From Sierpinski Triangle to Fractal Flowers

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Abstract

We describe an iterated function system consisting of transformations defined by a pair of complex parameters. Each of the transformations maps the unit circle into itself. For one assignment of the parameters the limit set is the famous Sierpinski Triangle. By "continuously" varying the parameters the limit set appears to erode into various shapes, some suggesting a dried river bed and others suggesting fractal flowers.

1. An iterated function system (IFS) defined by a simple replacement rule

We begin with an Iterated Function System (IFS) defined by a simple replacement rule. We will denote by C the unit circle, that is the set of points $z = (x, y)$ in the complex plane such that $|z|^2 = x^2 + y^2 = 1$. At step 1 we will replace C by three smaller circles that are tangent to C and a fourth circle centered at the origin and tangent to the other three as in Figure 1. At step n we replace each of the step n-1 circles with four smaller circles. Figure 1 shows the first three steps.

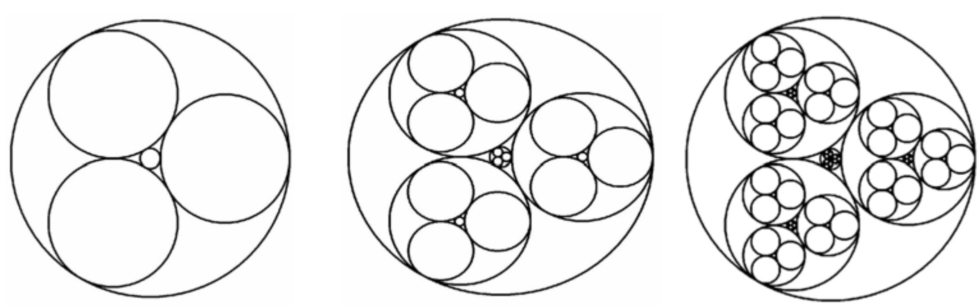

Figure 1: *The first three stages in the iterated function system*

Consider all compositions of these maps. Since these maps are contractions this forms an iterated function system (IFS). The *limit set* of the IFS is the set of points that are invariant under the semigroup generated by the four transformations. We note that the point of tangency of two of the Step 1 circles is not in the image of the second iteration and therefore is not in the limit set. Iterating again we observe that all the points of tangency of the circles will not belong to the limit set, and thus the limit set will be a "Cantor Set", or totally disconnected. In fact, the limit set is, except for the images of the small center circle, the famous Sierpinski Triangle.

2. Möbius Transformations and the Unit Circle Group

We need to define the four transformations T_i , $i = 1,\ldots,4$ that map the unit circle to the four small circles pictured in the first frame of Figure 1. The easiest way to do this is to let T_1 be the transformation that first shrinks the unit circle and then translates it so that it is tangent to the unit circle at (1,0). Then T_2 and T_3 can be obtained by multiplying T_1 by $e^{2\pi i/3}$ and $e^{4\pi i/3}$ respectively. T_4 is just multiplication by a scale factor.

A *Möbius Transformation* is a complex function of a complex variable of the form $f(z) = (az+b)/(cz+d)$. The image of a circle under a Möbius Transformation is another circle or a straight line. Because of this property Möbius Transformations are ideal for use in iterated function systems where we map circles into circles. The four transformations we have just described are all Möbius Transformations. But the limit set of this iterated function system is very predictable. To obtain more interesting limit sets we can compose the transformations from this IFS with another transformation that depends on parameters. Then, by changing the parameters, we can alter the pictures dramatically.

The subgroup of the group of Möbius Transformations called the *Unit Circle Group* consists of functions of the form $U(z) = (uz + v)/(\overline{v}z + \overline{u})$ where $|u|^2 - |v|^2 = 1$. Any transformation from this group maps the unit circle onto itself and maps the interior of the circle onto itself; by changing the parameters u and v we can distort the circle and/or rotate points on the circle before we map it into the smaller circles. By composing a transformation from this group with each of our original transformations we obtain a new iterated function system. As we iterate the transformations the distortions are repeated and, as Figure 2 shows, we can achieve a variety of images.

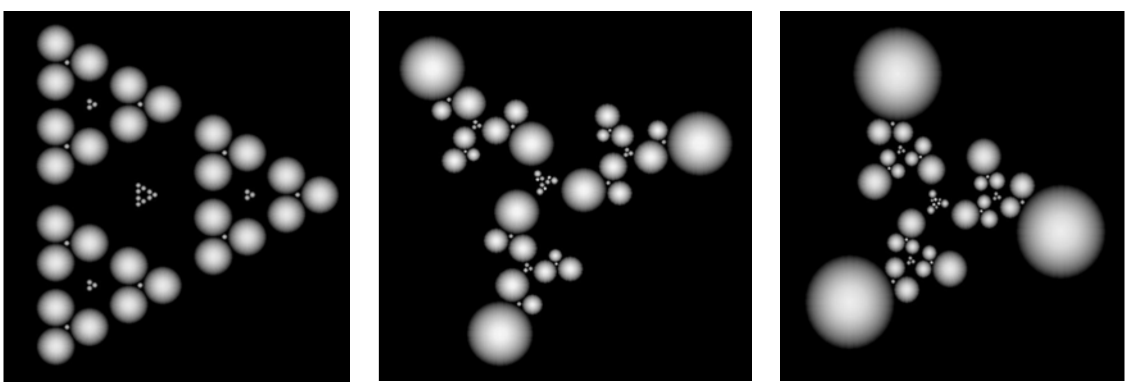

Figure 2: *After three iterations with a variety of parameters*

Figure 2 illustrates the third stage of iteration. In the first frame $|u| = 1.0$, arg $u = 0$ and $v = 0$. In the second frame $|u| = 1.1$, arg $u = .15$ and arg $v = .21$. In the third frame $|u| = 1.232$, arg $u = .476$ and arg $v = 0$. By changing the parameters u and v, "continuously", we can make an animation of the limit set. Of course we cannot change the parameters continuously, but if we change them in small increments, the animation will appear to change smoothly. In Figure 3 we can see a progression of images of limit sets of this iterated function system as we allow the parameters to change.

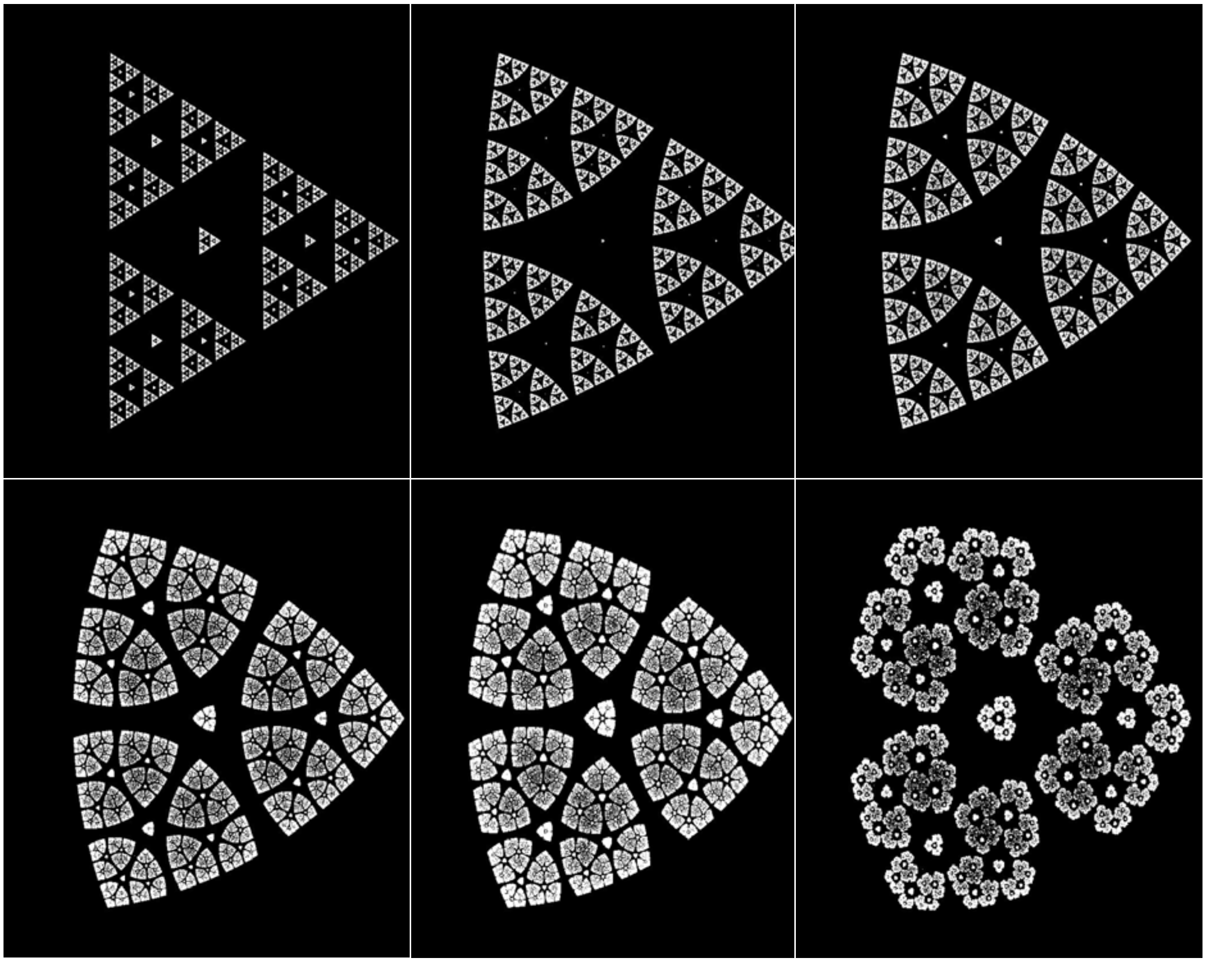

Figure 3: *Changing the parameters*

Figure 4, "Sierpinski Triangle Eroding", which was part of the Art Exhibit at the Joint Mathematics Meetings in January 2008, was made by superimposing a series of images on top of each other. The parameters in the bottom image were $|u| = 1.0$, arg u = arg $v = 0$. The limit set in that case is the Sierpinski Triangle. As the parameters change, the triangle appears to erode. The image bears a remarkable similarity to a picture that appeared in the New York Times, Dec. 3, 2007, of a dried reservoir bed in Thailand. (See Figure 5.)

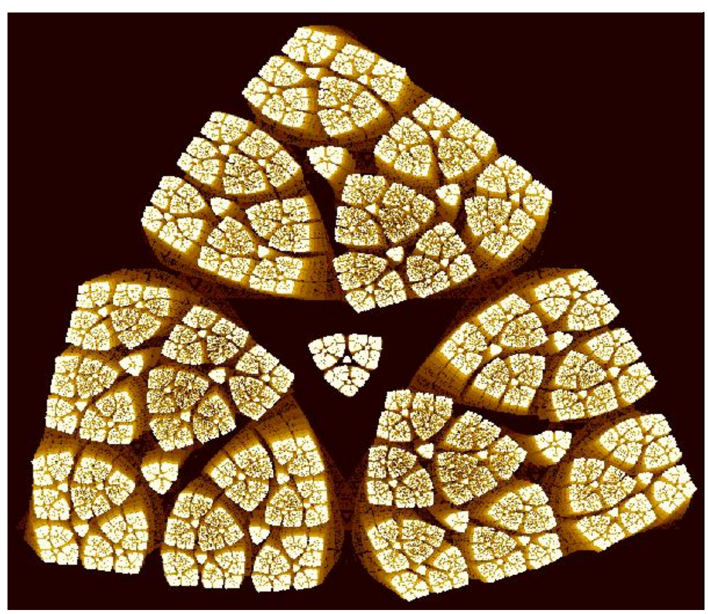

Figure 4: *"Sierpinski Triangle Eroding"*

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Figure 5: *Dried reservoir bed in Thailand (N.Y. Times, Dec.3, 2007)*

3. Increasing the Number of Circles

It is easy to modify the original IFS so that we can use any number of transformations to map the unit circle into circles that are tangent to the unit circle and to each other. Figure 6 shows two examples of iterating three times with five original circles and different choices of the parameters, u and v.

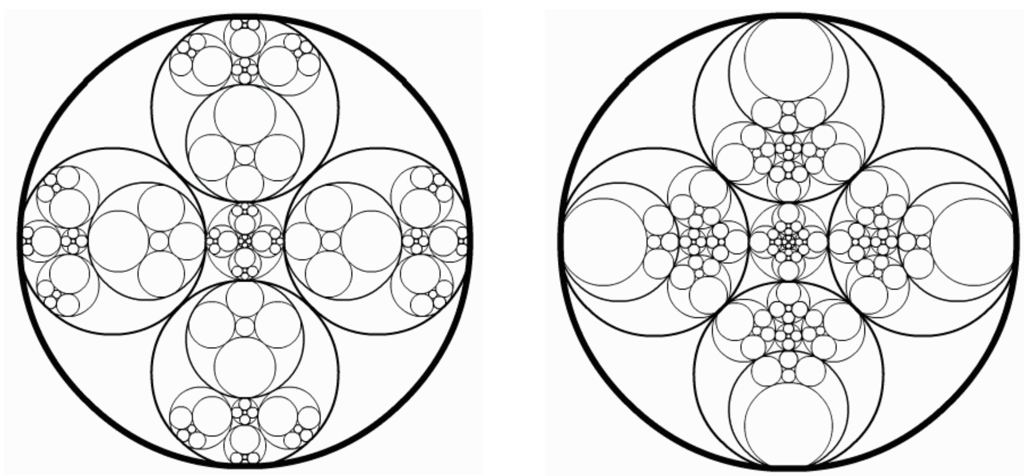

Figure 6: *Five circles at stage 3*

Even a small change in the parameters causes a dramatic change in the limit set. Figure 7 shows two examples of the limit sets for two choices of u and v with five original circles. The left one has parameters $|u| = 1.1$, $\arg(u) = 0$ and $\arg(v) = \pi/2$. The one on the right has the same parameters except for $\arg(v)$ which is $\pi/4$. Changing $\arg(u)$ and/or $\arg(v)$ to angles that are not multiples of $\pi/2$ rotates the original circles as in the right hand figure in Figure 7.

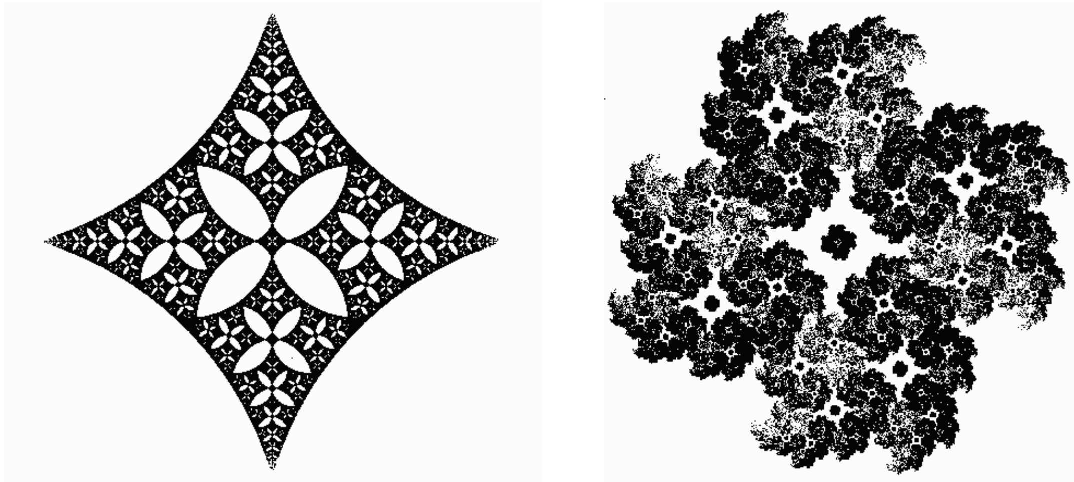

Figure 7: *Limit sets: five original circles*

In Figure 8 we illustrate a sequence of images obtained by changing the parameters in the same IFS with a different number of original circles.

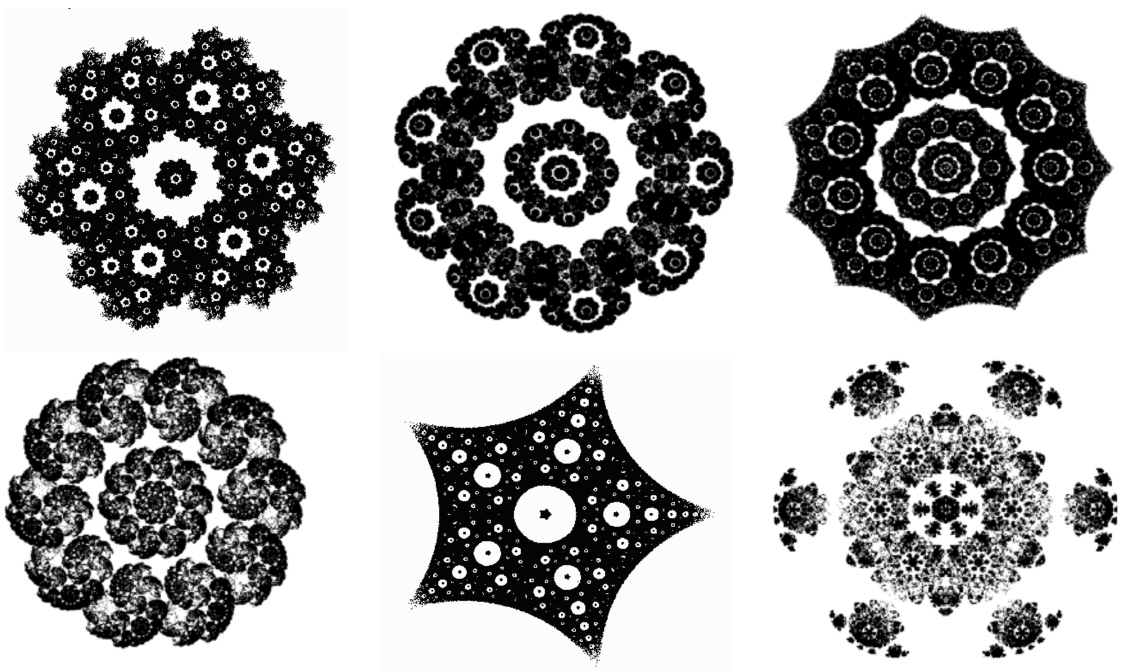

Figure 8: *More limit sets*

4. Conclusion

Iterated function systems make a great topic for an undergraduate research project. Iterated function systems that depend on one or more parameters are a good way to teach students about the effect of small changes in parameters. By incrementally changing parameters we can generate a series of images and then put them together to make an animation.

Another source for mathematical art is to show only early stages in the development of an IFS and make use of color in different and exciting ways. Finally: use your imagination! In Figure 9 a picture is drawn inside a unit circle. Then the circle is mapped to smaller and smaller circles using our same IFS.

Figure 9: *An IFS with flowers*

References

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