. They lie on the boundary between the two domains of attraction.

The process is readily understood using real numbers. In 1980 Mandelbrot followed the same process using complex numbers, that is, numbers in a plane rather than on a line. The starting point is driven to one attractor or the other or it is on the boundary and cannot decide.

The first experiment used a value of $c=0$. Figure 1 shows three circles satisfying the three conditions. The black dashed circle contains all the complex numbers at a distance 1 from zero. The points on the inner black circle are attracted towards zero and the points on the outer black circle are attracted towards infinity. The first experiment involved drawing a picture consisting of the three concentric circles. Software was written to calculate the subsequent position for each pixel. This involved decomposing the complex number into its real and imaginary components which were used to calculate the x and y components of the source and destination positions of the pixels. The colour of the corresponding source pixel was copied to the destination pixel. This was repeated for each pixel in the picture. The process was repeated three times. Figure 1 shows the starting picture and Figure 2, the result after three iterations.

The outer black circle quickly leaves the page on its way to infinity. The inner circle moves towards zero and the dashed circle stays put.

Figure 1: *Three circles before the application of the rule. The points on the dotted circle are where*

$$r_n = 1$$

Figure 2: *Three circles showing several applications of the rule for every point*

$$r_{n+1} = r_n^2$$

Figure 3: *Colours are conventionally assigned according to the number of steps itaken to get to infinity or to a fixed point*

Figure 4: *Many beautiful and intriguing pictures can be obtained for the rule*

$$r_{n+1} = r_n^2 + c$$

when c is not zero.

The conventional way to assign a colour to a pixel representing a point is to count the number of repeated applications of the rule it takes for a point to reach infinity or zero. In practice the reaching of infinity is defined as an arbitrary distance, say 100 units, and zero, say, anything less than 0.01. The grey tones in Figure 3 are all points diverging towards infinity. The grey tone is determined by the number of steps that it takes for it to move more than 100 units. The white inner region could be coloured in a similar way depending upon the number of steps it takes to be less than 0.01 away from zero. Figure 4 shows a more intriguing example for an application of the rule where $c=0.27334+0.00742i$

The colouring scheme chosen for these pictures has a profound effect upon the aesthetic appeal of the pictures. The main decisions that the artist can make are the colour scheme, the value of the constant c, the scale and the resolution of the final picture. Many further coloured examples of such fractals can be found at [1][2][3][4][5].

3. Manipulating Non-Geometric Images

My aim in this project was to develop software that would allow the manipulation of images. The approach taken was to start with an image and then to explore ways in which it could be manipulated using the sort of algorithms described earlier. In the first instance the simplest form of the rule was used i.e. with c=0.

$$r_{n+1} = r_n^2$$

In my first experiment I chose a picture of Al Gore surrounded by a green/ grey border because this reflected his green credentials and he has made some progress in convincing the United States that the science associated with global warming, whilst it might be wrong, is non-negotiable. Furthermore, the election that he narrowly lost came down to a few spoiled ballot papers in Florida; a result that may well have changed the course of modern world history; a situation akin to a butterfly causing a hurricane. Finally, adding the word "Rhythmic" to his name it becomes an Art form! (pun intended)
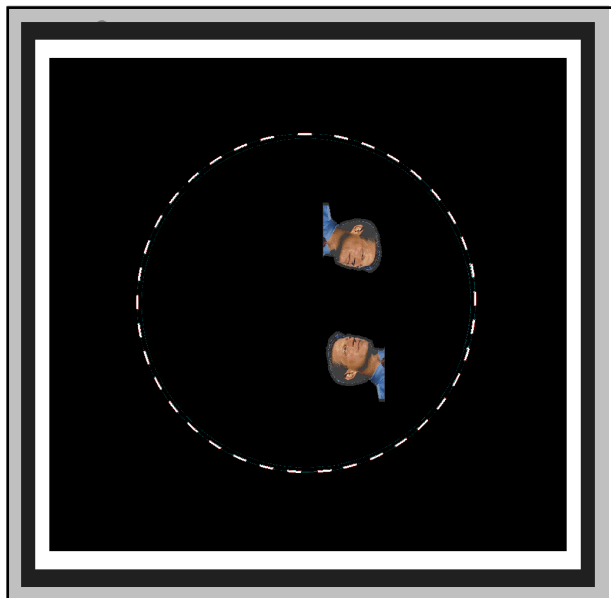

Figure 5: *The starting image for the first experiment was two images of Al Gore with a black,white and grey border. The white dotted circle is the unit circle.*
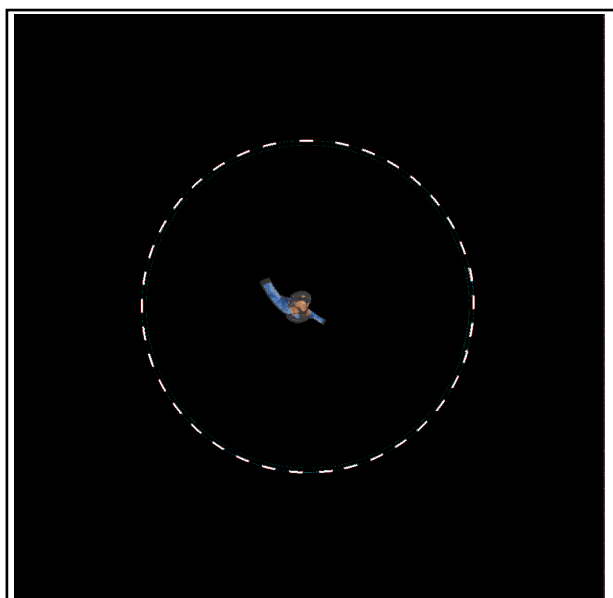
Figure 6: *Using the simple rule we see that the border disappears from the image and the Gore images are attracted towards zero.*
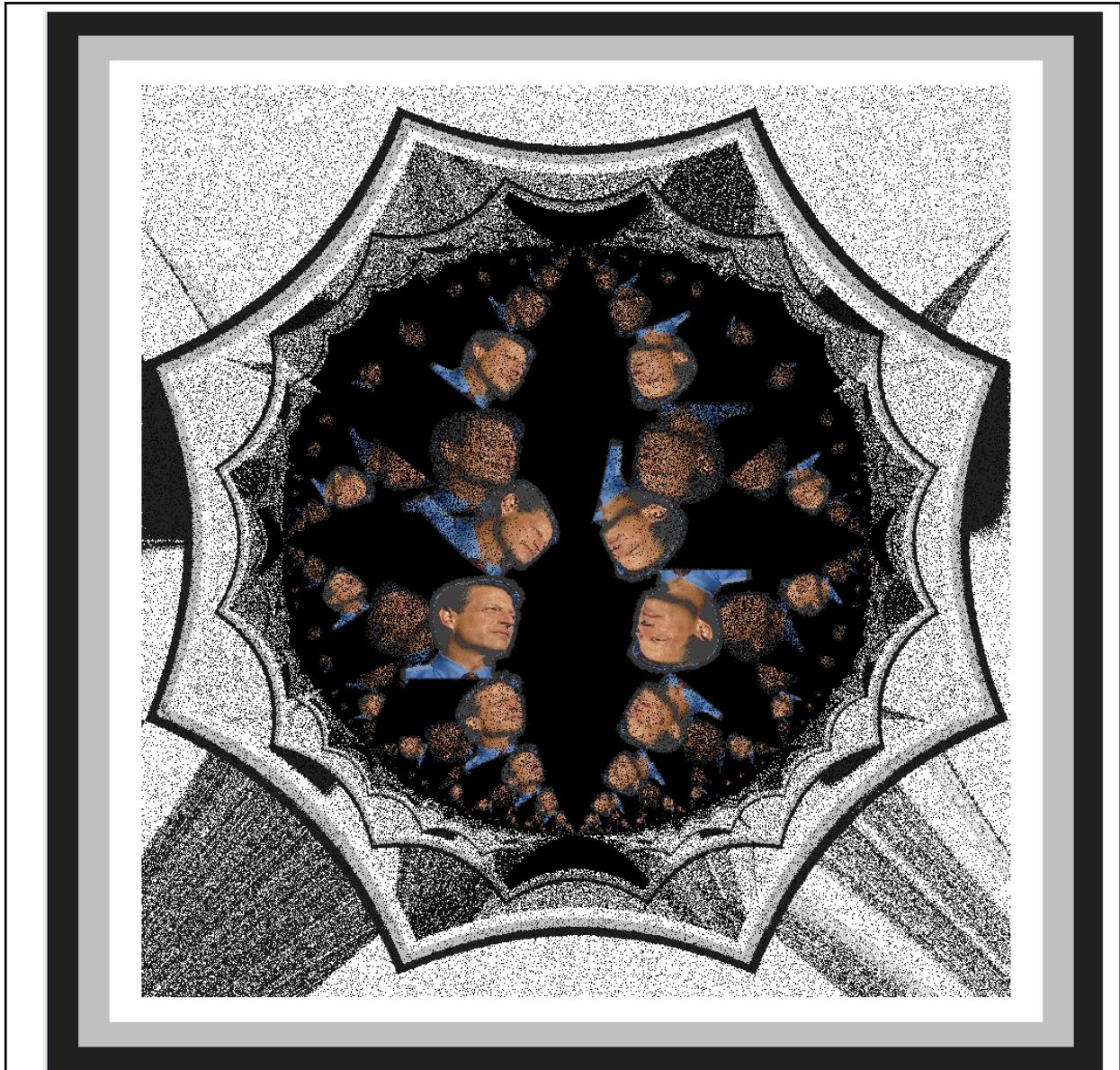
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Figure 7: *The result of approximately 1 million inverse applications of the rule*

The first experiment aimed to move points of an image (see Figure 5) using the simple rule. The border, being well outside the unit circle, disappeared from the screen as it made its way to infinity. The images of Al Gore, being inside the unit circle, were attracted to zero. They became distorted and disappeared after a few iterations. (see Figure 6). The experiment was repeated using non-zero constants with similar outcomes. Generally, I didn't regard the resulting images as having much artistic potential.

The next experiment aimed to turn the problem on its head. What picture would be needed so that when the rule was applied the picture in Figure 5 would be produced? Software was written to calculate the subsequent position for each pixel as it followed the rule. This involved decomposing the complex number into its real and imaginary components which were used to calculate the *x* and *y* components of the source and destination positions of the pixels. In this case, however, if the destination pixel was actively coloured then the source pixel took on that colour. In simple terms a colour was defined as active if it wasn't black.

Figure 8: *The result of approximately 12 million inverse applications of the rule*
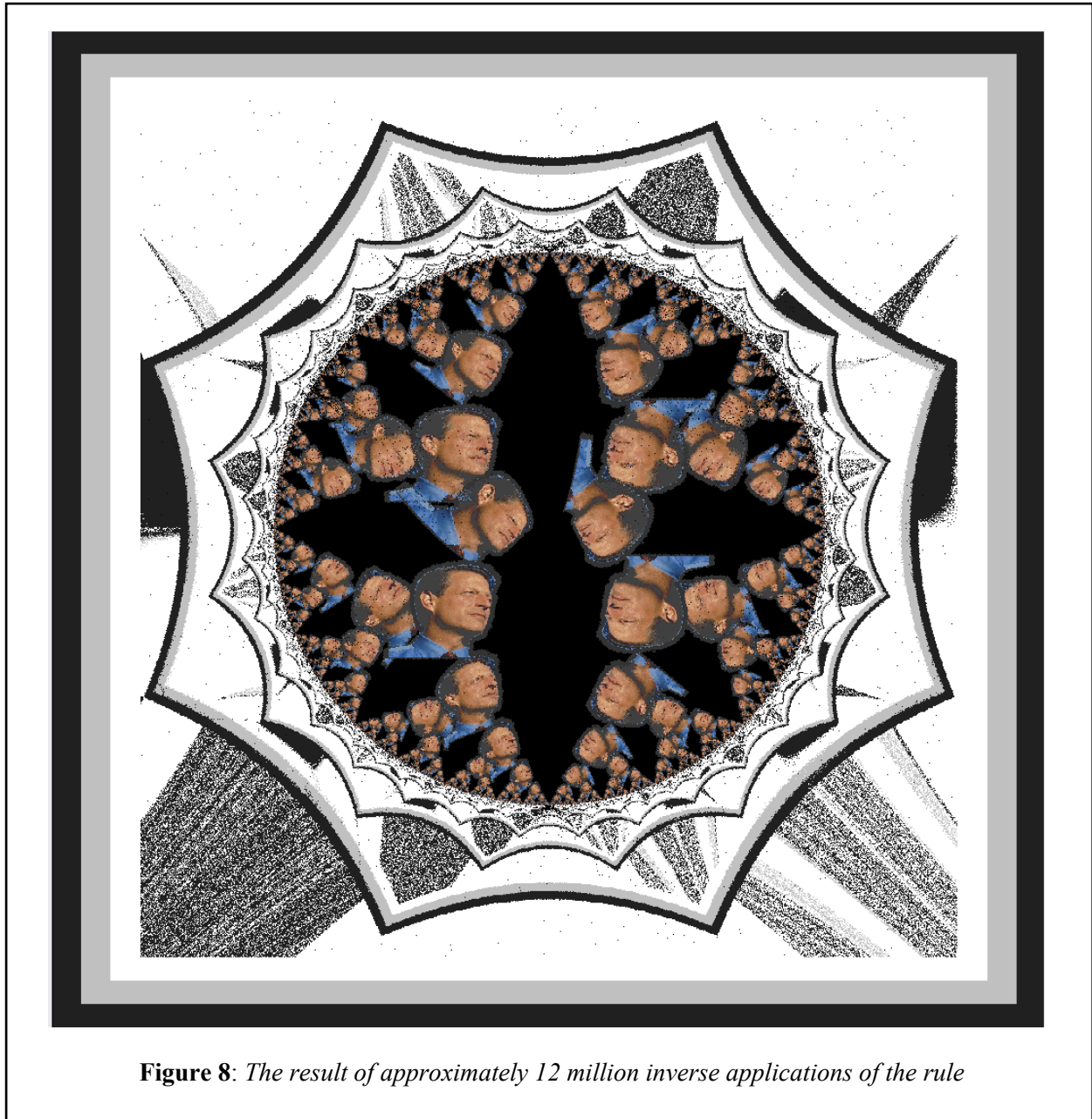
The resolution of the original picture was approximately 1 megapixel. Figure 7 is the result of approximately 1 million inverse applications of the rules, one for each pixel. It needs to be noted that this was a dynamic process. Each pixel coloured may influence subsequent colouring. I use the term inverse to describe the process although I accept that in the strict mathematical sense this may not be true. Figure 8 is the result of approximately 12 million inverse applications. The resulting picture has much more artistic potential. It is interesting to note that in the region where the points would be attracted to infinity they are actually repelled away from infinity. In the inner region where the points would normally be attracted towards zero they actually move away from zero, the inverse of the original process.

Encouraged by the apparent potential for this inverse approach I developed software which would handle examples from the Julia set,Yang Lee singularities and Newton's method for complex polynomials The work in this paper is limited to examples from the Julia set.

4. A Brief Description of the Software

As in most art work, colour is an important feature as is the scale of the picture. The software was designed to develop pictures up to 10 layers each of up to 120 megapixels and 256 colours. This being the limit of the hardware available. The length of these calculations necessitated regular dump and restart facilities in the software. The time involved in the creation of a high resolution picture can be considerable, up to 20 hours, so the software allowed the development of a low resolution picture prior to the final version. Frames of an animated sequence can be saved so that the dynamics of the process can be studied.

5. Example for the Julia Set with a Non-Zero Constant

The previous example shows the result of an inverse application of the rule

For a value of c=0
$$r_{n+1} = r_n^2 + c$$

The picture in Figure 9 is one from the group of pieces I call "every girl". In this case the value of $c = -0.12 + 0.74i$. The starting point is the picture at the bottom right of figure 9. The black and white border is positioned in the region which would normally be attracted to infinity. The two small images made from a number of overlapping unicursal curves are inside the inner region and would normally move towards one or more attractors. The picture is an example from the Julia set called a basin of an attractive cycle of period 3. A discussion can be found at [1] (page 10).
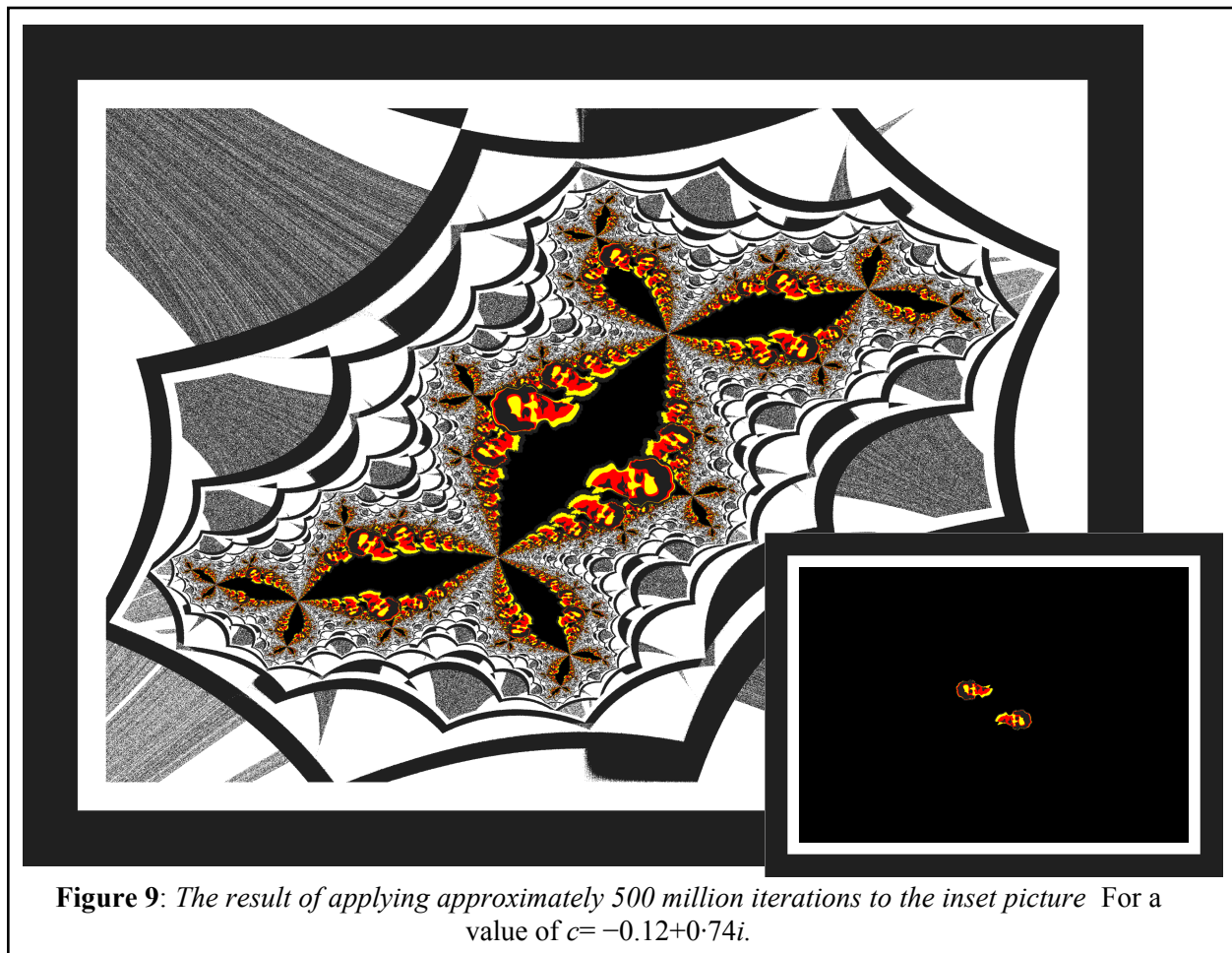

Figure 9: *The result of applying approximately 500 million iterations to the inset picture* For a value of $c = -0.12 + 0.74i$.
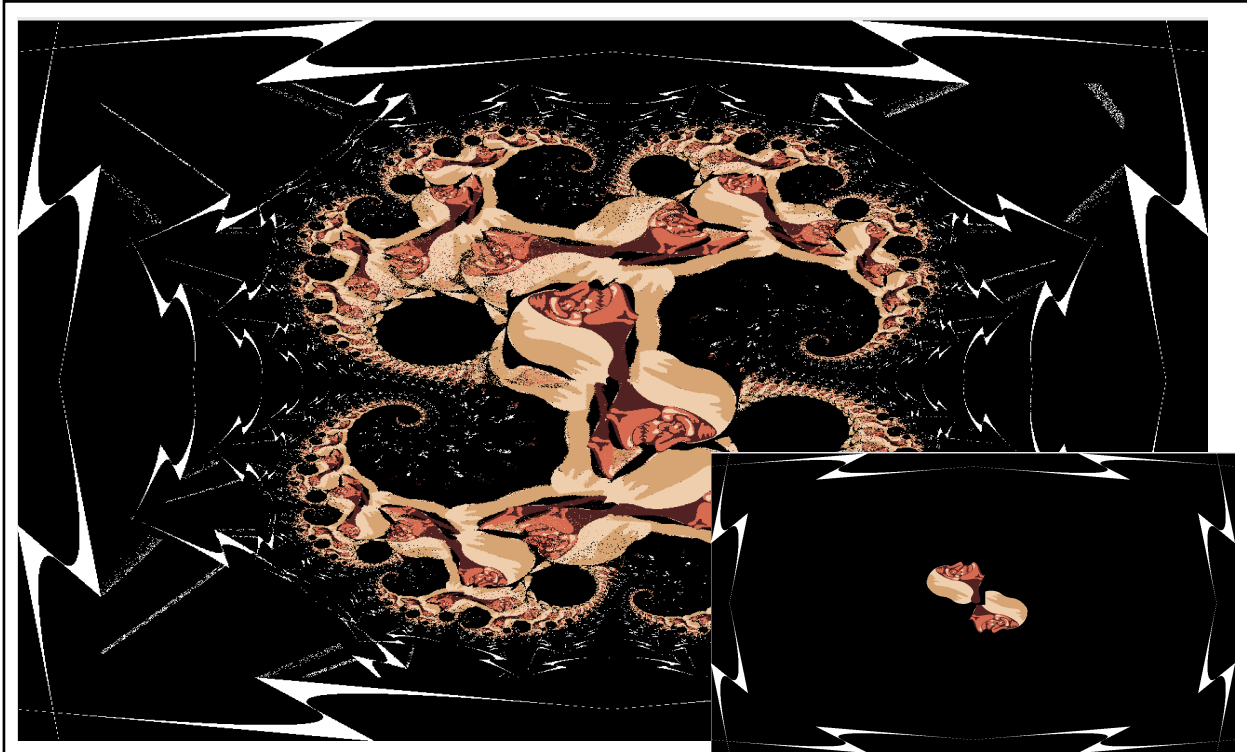
Figure 10: *Picture entitled "When the light goes out" from the Julia Set with C=0.2733+0.00742i*
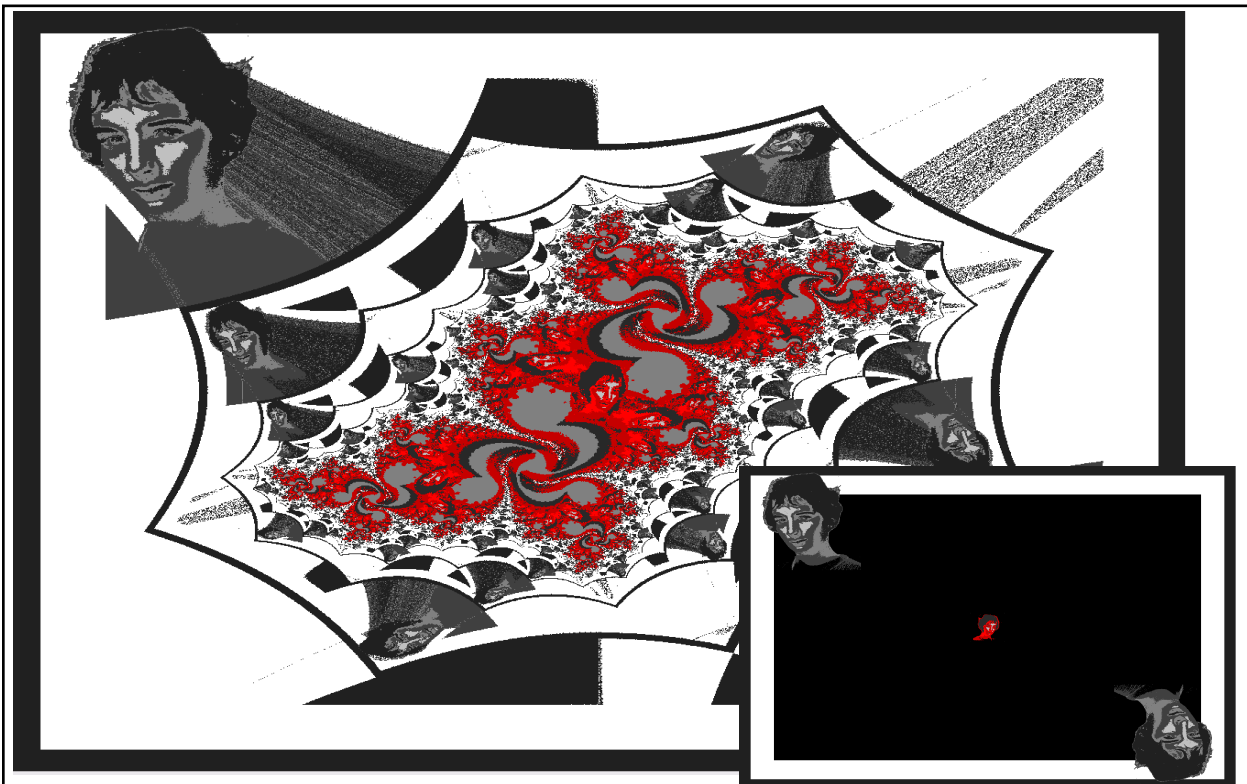

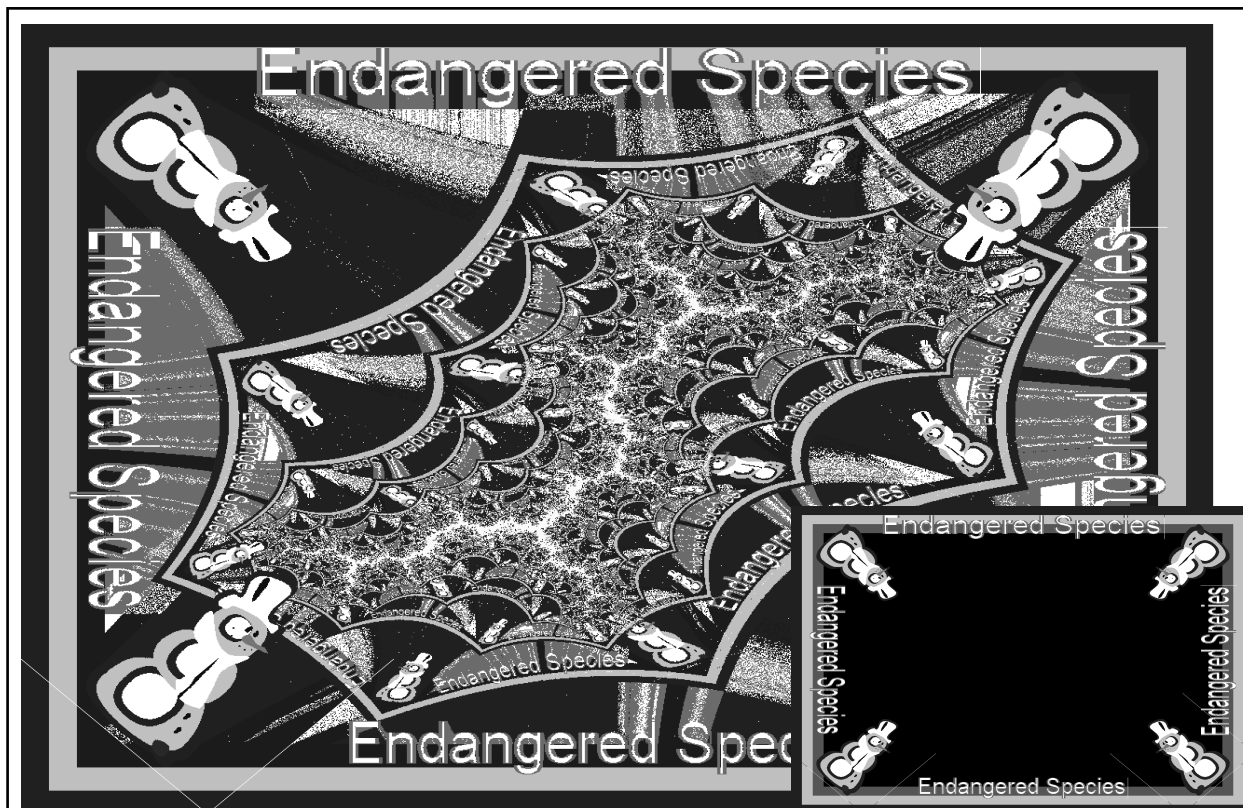**Figure 11:** *This picture from the Julia set with C=−0.11+0.6557i*

Figure 12: *This picture entitled "Endangered Species" is from the Julia set with c=+i. It is known as a dendrite. The inner region has a border but no area.*

6. Further Examples Using Images in the Outer and Inner Domains of Attraction

Figure 10 has an attractive cycle of period 20 [1] (page 14). The picture entitled "When the light goes out" is from the Julia Set with the constant c=0.2733+0.00742i. Figure 11 has an attractive cycle of period 3 shortly before it decays into a Cantor set [1] (page 14). The picture from the "every girl" group is from the Julia set with constant c=-0.11+0.6557i. Figure 12 is an example of a dendrite. Dendrites have no interior; there is no attractor other than the one at infinity [1] (page 14).The picture entitled "Endangered Species" is from the Julia set with c=+i.

7. Animated Development

Most of the pictures have animated versions which show the dynamics of the chaos theory in action, especially if played in reverse. Some examples of the animations as well as other pictures will be brought to the conference. The conflicts in the border regions can take many millions of iterations to resolve. In some cases the conflicts have no clear outcomes because of the nature and ambiguities of the pictures. The analysis of these elements of uncertainty is the subject of on going study.

References

[1] Peitgen and Richter, *The Beauty of Fractal*s, ISBN 3-540-15851,-0 Springer, 1986 - actual pages are mentioned in the body of the paper.
[2] Hans Lauwerier, *Fractals Images of Chaos,* ISBN 0-14-014411,-0 Penguin, 1991
[3] David Wells, *Curious and Interesting Geometry,* ISBN 0-14-011813-6, Penguin, 1991
[4] Chaos Theory, <http://en.wikipedia.org/wiki/chaos_theory>, (Accessed 28.12.2007)