A Computer Aided Geometric Model of a
Ten-Plane Polyhedral Transformation

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Abstract

A computer aided geometric model for a polyhedral transformation was written in C++, OpenGL and GLUT. This geometric model allows a user to interactively change the shape of a closed convex polyhedron derived from 10 face planes in a fundamental region. Given different inputs, a different shaped polyhedron was output. For some inputs, the output was a Platonic or Archimedean polyhedron. For other inputs a more complex polyhedron was output. This transformation is based on a fundamental region that is formed by 3 symmetry planes intersecting at the centroid of a polyhedron. This fundamental region is combined with 3 different sets of transformations to produce a polyhedron for either a tetrahedral, an octahedral, or an icosahedral family of polyhedra. Using the interactive software, a polyhedron transforms smoothly and continuously from a *regular* polyhedron, to a *truncate* polyhedron, to a *dual* polyhedron, to a *stellate* polyhedron along with many transitional polyhedra.

1. Introduction

This paper extends a previous paper presented on Coxeter Day at Bridges 2005 [1] where a 3 *plane* computer aided geometric model was presented. This 3 *plane* model was presented using 10 minutes of computer animation. More recently work has produced interactive software to extend that 3 *plane* computer model to a 10 *plane* computer model. Software using Microsoft Visual Studio C++ for modeling, OpenGL for graphics, and GLUT for interaction has resulted. A video documentary was produced to show an interactive experience with this software. Subsequently, a DVD was authored to include this documentary [2].

The *3 plane* model produces both Platonic and Archimedean *polyhedra,* as well as, many transitional *polyhedra*. The 3 *plane* model uses 9 *vectors* from 3 *foundational polyhedra*; a *regular polyhedron*, its *dual* and their *mid-figure*. For example: the *tetrahedronal* family uses 9 *vectors*, 3 each, from a *tetrahedron*, a *dual tetrahedron* and an *octahedron*. In addition the *octahedronal* family uses *vectors* from an *octahedron*, a *cube* and a *cubeoctahedron*, where the *icosahedronal* family uses *vectors* from an *icosahedron*, a *dodecahedron* and an *icosidodecahedron*. A tri-linear combination of the 9 *vectors* from each of these 3 families produces 3 *vectors* for a *polyhedron* that transforms it shape in the computer animation.

The 10 *plane* model uses 40 *vectors* from 4 *foundational polyhedra*. The *foundational polyhedra* are a *regular polyhedron* produced by adding 7 *planes* to the initial 3 *plane* model, a *truncate polyhedron*, a *dual polyhedron*, and a *stellate polyhedron*. Many *polyhedra* are formed when transitioning between these 4 *foundational polyhedra*.

In 1982 Haresh Lalvani, an architecture faculty at Pratt Institute in Brooklyn, New York, exhibited his polyhedral transformations. Many paper models were displayed along with descriptions and drawings. This work appeared in his Ph.D. dissertation completed under Buckminster Fuller at the University of Pennsylvania [3]. His *10 plane* transformation was depicted by a series of drawings in his dissertation and physical models in his exhibit.

The initial *3 plane* model and the subsequent *10 plane* model use C++ code *LIB3D* written by Pat Hanrahan, who is currently a computer science faculty at Stanford University. This code was first written in C when we were colleagues at NYIT Computer Graphics Laboratory. The inputs for the procedures are *vectors, points, lines,* and *planes.* The outputs are also *vectors, points, lines,* and *planes* derived from geometric relationships between the inputs. For example, the intersection of 3 *planes* is a *point.* The *LIB3D* code leads the computer aided geometric design model to be a series of geometric relationships, with the final output being a set of *points* for *polygons* that are displayed as a *polyhedron.*

2. *Explosion-Implosion* Polyhedral Transformation

Haresh named a polyhedral transformation as an *Explosion-Implosion*. In **Figure 1 an Explosion-Implosion** is seen as a *vertex exploding* to a *face,* then *imploding* to a *vertex.* An *edge explodes* to a *rectangle,* then to a *square*. A *square implodes* to a *rectangle,* then to an *edge*, that is orthogonal to the initial *edge*. A *face implodes* to a *vertex,* then *explodes* to a *face.* In **Figure 1** a *vertex* of a blue icosahedron **(a)** *explodes* to a red *pentagon* **(b,c,d,e,f)** of a red dodecahedron **(g)**. An *edge* of an blue icosahedron **(a)** *explodes* to a green *rectangle* **(b,c),** then a green *square* **(d)** and *implodes* to a green *rectangle* **(e,f),** and then to an *edge* of a red dodecahedron **(g).** Finally, a blue icosahedron *face* **(a)** *implodes* to a blue *vertex* of a red dodecahedron **(g).** Note that an *edge* of the blue icosahedron **(a)** is orthogonal to an *edge* of the red dodecahedron **(g)**.
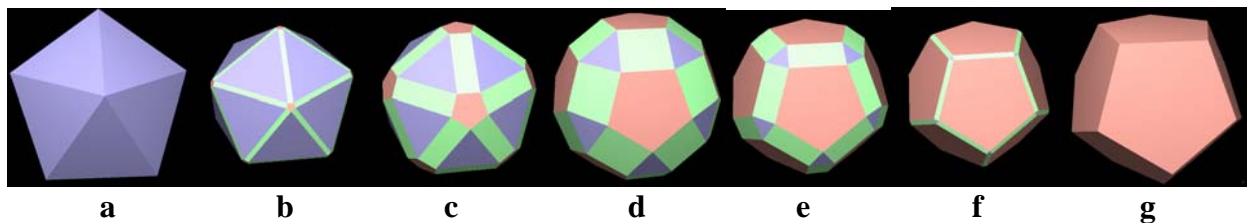

Figure 1: *Vertices, edges,* and *faces* are *exploding* and *imploding.*

3. *A Hypercube* with a Unit Edge Length as a Reference Space

The 3 *plane* model has a *cube* with unit *edges* for a reference space for its transformation as there are 3 Explosion-Implosion transformations for a vertex, an edge and a face **Figure 2a**. The 10 *plane* model has 4 *foundational polyhedra* using a *hypercube* with a unit *edge* as a reference space for its polyhedral transformation, **Figure 2b**. A point in a unit edge length reference space has coordinates that vary between 0 and 1 and represents a different *polyhedron.* A *regular polyhedron* from these models has edge lengths that vary between 0 and 1. Corner (1,1,1) of the reference *cube* is the same polyhedron as corner (1,0,0,0) of the reference *hypercube*. Corner

(1,1,1,1) of the reference **hypercube** has 10 faces visible in the fundamental region, 1 face for each of the 10 **vectors** and 10 **planes** in the fundamental region of the model **Figure 2c** and **Figure 3a**.
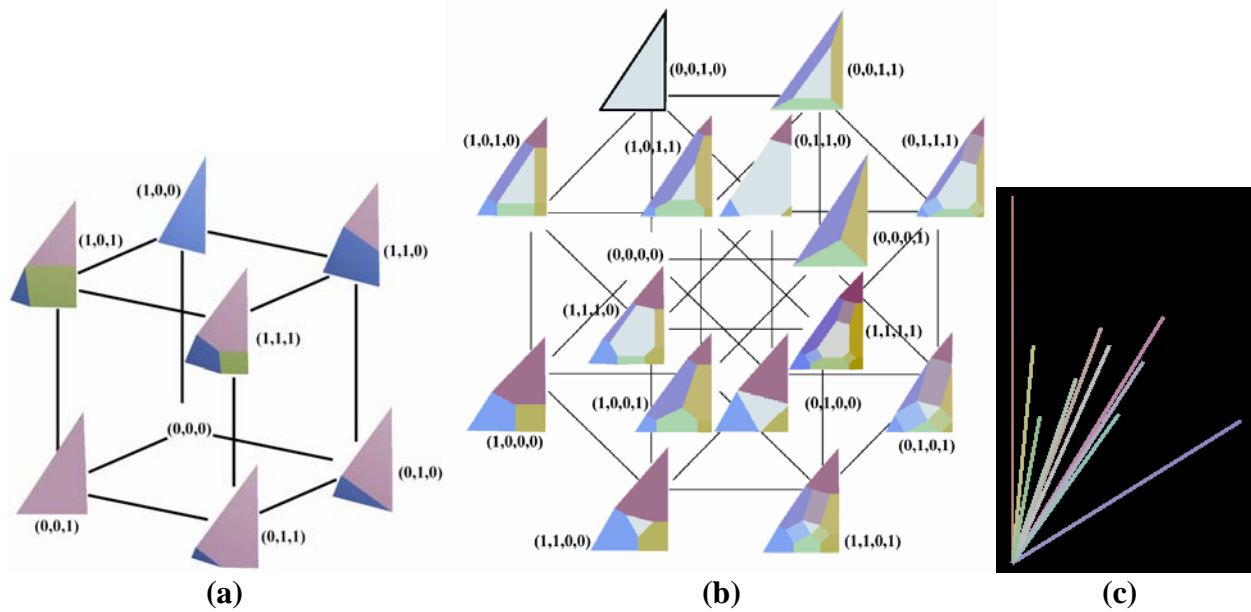

Figure 2: A reference **cube (a)**, a **hypercube (b) 10 normal vectors** of **fundamental region(c)**.

4. *Fundamental Region* for a 10 Plane Model

Three **primary faces;** red(1), green(2) & blue(3) for a **regular polyhedron 3(b)** are orthogonal to symmetry **vectors**, at the corners of the **fundamental region**. Three **secondary faces;** yellow(4), cyan(5), and magenta(6) for a **stellate 3(e)** are orthogonal to **vectors** at the mid-sides of the **fundamental region**. A **tertiary** pink **face** (7) is **orthogonal** to a **vector** between a red & white **face** and a magenta & yellow **face**. A **tertiary** lime **face** (8) is **orthogonal** to a **vector** between a green & white face and a yellow & cyan face. A **tertiary** aqua **face** (9) is **orthogonal** to a **vector** is between a blue & white **face** and a cyan & magenta **face**. In the center is a white **face** (10) for the 3 **truncate faces Figure 3c** & a single **dual face Figure 3d.**
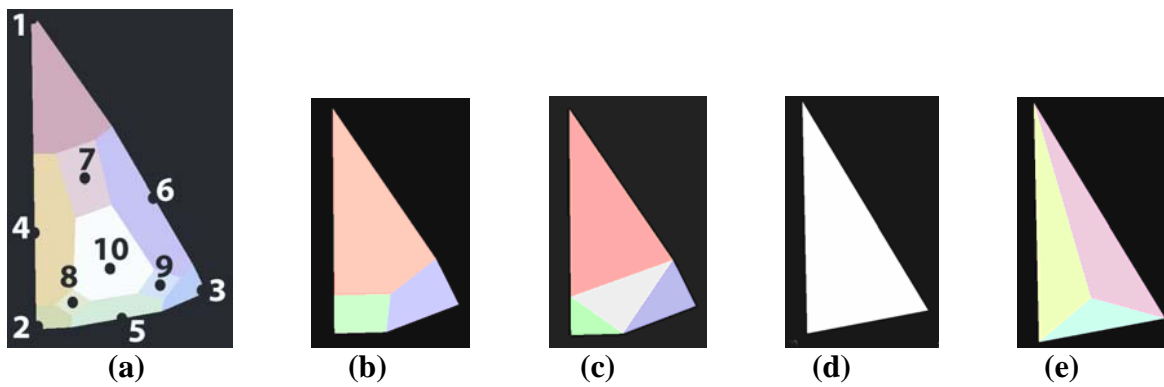

Figure 3: *faces* **(a)**, *regular* **(b)**, *truncate* **(c)**, *dual* **(d)**, *stellate* **(e)** of *fundamental region*.

5. Polyhedra from the 10 Plane Model with *Hypercube* Reference Space

A *polyhedron* with different *edge* lengths is considered a different polyhedron. *Point* (1,0,0,0) represents a *regular polyhedron* with icosahedronal symmetry (5,2,3), **Figure 4a**. A *truncate polyhedron* is referenced at *point* (0,1,0,0), **Figure 4b,** a *dual polyhedron* is referenced at *point* (0,0,1,0), **Figure 4c,** and a *stellate polyhedron* is referenced at *point* (0,0,0,1), **Figure 4d**. The polyhedra referenced at the other corners of the *hypercube* appear in **Figure 4e, Figure 5,** and **Figure 6.** There are 15 non-null *polyhedra* at the corners of the *hypercube* reference space.

Interacting with software to observe the display of polyhedra continuously changing shape, has been very stimulating for those who are interested in polyhedra. Will and I plan a video documentary of an instructive experience as well as author it to a DVD.
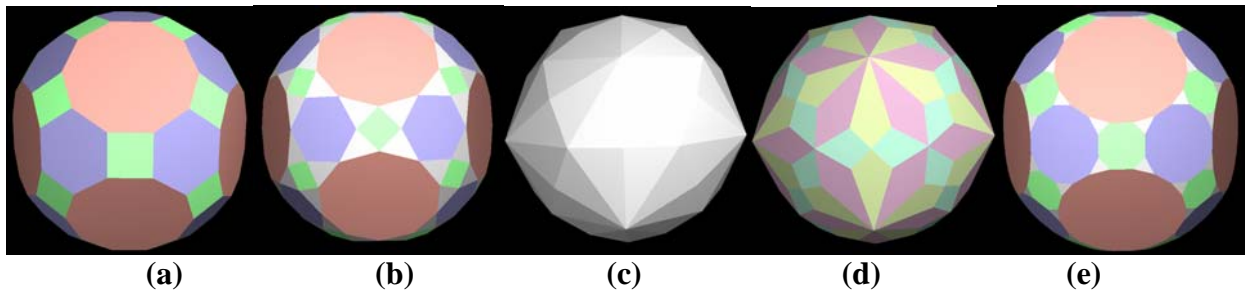

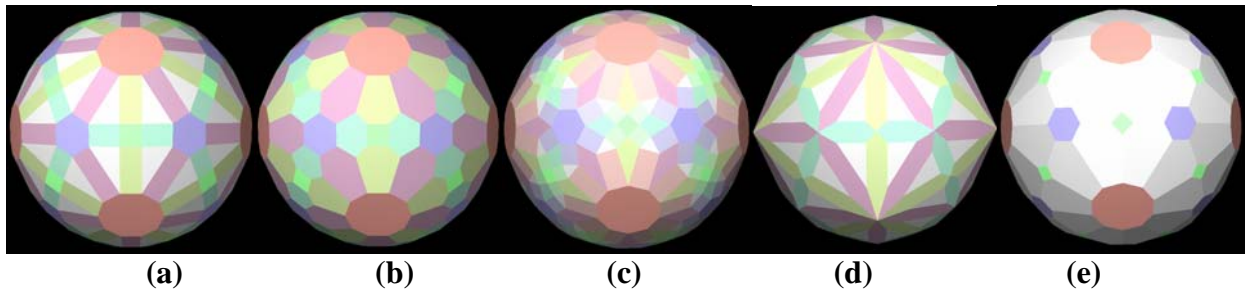**Figure 4**: *Regular*(1000) **(a),** *truncate*(0100) **(b),** *dual*(0010) **(c),** *stellate*(0001) **(d),** (1100) **(e).**

Figure 5: (1010) **(a),** (1001) **(b),** (0101) **(c),** (0011) **(d),** (0110) **(e).**
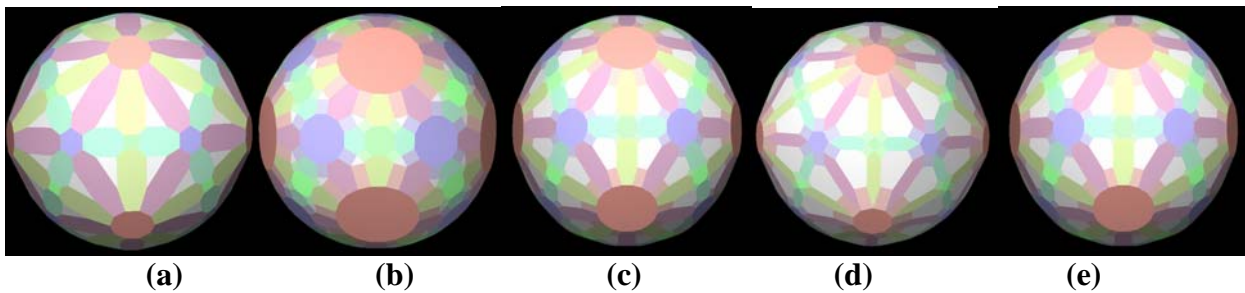

Figure 6: (1011) **(a),** (1101) **(b),** (1110) **(c),** (0111) **(d)** (1111) **(e).**

6. Implementation for a 10 Plane *Explosion-Implosion*

6.1. 4 Foundational Polyhedra. This implementation uses 10 ***normal vectors,*** **Figure 2c,** to position 10 ***planes*** for each foundational ***polyhedron.*** Once 10 ***planes*** have been positioned, sets of 3 ***planes*** intersect in a ***point***. Sets of these ***points*** combine to form ***face polygons.*** A set of 10 ***faces*** are numbered and displayed in **Figure 3a.** These ***point*** coordinates for the ***faces*** are transformed with symmetry transformations to display full closed convex ***polyhedra*** in **Figures 4, 5, and 6.**

6.2. *Vectors*. Four sets of 10 ***vectors*** were computed from on a ***regular***, ***truncate, dual*** and ***stellate*** polyhedron [4]. For each foundational polyhedron, 1 ***normal vector*** was computed for each of the 10 ***faces*** in the ***fundamental region,*** **Figure 3b**. These ***normal vectors*** lie on the boundary or in the interior of the ***fundamental region*** **Figure 2c**. These 40 vectors are blended in 4 quad-linear equations with 4 coordinates from a ***point*** within the unit ***edge hypercube*** to produce a single set of 10 ***vectors***. These 10 ***vectors*** are normalized so that they become meaningful ***vectors*** for 10 ***face planes.***

6.3. *Planes, Points, Faces*. With this set of 10 ***face planes*** and the 3 ***symmetry planes*** on the sides of the ***fundamental region,*** a set of 21 ***points*** are computed to display 10 ***faces***. Three ***planes*** intersect to compute each ***point.*** This implementation works correctly for ***p, q,*** and ***r*** values of (1,1,1). A solution for the other values of ***p, q,*** and ***r*** between 0 and 1 is currently under development.

Viewing the colored ***normal vectors*** in **Figure 2c** led to an understanding of the locations of ***faces*** for this ***10 plane computer model.*** The ***fundamental regions*** in **Figure 7a** displays ***implosion-explosion*** transformations of ***vertices, edges,*** and ***faces*** by the 10 plane model.

7. Display and Interaction

7.1. Display. In addition to displaying a ***polyhedron*** and a variety of parameters and their values are displayed in two windows to observe while interacting with the 10 ***plane explosion-implosion*** computer model, **Figure 7b**. One window gives the user a view of polyhedra continuously changing shape and rotating to change the view. Some parameters change value continuously while other parameters change value discretely. The **Modern Computational Geometry Visualizer** window and the **Controls** window **Figure 7b** displays values and provide an opportunity to change values. The P:5, Q:2, and R:3, are the symmetry parameters for the ***vertex, edge,*** and ***face*** symmetries for the icosahedral family of ***polyhedra***,. The p:1, q:1, and r:1, are 3 parameters for the 3 plane model [1]. The r:1, t:1, d:1, and s:1 are the 4 parameters for the 10 plane model. The zoom:value of 294.73%, provides the user with the relative size of the polyhedron being displayed; when the zoom factor is the same for two different displays the two polyhedron are in the same size relative to each other. In general, the 10 ***plane polyhedra*** are larger than the 3 ***plane polyhedra*** so that it needs a larger zoom factor to be seen in the display window. A mouse is used to either zoom the view in or out with the wheel, or rotate the displayed polyhedron for a different view by moving the mouse in the **Visualizer** window as if rotating a roller ball.
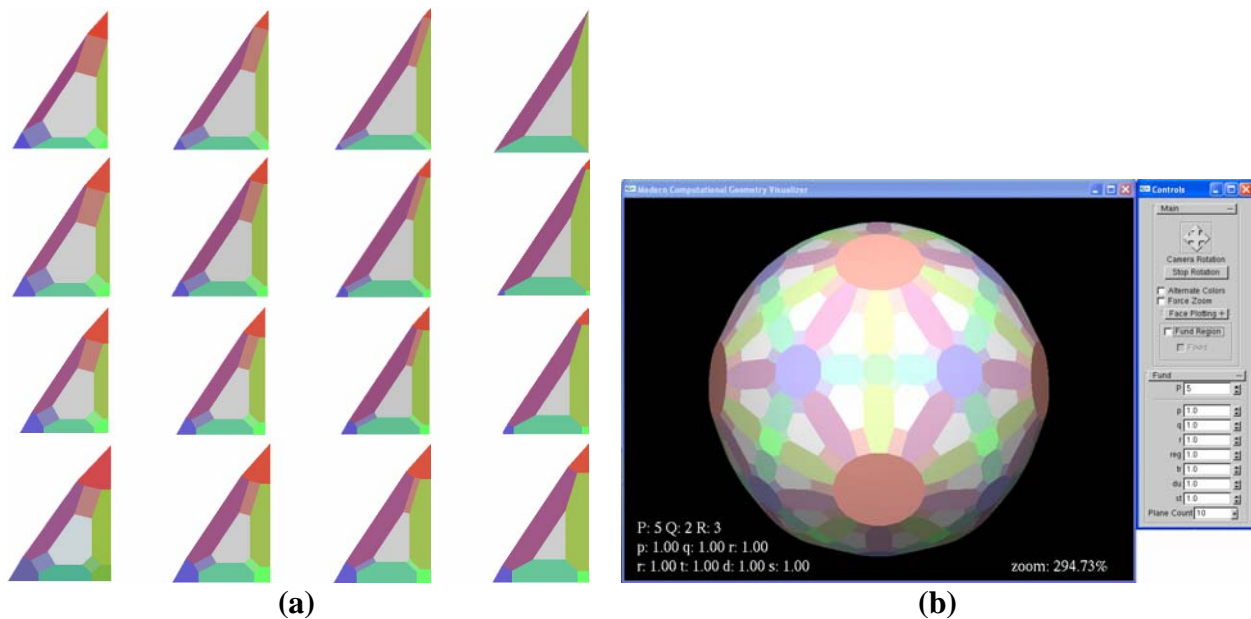
Figure 7: 16 *fundamental regions* **in 4** *squares* **of a** *hypercube* **(a) & model windows (b).**

7.2. Controls. The **Controls** window provides handles to interact with the 10 plane *explosion-implosion* computer model. The **Camera Rotation** continuously rotates the polyhedron, where the **Stop Rotation** ends this continuous rotation. The **Alternate Colors** uses two tints of a hue to accentuate the *faces* of two adjacent *fundamental regions,* **Figure 8a.** Polyhedra are considerably different in size because of the edge length of the different *polyhedra*. The **Force Zoom** control helps the user see a full polyhedron quickly. The **Fund Region** control displays only the *faces* of the *fundamental region,* **Figure 8b,** where the **Fixed** control displays the *fundamental region* with the lower left corner of the green *face* is in the center of the screen and the bottom edge and the right edge are parallel to the sides of the screen, **Figure 8b.** The input for symmetry parameters of the 10 plane model can be changed from **P** being 3 for tetrahedral, to 4 for octahedral, and to 5 for isocahedral families of *polyhedra*. The **p, q,** and **r** are the coordinates of a *point* in a unit edge *cube* for the 3 *plane* model. The **reg, tr, du,** and **st** are the coordinates of a *point* in a unit edge *hypercube* for the 10 *plane* model. Plane count allows the user to choose between the 3 *plane* model and the 10 *plane* model.
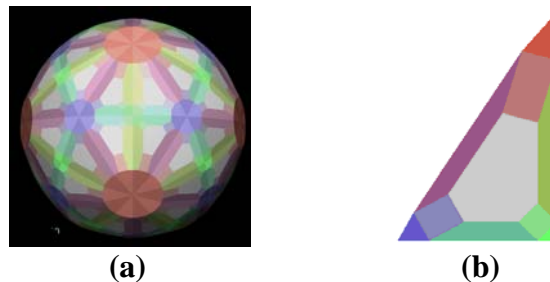

Figure 8: Alternate colors for polyhedron (a) and a fundamental region (b).

7.3. Colors. The 10 *faces* were organized into 4 sets for coloring. A first set of 3 *faces* was the primary colors of red, green and blue for a *regular polyhedron*, **Figure 4a**. A second set of 3

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faces were the secondary colors of yellow, cyan, and magenta for a ***stellate polyhedron***, **Figure 4d**. The *face* for a ***dual*** polyhedron, **Figure 4c,** was white and *faces* for a ***truncate polyhedron***, **Figure 4b** were red, green, blue, and white. The final 3 *faces* were pink, lime, and aqua as blends between red and white, green and white, and blue and white **Figure 8b.** The hue of each color has been adjusted, attempting to produce displays, where the *faces* appear to be adjacent to each other in their 3-dimensional viewing space.

7.4 Plotting. The **Face Plotting** option in the **Control** window displays *faces* that could be cut out to build a physical model of a **polyhedron** from the 10 **plane** model, **Figure 9**.
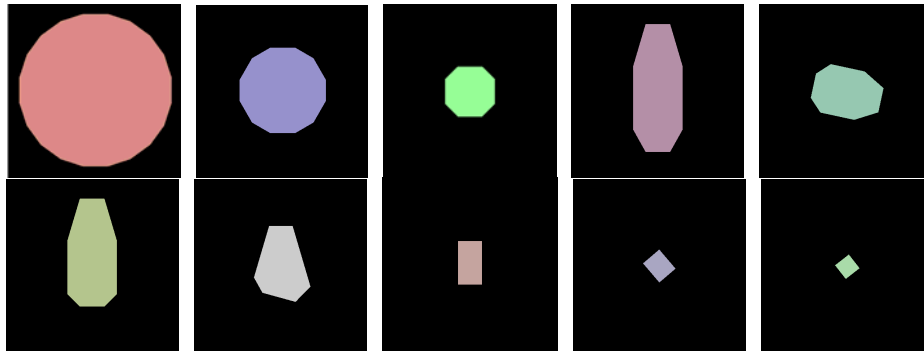

Figure 9: 10 Flat faces for plotting.

8. Animation, Documentary, Interaction.

8.1. Animation. While a staff scientist at NYIT Computer Graphics Laboratory from 1980 to 1990, a 3 *plane* computer model for an **explosion-implosion** polyhedral transformation was animated. A range of shape changes for this computer model was presented in 10 minutes. There was a positive response to the first showing of this animation at a conference of geometers at Smith College in 1989. It was very excited for me to have Coxeter view the animation as part of a solution for the computer model. The algebra for this polyhedral geometric model started with his writings [4]. After viewing the animation he said, "This work is for a next generation of geometers". The animation introduced viewers to a dynamically changing shape displayed for the *explosion-implosion* of a 3 *plane* computer model.

8.2. Documentary. In the fall of 2006 an interactive program was started that would further assist in the understanding of this 3 *plane* model by utilizing an interactive display instead of an animation. We started with old C code and knowledge of C++ code, OpenGL and GLUT to produce an interactive program exhibiting a solution to a 3 *plane explosion-implosion* computer model. At first we aspired to tell the same story that was told in the animation. We changed our thinking to fit an interactive display to be video taped. A documentary of Will interacting with the code was produced. This documentary can be seen on a DVD and is considerably different from the animation, but it displays a similar range of shapes that appear in the animation, however it produced a considerably higher level of energy utilizing an interactive display.

8.3. Interaction. The 3 *plane* model has been extended to a 10 *plane* model. While at NYIT in 1989, while visiting with Pat Hanrahan, when he was faculty at Princeton, an initial result was

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obtained [5]. This initial result has been extended into an interactive program for the 10 *plane* model. As the 10 *plane model* work developed values for parameters were displayed as well as the opportunity to interactively change values for parameters. Viewing this interactive *explosion-implosion* polyhedral transformation has proven to be considerably more stimulating. As the complexity of the polyhedron grows so grows the stimulation from viewing the changes of shape. Initially animation was helpful in telling a story, eventually interactivity became a more powerful experience to tell the story of a dynamic phenomenon of polyhedra changing shape.

9. Conclusion

A 10 *plane explosion-implosion* polyhedral transformation was implemented to provide an interactive display of polyhedra that are well-behaved structures in a 3-dimensional space. Each polyhedron is referenced by a 4-dimensional *point* from a unit edge *hypercube*. Lalvani's *explosion-implosion* polyhedral transformation was conceptually clear from his dissertation and his exhibit. A computer model was derived for this concept by writing C code, in the late 1980's, producing computer graphics animation as a result in 1989. More recently, Will Hawkins and I have been able to update the 1980's code for an interactive display.

Animation, documentation, and interaction are different media for telling a story. When attempting to tell a similar story the different medium accentuate their strengths. Given a choice, interaction is the media that generates the highest level of energy. It does not mean that the others media are less valuable, only that they are different in that the technology to support each is very different. Animation, documentation, and interaction each have a significant value and all 3 help students to better understanding a 3-dimensional dynamic computer aided geometric model.

Acknowledgement

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