A Sangaku Revived

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Abstract

Abstract In this paper we give an account on our mathematical and visual explorations inspired by a sangaku. First we introduce sangakus – traditional Japanese mathematical tablets. Then we give four examples of our static contemporary variants. Finally, we discuss in detail how a fifth sangaku led us to simulate the growth of water lilies, as a means of visualizing the problem. This approach lead to the mathematical field of circle packing, and made it possible to experience the visually intriguing process with different settings of the algorithm.

Keywords: sangaku, history of mathematics, circle packing, simulation.

1. About sangakus

The word Sangaku means 'mathematical tablet' in the Japanese. We know of some 800 sangakus preserved from the many more produced in the country from the beginning of the Edo period (1603-1867). Today these tablets are kept in museums, but originally, they were hanging under the roof of Shinto shrines and Buddhist temples. The tablets depict one or more mathematical problems or theorems with drawings which are not pure illustrations as one sees in modern geometry books. Much care was taken for the visual presentation, considering arrangement, size, usage of color and style – a sangaku should look beautiful and appealing, see Figure 1 taken from [3]. Often there was very little description added – one had to understand what the drawing was stating: a theorem in planar geometry, most often about circles and ellipses. The viewer, once caught by the sight, had to formulate the problem himself, and afterwards, felt challenged to find a proof. So the tablets were like a public display of geometrical problems. But by whom were they made, and for whom?

There are bits and pieces of facts and speculations about the original ritual of placing problems on display at religious locations. In the Edo period Japan was sealed off from the outside world. The Japanese were on their own, without any knowledge of the European developments when doing sciences, and particularly, mathematics. Shrines and temples served as centers of knowledge and education, and attracted pilgrims from all over the country. Thus they were the best locations to 'announce' new findings in geometry. But it also possible that the placing of the theorems was meant as an acknowledgement for the intellectual power that made their discovery possible. Or, yet a step further, the sangakus could be offerings to the gods: items of beauty and eternity. Indeed, sangakus are beautiful not only from the point of view of visual aesthetics, but because of their mathematical nature. Many of the problems invite the viewer to further ponder and explore the topic raised. Sangakus are real gems of mathematics, problems which come to a life by the intellect of the beholder.

The making of sangakus was not a privilege of the scholars. According to signatures, there are sangakus made samurais and housewives, and even by children [1]. The habit of placing sangakus lived up to the beginning of the 20th century. In the 19th century, sangakus started to appear in printed collections in Japan – some only survived thanks to these publications. The Japanese high school teacher Hidetoshi Fukagawa traced and deciphered many of the original sangakus, and made many of them available in English too [2]. The interpretation of the problems, even with written annotations, is far from a trivial task, as the ancient scholarly language of the tablets is very different of today's Japanese. There are a few more resources available for the Western public in print [3-4] and on-line [5-9].

Figure 1: *Original sangauk, photo from [3]*

2. Four contemporary Dutch sangakus

Sangakus, even if out of their original context, can be engaging for people from a distance of hundreds of years and thousands of miles [10]. This has happened to us too. In 1999 with the Dutch digital artist Ineke Lambers we set out to make a series of end of 20th century sangakus. I selected 5 problems which I found intriguing, and yet simple enough for a general public. Ineke created the visual representation, by processing digitally photos taken near to her home in the North of the Netherlands (not far from the location of Bridges 2008). You can see the result of a series of 4 sangakus in Figure 2. You, just as those looking at the originals under the protection of the roof of a shrine, are invited to formulate the message of them, and find an – of course beautiful and elegant – proof too. Solutions (in Dutch) are given at [11].

3. Revival of the fifth sangaku

We could not just forget about the fifth sangaku 'which did not make it', as the highly symmetrical drawing did not lend itself to a visually exciting presentation, see Figure 3. In the rest of the paper we tell about our explorations of it.

3.1 Some mathematical investigations

What is the mathematics behind our sangaku? What is it asking about? Is shows 5 circles of the same size, arranged within the square without overlap – this is called shortly a packing, in general. In our special case, what is the radius of the circles in the drawing? Figure 4 shows the solution.

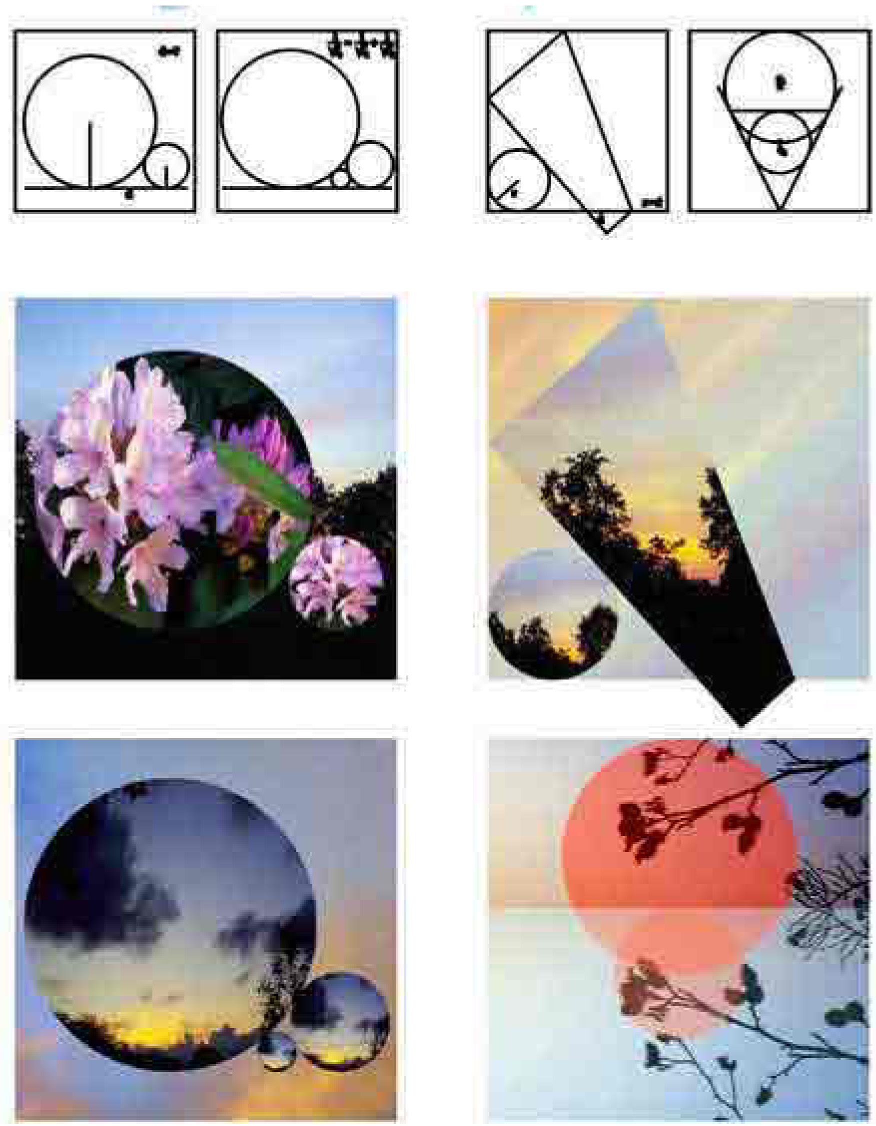

Figure 2: *Four contemporary sangakus. Above the geometrical problems, below the artistic graphical designs by Ineke Lambers.*

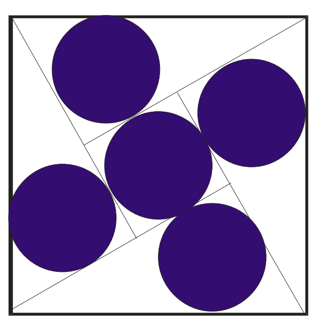

Figure 3: *The fifth sangaku.*

Figure 4: a) *Computing radii of identical circles. b) The case when only the corner circles are identical.*

How can one construct such an arrangement, with ruler and compass? Thinking of the culture of origami, the art of paper folding in Japan, one may be interested in folding it from a piece of square paper, like a paper serviette. And how about if we want a given coin to be the size of the circles – can we fold the cage surrounding a given size of circles? One may ponder what happens if we draw arbitrary, but identical triangles on the sides of the square? In the center we always get a square, and we can define the radii of the internal and the four outer circles, see Figure 4.b.

3.2 Simulation of growth of water lilies

The above sangaku, though interesting for one with an eye to geometry, was not inspiring from a visual point of view, due to its high symmetry. And that is where this sangaku started its own life … In search for a visual analog, the circles touching each other reminded me of water lilies floating on the lake near to my home, and also, on the lake of Monet's garden in Giverny [12]. What if the sangaku is the snapshot of a dynamical process: water lilies growing in a pool, as long as there is room for them? Just as on a real lake, they should push each other out of their way in order to gain space to grow. This idea was inspiring, both mathematically and artistically. For the visual simulation, processed photos were used for the grey water as background, the lilies to be scaled to size, and the reed in the foreground.

From a mathematical point of view, the growth of water lilies in a square pool can be described by the following steps:

1. Select 5 points in the square – these will be the *seeds* of the lilies.
2. Check if all lilies are *free*, that is they have some space to grow.
    a. If yes, let them grow a little, and go to Step 2.
    b. If not, check if any of the lilies got *stuck*
        i. If yes, the growth process is ceased – STOP.
        ii. If not, move the lilies so that they get free space around, and go to Step 2.

The simulation is based on two assumptions: each water lily keeps growing as long as has the space to do so, without overlapping other lilies; and a water lily moves in the direction of the sum of the forces emerging due to touching other lili(es) and/or wall(s) of the pool. Before going into the details of the algorithm, let's have a look at the possible outcomes in a qualitative way. In the first 3 cases, all the 5 lilies got stuck: none of them could grow further, neither could any of them move to gain space. In Figure 5.d) and e) the process ended when 4 lilies got stuck. Figures 5 f) and g) show ways of ending up with 3 lilies stuck.

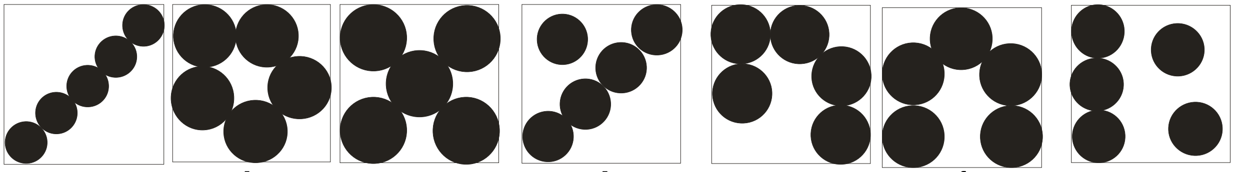

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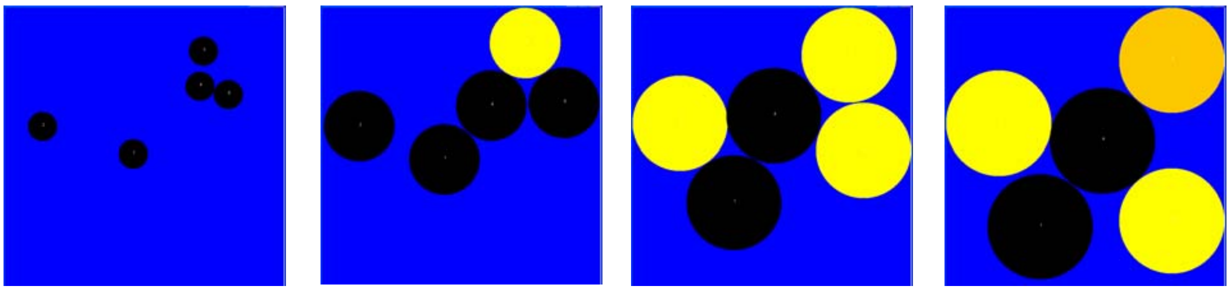
Figure 5: *Possible end situations of the growth of 5 circles.*



a b c d

Figure 6: *Snapshots form a single dynamical growth process. Thelight or dark yellow color indicates if a circle is touching 1 or 2 walls.*

These possible end results raise a number of mathematical questions which we will address in more detail in the next chapter. For discussing the end-results of our simulations, it is enough to remember at this point that depending on the choice of the initial seeds, different end configurations may arise.

So let's run our simulation with randomly selected 5 seeds, and watch what is happening. Unless the seeds are selected to be one of the simple and regular arrangements shown in Figure 4, there will be some pushing going as soon as lilies grown big enough and touch each other and/or a wall. This process is fun to observe. Some intermediate, asymmetrical configurations are pleasant to look at as snapshots – see Figure 6. But also the slow movement of the lilies is a pleasing, even engaging experience. In some cases it is hard to predict what the final configuration will look like; there is an element of surprise.

Figure 7: *Snapshots of a growth process – with rendering as water lilies.*

3.3 On circle packing

Our sangaku, and especially the interpretation we found for a pleasing graphical representation, gave rise to another path to explore. Given a square, one can pack into it 5 circles of identical size in different ways. Which is a maximal packing, that is a packing when the size of the circles is maximal? Our sangaku shows a packing, but this is not a maximal packing: the 4 circles can be moved in such a way that each circle can grow further. The maximal packing is shown in Figure 5.c , and it is easy to justify that the shown arrangement is indeed the best possible packing. However, things get difficult if one asks the same question but for a larger number of circles:

> Given a square with a unit sides, we pack n identical circles into it. What is the maximum of the radius of the packed circles?

The solution is also called the *densest packing.* It is obvious that some (but not necessarily all) circles in a densest packing are blocked: touch other circles and possibly some of the sides of the square [15].

The issue of densest packing has kept mathematicians busy. One can find many real-life tasks which can be formulated as a packing problem, and hence, the 'best packing' has practical relevance. Farkas Bolyai studies the problem as a basis for the optimal planting scheme for trees [18]. When cutting out circles from paper or metal sheets to make lids for jam bottles, the densest packing will yield in the least loss of material. When transporting e.g. barrels on the plateau of a van, they should be arranged according to the optimum packing, in order to use the capacity the best. When looking at the 3d equivalent of the question, we find that already Kepler conjectured in 1611 that the most efficient packing of balls (in his time: gun balls) in a box is the one used intuitively on markets, when piling oranges [13]. Interestingly, this assumption was proven only in 1998 by Hales and Ferguson, with the aid of computer programs [14]. As the correctness of the computer proof could not be fully verified by a team of 12 people in more than 4 years, there is still some uncertainty if the conjecture has been really solved. In order to come to a certainty, Hales launched the Flyspeck project (Formal Proof of Kepler) in 2003, this time to use computers to automatically verify every step of the proof. In spite of the fact that the computers will be doing the real job, this project too requires 20 person-years of labor, according to Hales [14].

But it is not the practical value but the intriguing nature of the 'hunt for the optimum' which keeps the minds – and computer programs – running even today, in pursuit of the best packing of circles in a square, and containers of may other shapes [16, 17, 18].

But what has this to do with our simulation of water lilies? Well, whenever our algorithm stops, we end up with a packing, albeit not necessarily with an optimal one. So running the simulation many times with different, randomly selected starting points as seeds, we can select the best of the resulting packings. If for a huge number of further simulations we do not find a better arrangement, we may assume that what we have found so far is the best packing. However, this remains a conjecture which needs to be proven. The found arrangement can help to 'see' why it is optimal, or where it could be improved. Actually, this very idea, as a kind of stochastic search was used by mathematicians to find promising dense packings [18].

Taking a closer look at the computer program outlined in 3.2, we find challenging issues there too. The final force pushing a lily can be computed as the vector some of forces emerging between contacting lilies and the walls. If there are forces, but the sum is 0, the lily is stuck. Otherwise it will move in the direction of the total force. In our simulation, we used small unit movements, decreasing with time and not bigger than the space allows, for displacement of all lilies, at this point deviating for the physical analogy where displacement would be proportional to the forces. Also the growths factor is chosen uniformly, and decreasing with time. This assures that at the beginning the growth process is fast, but as the packing becomes denser, the speed decreases. Another point which had to be taken care of is the effect of computers working with approximating values for irrational numbers like sqrt(2). In large simulations, small intervals are taken as representations of absolute values of numbers. We solved this problem by allowing a small error margin for the 0 distances. Finally, the algorithm requires in each step to check if any two lilies are touching. This, computationally expensive operation has to be cut down for experimenting with a large number of circles, by keeping track of circles in the close neighborhood of each other, and thus checking collision only for those.

We have also experimented with non-parallel growth, and with more than 5 circles. In the original version one single lily which is blocked delays the growth of all other ones. In a non-parallel version, lilies grow individually as long as they have the space. This variant results in visually appealing arrangements, see examples for different number of circles in Figure 8.

The reader is invited to experiment with different number of circles, way of growth an visualizations, by using the Java applet at [19].

Figure 8: *Snapshots from non-parallel growth, with 5, 7 and 9 lilies.*

Acknowledgement

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