Hankin's 'Polygons in Contact' Grid Method for Recreating a Decagonal Star Polygon Design

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Abstract

The crafts tradition that produced Islamic geometric designs is most likely based on methods requiring a considerable amount of practical geometrical knowledge. This methodology was understood and practiced by master builders using the traditional tools of the medieval period, such as the compass with a fixed opening (a "rusty compass"), straightedge, and set square for initially creating new designs; and then memorized repeat units for recreating already familiar and well-established patterns. This paper will discuss the re-creation of a specific decagonal star polygon design, using a method proposed by E.H. Hankin, called a "polygons in contact" grid.

Introduction

How did artisans centuries ago create the beautiful geometric designs still found today on historic buildings and monuments throughout the Islamic world? According to J. L. Berggren, the generation of elaborate geometrical designs most likely involved "a considerable amount of geometrical knowledge," which suggests that some degree of mathematical literacy may have existed among artisans, or at least among the master builders, architects and master engineers [1]. There are few written records to definitively answer our question and it is quite likely that several different methods requiring practical geometrical knowledge were actually employed, because no one method was ideal in all situations. This methodology was understood and practiced by master builders using traditional tools of the medieval period – the compass with a fixed opening (a "rusty compass"), straightedge, and set square for initially creating new designs, and then memorized grids for recreating already familiar and well-established patterns. This information was jealously guarded and passed on through the generations from father to son, or from master builder to apprentice.

Teaching Compass and Straightedge Construction Techniques

As Muslim scholars became aware of this tradition and the problems confronting craftsmen, an exchange developed between the two groups; meetings were held and ideas discussed. Theoretical mathematicians (al-Sijzī, Abū Nasr al-Farabī', Abu'l-Wafā' al-Buzjani, Al-Kashi, Umar al-Khayyami, and Abu Bakr al-Khalid al-Tajir al-Rasadi, among others) developed and wrote about geometric techniques useful to artisans interested in creating geometric ornamentation [2, 3]. Manuals were written as a result of the meetings between these two groups. For example, al-Buzjani's manual, *Kitāb fīmā yahtāju ilayhi al-sāni' min a'māl al-handasa* (About that which the artisan needs to know of geometric constructions) provided simplified "how to" instructions for artisans on using three tools, (the straightedge, set square and compass with a fixed opening) to perform basic Euclidean constructions, such as

> the construction of a right angle; the bisection of a square or circle; the division of a right angle into equal parts; the trisection of an angle; drawing a line parallel to, perpendicular to, or at a certain angle to a given line; determining the center of a circle or its arc; dividing the circumference of a circle into equal arcs; dropping a tangent to a circle from a given point; drawing a tangent to a circle through a point on it; and trisecting the arc of a circle. … From these general problems al-Buzjani moved on to the construction of regular polygons inscribed in circles, other constructions involving circles and arcs, and the constructions of polygonal figures inscribed in various figures. The circle is used in al-Buzjani's treatise to generate all of the regular polygons in a plane [4, p. 138].

These geometric constructions formed the basis for creating many of the symmetric geometric patterns of the time.

Appended to a copy of the Persian translation of al-Buzjani's manuscript in the Bibliothèque Nationale in Paris, is an anonymous, 20-page Persian manuscript, *Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa* (On Interlocking Similar and Congruent Figures) usually referred to by its shorter title, *A'māl wa ashkāl* (Constructions and Figures). Believed to have been prepared by an artisan sometime during the 11th – 13th centuries, it is the only known practical manual that provides "how to" instructions for drawing 61 repeat units of planar geometric patterns. The instructions in Persian usually start with the standard phrase: 'The way of drawing and the ratio or proportion of this construction is as follows' [4]. Abu Bakr al-Khalid al-Tajir al-Rasadi, an otherwise unknown mathematician, is mentioned twice in the *A'mal wa ashkal.* "Master craftsmen had questioned [Abu Bakr al-Khalid] about the different ways in which a particular geometric construction could be drawn; one of his solutions is explained in an accompanying diagram" [4, p. 168]. *A'mal wa ashkal* also contains another geometric construction by Abu Bakr al-Khalid, which shows how to draw a pentagon inscribed with a five-pointed star by using the chord of an arc as the module.

A 17th century practical geometry manual by Diego López de Arenas entitled *Breve compendio de la carpintería de lo blanco y tratado de alarifes* is the only other surviving document on the generation of Islamic patterns. Written with Arab and technical terms, and accompanied by geometric constructions drawn with the basic tools of straightedge, set square, and "rusty" compass, *Breve compendio* records the methods of master builders in Spain that were continued in the *Mudejar* style after the Reconquest of *al-Andalus* [5].

Since meetings were held between mathematicians and craftsmen starting in the 10th century, and "practical geometry" manuals instructing artisans how to perform basic geometry constructions were written, it seems plausible that some master builders were capable of using geometric construction techniques to generate patterns. Using a compass and straightedge allows for total flexibility when designing a pattern – there are no artificial constraints imposed as is the case with the use of polygonal grids. In addition, compass and straightedge techniques are quite useful for the accurate reproduction and reconstruction of patterns to their true design proportions without the need to measure dimensions. However, mastering these tools may have been unwieldy and beyond the capabilities or interests of the typical medieval artisan. And, even though the *A'mal wa ashkal* provided very complicated, scientifically correct geometric constructions, it also provided simpler methods for constructing the same patterns [4]. So perhaps a compromise in the use of compass and straightedge techniques was necessary for the generation of these geometric patterns.

The Use of Grids

Architectural scrolls provide evidence that grids may have been used by Islamic artisans for the creation of geometric Islamic patterns in Persia between the 10th and 16th centuries. The *Tashkent Scrolls* compiled in the 16th or 17th centuries by a master builder and kept at the Institute of Oriental Studies at the Academy of Sciences in Tashkent and the 15th century *Topkapi Scroll* (a manual of architectural designs housed at the Topkapi Palace Museum Library in Istanbul) contain geometric design sketches with no measurements or accompanying explanations for how to create them. The scrolls show that square and triangular grids were used mostly for generating calligraphic and brickwork designs, and polygonal and radial grids composed of concentric circles may have been used for generating other two- and three-dimensional patterns. These consist of black and/or red inked construction lines (some may also be shown as dotted lines) along with uninked "dead" lines lightly scratched on the paper with a sharp metal tool (such as the pointed end of a compass).

Unfinished architectural decorations also seem to show that polygonal grids—later called by E. Hanbury Hankin "polygons in contact" —may have been used as templates to generate and then transfer complex two-dimensional star-and-polygon patterns onto flat plaster walls [6]. In 1905 he reported that several interlaced geometric patterns had apparently been drawn with the aid of "triangular and polygonal grids faintly scratched on the plastered walls of two baths in the late 16th century palace complex in Fatehpur Sikri" [6]. One design in the Hakim Bath contained interlocking rectangles generated from a grid of equilateral triangles. The other, in the bath of Jodh Bai's palace, contained an interlocking stars and polygons pattern drawn on an underlying polygonal grid. Hankin concluded that these grids of polygons in contact were probably applied to the walls to serve as the "construction lines used in producing the pattern" [4, p. 49]. Addition architectural evidence includes a square grid containing a series of concentric circles that was scratched on the plaster of an unfinished vestibule with large compasses at the 14th century Masjid-i Safid in Turbat-i Jam [7].

Using polygonal grids may well have been less complicated and expeditious, especially for generating those designs already known to the artisans. The use of grids may also have required less ingenuity or original thought, since complex designs may easily be produced by accentuating some segments of the generating grid lines while erasing others. Gülru Necipoğlu, has noted, "It would be easier to transfer such polygons onto wall surfaces than the multiple concentric circles used in generating the patterns on paper" [4, p. 49]. The use of underlying grids can produce some of the existing designs to the correct proportions, but it does not work universally for all patterns; for example, most designs with curvilinear elements could not be generated from grids made of line segments.

Hankin's "Polygons in Contact" Grids

In [8], Hankin states that builders in 19th century India "were incapable of drawing the particular class of pattern about to be described as "geometrical arabesques" and so he postulates that, instead, an underlying polygonal grid was used as an aid. He describes the technique:

> …In making such patterns, it is first necessary to cover the surface to be decorated with a network consisting of polygons in contact. Then through the center of each side of each polygon two lines are drawn. These lines cross each other like a letter X and are continued till they meet other lines of similar origin. This completes the pattern. The original construction lines are then deleted and the pattern remains without any visible clue to the method by which it was drawn [8, p. 4].

This is supported by a note scribbled on a scroll acquired in Tehran by C. P. Clarke (Superintendent of Her Britannic Majesty's Works in Persia) and attributed to Mirza Akbar, a 19th century architect: "The uninked drypoint tracing [which were scratched on the paper with a pointed tool] indicates the basis of the formation of figures." [4, p. 14].

In the next section, we illustrate Hankin's method for recreating the 10-pointed star polygon design found in Plate VII, Figure 32 in [8]; this was recreated by the author using *The Geometer's Sketchpad* software.

Hankin's Instructions for Recreating a Decagonal Star Polygon Design

In [8, p. 14-15], Hankin states that "the repeat of the pattern we are considering fits a rectangle whose diagonal makes an angle of 36 degrees with one side and 54 degrees with the adjacent side." Hence, to create this pattern using his method, an artisan had to first find a way to construct the requisite angles. Although Hankin does not suggest how to do this, constructing an angle of 36° is not difficult. For example, a "golden triangle" with apex angle of 36° can be constructed with straightedge and compass; also, Euclid shows how to construct a regular pentagon in a circle, and two non-intersecting diagonals of that pentagon form a 36° angle. Hankin then continues, "A decagon is required to be drawn round E [the midpoint of the diagonals] and quarter decagons will have to occupy each of the four corners of the rectangle. They must be of such size that the distance between any two of them will be equal to the length of one of their sides. To discover the requisite size the following procedure should be adopted…" This includes the construction of a midpoint of one half of a diagonal and then a perpendicular line segment through that point, as shown in Figure 1. Next, two angle bisectors are constructed as in Figures 2 and 3. The point where the second angle bisector intersects the diagonal is a point on the circle that will eventually circumscribe a decagon at that lower left corner vertex (Figure 4.)

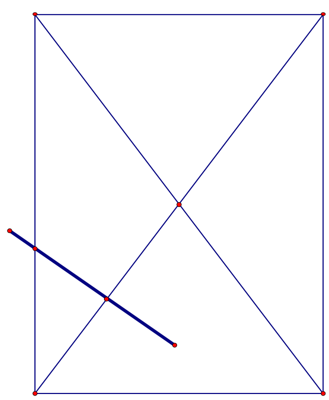


Figure 1

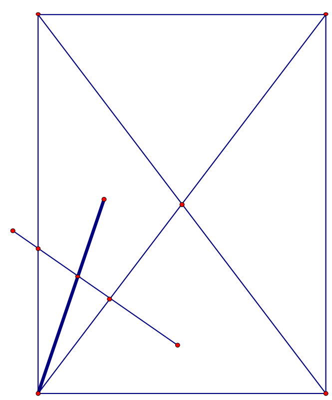


Figure 2

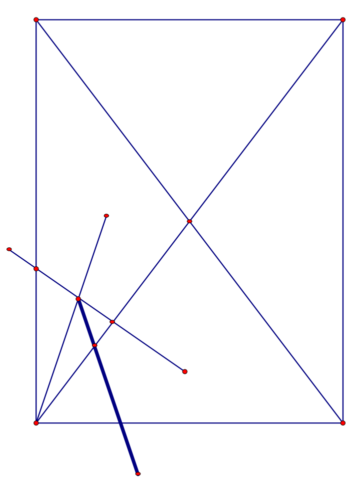


Figure 3

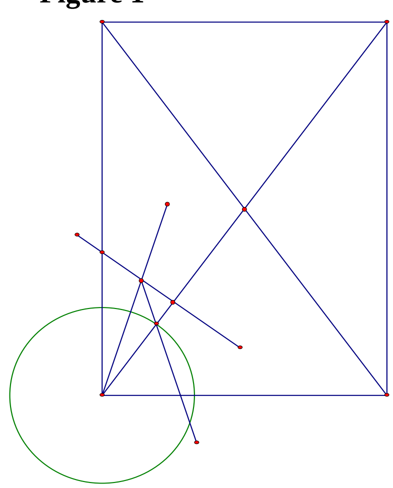


Figure 4

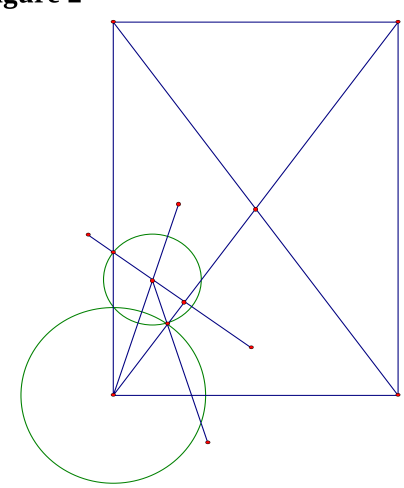


Figure 5

The smaller circle in Figure 5 will circumscribe one of the pentagons, which may be constructed within the smaller circle using the points shown in Figure 6.

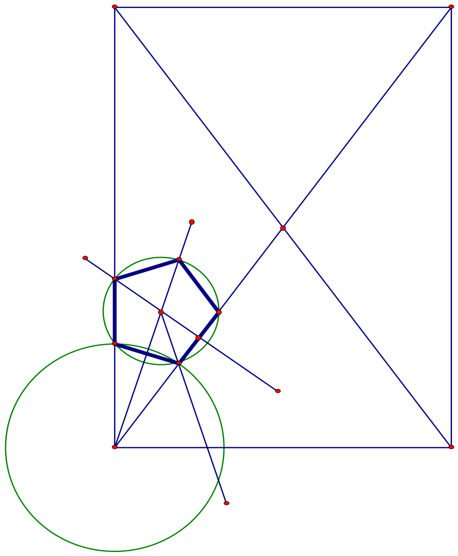


Figure 6

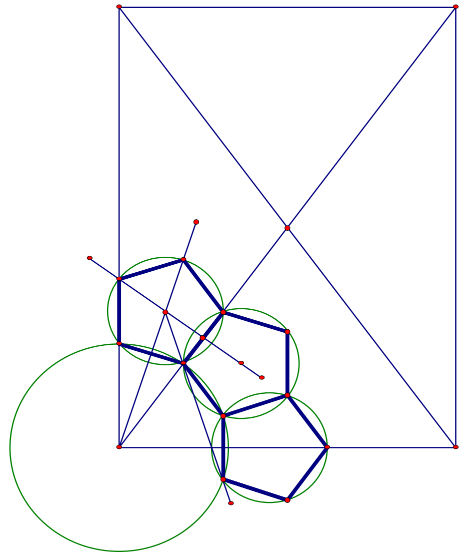


Figure 7

Hankin doesn't specify how to construct the remaining pentagons but he states that the other pentagons may be constructed. Since two regular pentagons and one regular decagon fit exactly around a common vertex of the three polygons, this is easily done. Figure 7 shows two and a half pentagons positioned within the rectangle about the lower left corner vertex. Repeating this process about the other three corners of the rectangle leads to the configuration in Figure 8, with twelve pentagons (two pairs of them overlapping). Joining the points to form a decagon and erasing the circles, the overlapping pentagon edges, the diagonals of the rectangle, and all the points and segments outside the rectangle, we obtain what Hankin calls a "polygons in contact" grid (Figure 9.)

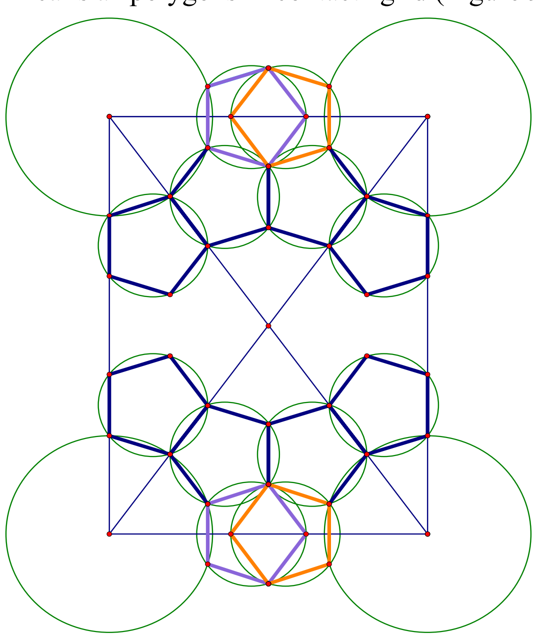


Figure 8

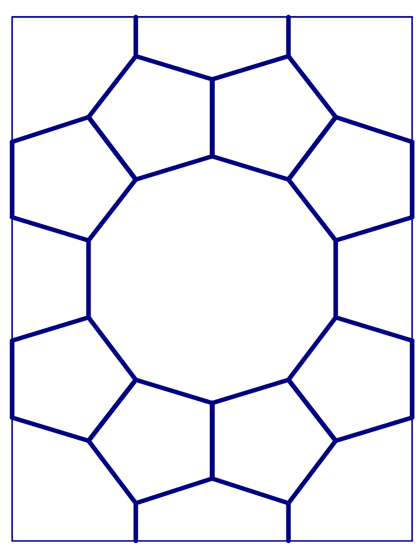


Figure 9

To construct a 10-pointed star polygon within the central decagon and 5-pointed star polygons circumscribed by pentagons, we connect the midpoints of the sides of the pentagons with line segments that pass through these points, are parallel to the sides of the pentagons, and end at the enclosing rectangle, as shown in Figure 10.

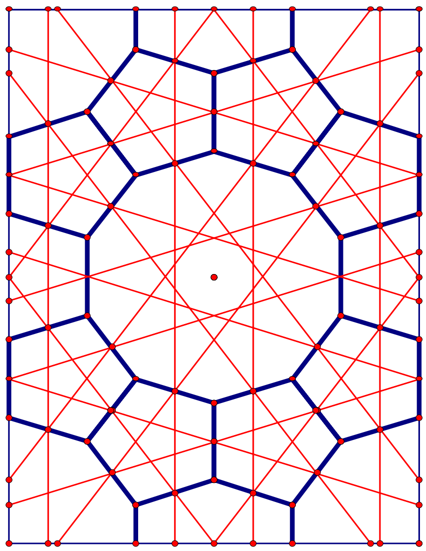


Figure 10

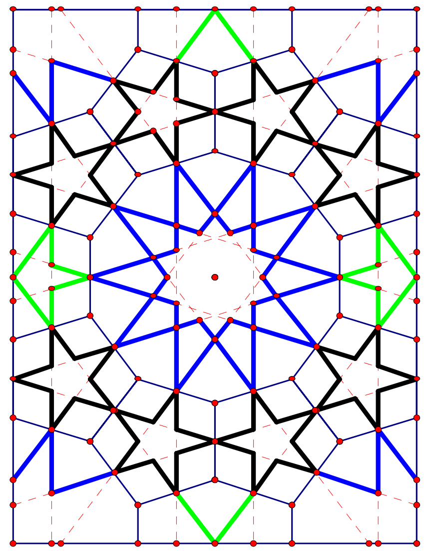


Figure 11

Highlighting appropriate line segments yields the 5- and 10-pointed stars and other polygonal shapes of the design, as shown in Figure 11. However, to complete the design we must construct additional line segments between existing points, as shown in Figure 12 in the corners of the rectangle and Figure 13 on the center of the top and bottom of the rectangle – an issue that is not addressed by Hankin. Figure 14 shows the completed design with the polygons in contact grid along with the dotted line segments drawn between existing points.

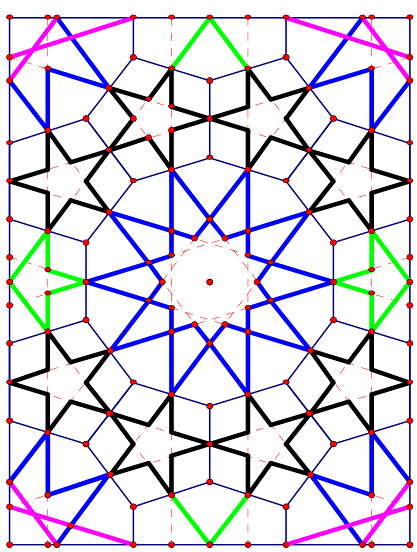


Figure 12

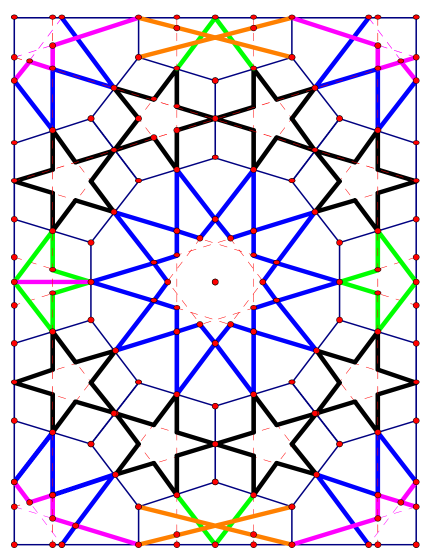
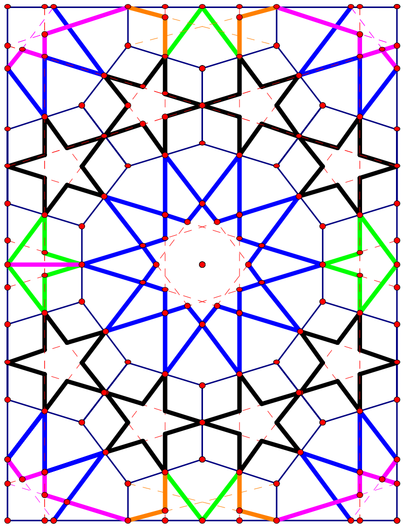


Figure 13



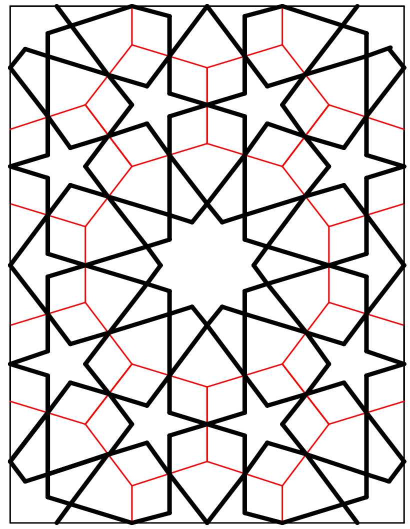


Figure 14 **Figure 15**

Figure 15 shows the completed design with the underlying polygons in contact grid, and Figure 16 shows the completed design. The photograph in Figure 17 shows a portion of an extant 10-star pattern.

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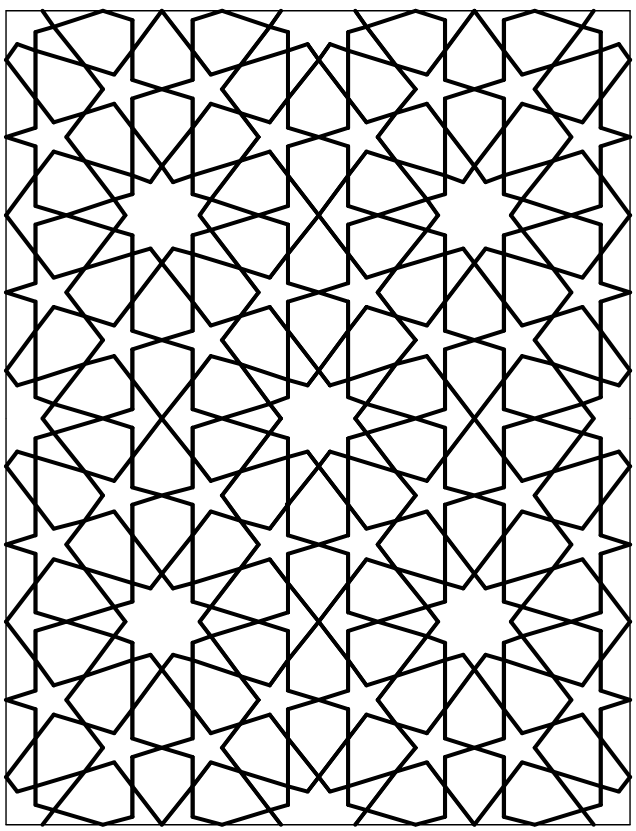

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Figure 16

Figure 17

Discussion

Although there are few written records documenting how medieval artisans created geometric Islamic patterns, it is quite likely that practical geometrical knowledge was necessary. Evidence from architectural scrolls and unfinished architectural decoration also exists to support the contention that underlying polygonal grids may have been used, such as Hankin's "polygons in contact." The use of these grids was much less complicated and expeditious than using compass and straightedge construction techniques, especially for generating those designs already familiar to the artisans. Hankin does not provide all the necessary details to complete the design, but with some improvisation, his method produces the pattern in a relatively easy and straightforward manner. Because the angles formed by the diagonal and the adjacent sides of the rectangle are 36° and 54°, the constructed decagons (or partial decagons) and pentagons are *regular* decagons and *regular* pentagons and so there is no distortion of the final pattern. Other possible techniques to create this same pattern with straightedge and compass are discussed by the author in [9] – [11].

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