Patterning by Projection: Tiling the Dodecahedron and other Solids

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Abstract

This paper reports on the use of projections from duals to the surface of the Platonic solids, in particular the dodecahedron, in order to pattern or tile the faces of the polyhedra.

Introduction

Recent research has considered the patterning (or tiling) of regular solids in a systematic and complete way, avoiding gaps or overlaps and ensuring precise registration. The investigation identified which of the seventeen pattern classes were capable of regular repetition around the Platonic solids, applying only the restriction that the unit cell must repeat across each face in exactly the same way that it does in the plane pattern. It was shown that only certain pattern types, with particular symmetry characteristics, are suited to the precise patterning of each Platonic solid [1, 2, 3]. The investigation focused on the application of the pattern's unit cell to act as a tile when applied to the faces of the polyhedra, placing emphasis on the underlying lattice structure and the symmetry operations contained within it. Polyhedral faces were matched to suitable lattice types. Patterns applicable to the tetrahedron, octahedron and icosahedron were readily constructed on a hexagonal lattice, where the unit cell is comprised of two equilateral triangles. Pattern classes constructed on square grids were suited to repeat around the surface of the cube. This was reported by the authors at Bridges 2007 [1].

The principal concern of this paper is with patterning the dodecahedron. Compared to tiling the faces of the other regular polyhedra, patterning the dodecahedron is more difficult: the dodecahedron's faces are regular pentagons, which because of their five-fold rotational symmetry, do not tile the plane. There are, however, equilateral convex pentagons that do tessellate the plane, such as the well-known *Cairo tessellation* shown in Figure 1. Using knowledge of the Cairo tessellation, the method presented by Schattschneider and Walker [4] provides one solution to the problem of applying a regularly repeating pattern to the dodecahedron. Figure 2 illustrates the manipulation of a pattern based on the Cairo tiling in order to tile the dodecahedron, in which the pattern is projected outwards from the faces of an inscribed cube.

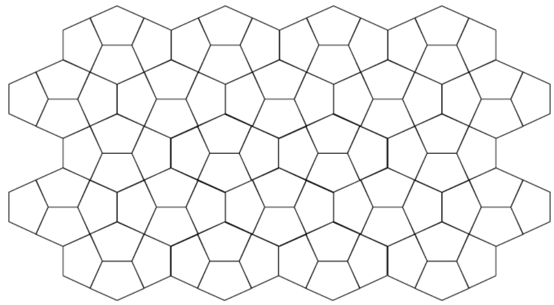

Figure 1: *Illustration of the Cairo tessellation*

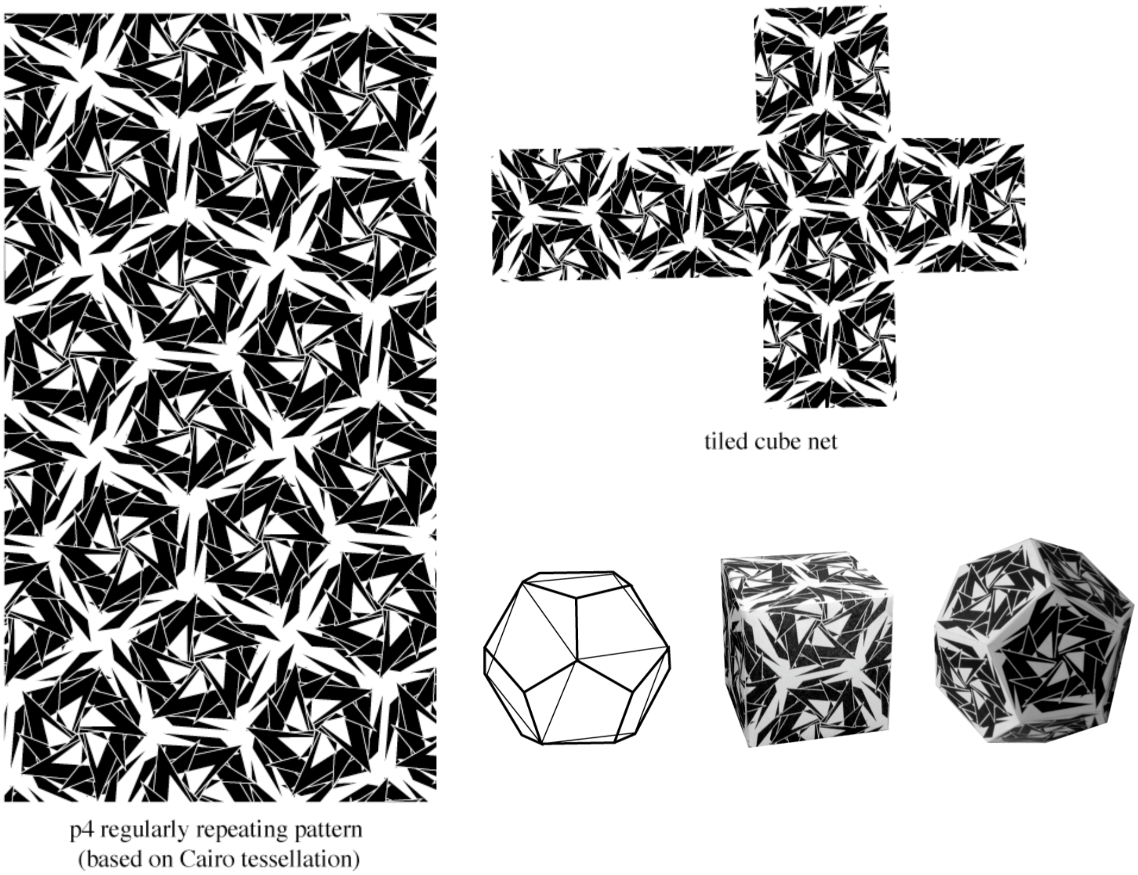

Figure 2: *Illustration of a design for the dodecahedron tiled with a class p4-derived pattern, based on the Cairo tessellation, following projection from the cube*

Patterning the Dodecahedron

The projection method, shown by Schattschneider and Walker [4], for tiling (or patterning) the dodecahedron from the cube suggests that, dependent on the inter-relationships between the solids, polyhedra may be tiled through projection of a pattern from another related solid. As the dual of the dodecahedron, the icosahedron may be inscribed into the dodecahedron, so that the vertices of the icosahedron correspond with the centres of the faces of the dodecahedron. A pattern can be projected outwards, in an equivalent procedure to that of the cube onto the faces of the dodecahedron, as shown in [4]. The use of the Cairo tessellation in patterning the dodecahedron permits the application of patterns derived from classes *p4* and *p4gm*, to repeat across the pentagonal faces. The projection of a pattern from the faces of an icosahedron permits the application of patterns derived from classes *p6* and *p6mm* to the dodecahedron. Figure 3 illustrates an icosahedron, regularly tiled with a pattern derived from class *p6*, inscribed within a dodecahedron. The application of a *p6*-derived pattern to repeat across the faces of an icosahedron is shown in Figure 4. A patterned dodecahedron, which results from the projection of the *p6*-derived pattern outwards from the surface of the icosahedron, is illustrated in Figure 5, alongside a net for the resultant patterned dodecahedron.

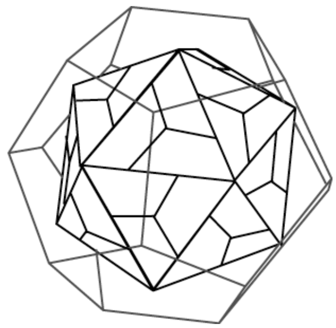

Figure 3: *Illustration of the icosahedron tiled with a class p6-derived pattern inscribed within the dodecahedron*

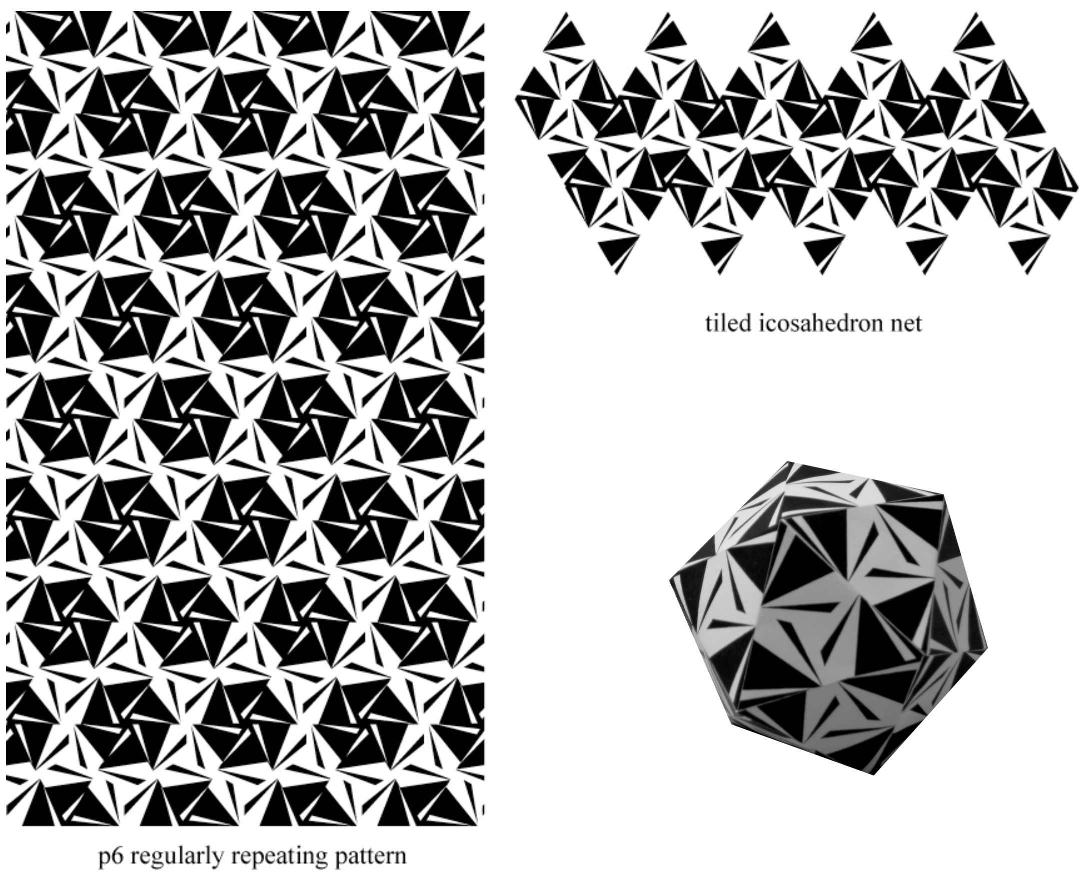

Figure 4: *Illustration of a design for the icosahedron regularly tiled with a class p6-derived pattern*

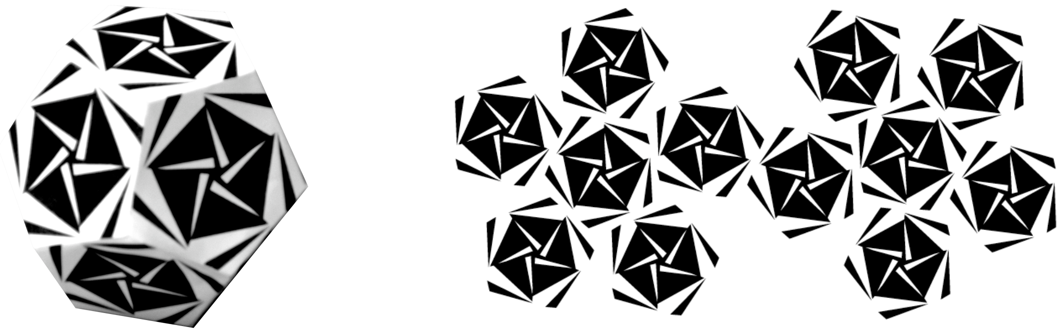

Figure 5: *Illustration of the dodecahedron and corresponding net, tiled with a class p6-derived pattern following projection from the icosahedron*

Patterning the Platonic Duals

It is well established that the Platonic solids exhibit duality, a characteristic that pairs them with one another. The cube and the octahedron are considered to be duals, as are the dodecahedron and the icosahedron. The tetrahedron is considered a self-dual. Projection of a pattern can also occur inwards from the surface of a patterned polyhedron onto its inscribed dual. Figure 6 shows a cube, tiled with a class *p4*-derived pattern, with an octahedron inscribed within it. The application of a *p4*-derived pattern to repeat across the faces of a cube is shown in Figure 7. The patterned octahedron, which results from the projection of the pattern inwards from the surface of the cube, is shown in Figure 8, alongside a net for the resultant patterned octahedron. Table 1 provides a summary of the pattern classes that are directly applicable to tiling the Platonic solids, through the application of an area of the pattern's unit cell as a tile, and also those pattern classes that can be applied through the projection method.

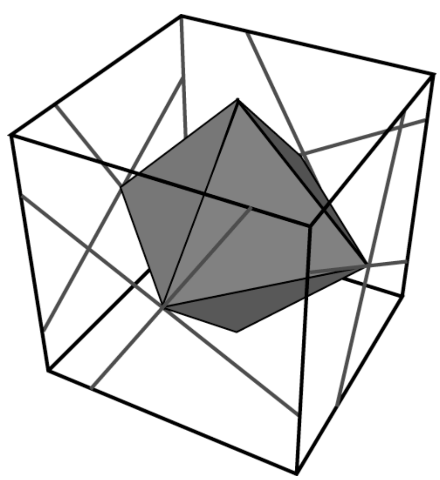

Figure 6: *Illustration of the octahedron inscribed within a cube tiled with a class p4-derived pattern*

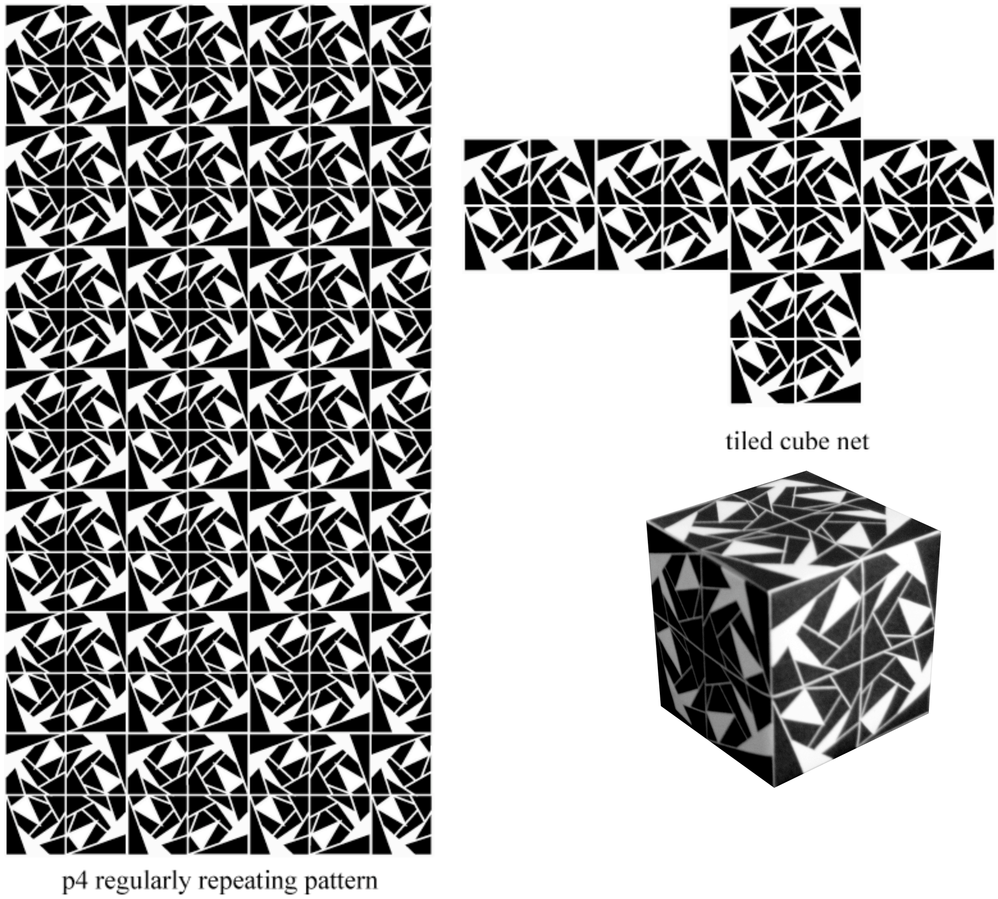

Figure 7: *Illustration of a design for the cube regularly tiled with a class p4-derived pattern*

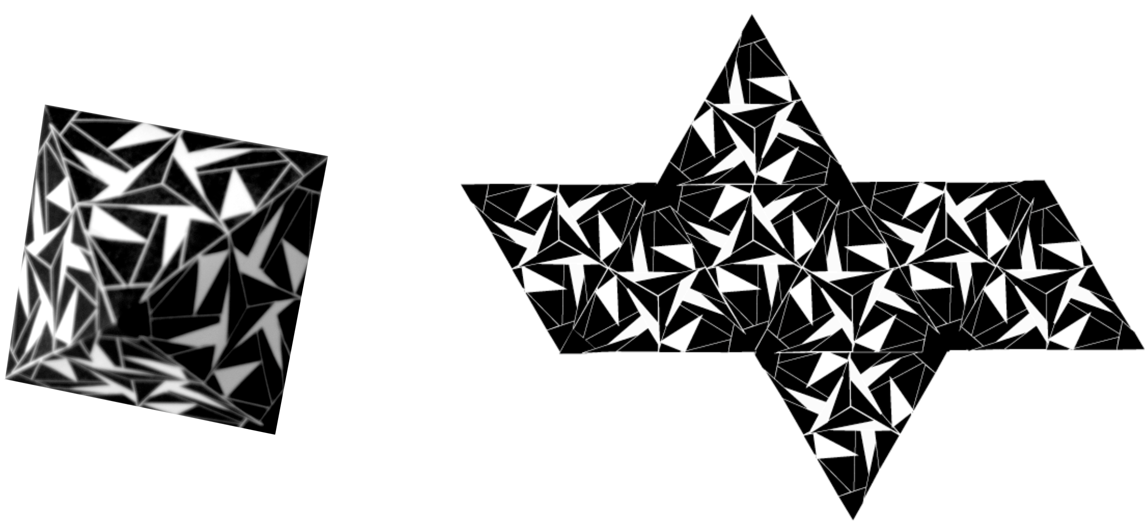

Figure 8: *Illustration of the octahedron and corresponding net, tiled with a class p4-derived pattern following projection from the cube*

Table 1 Summary of the all-over pattern classes applicable to tiling the Platonic solids

Platonic solid	Direct patterning from plane pattern			Patterning through projection from polyhedron	
	Pattern class	Lattice structure	Area of unit cell on face	Inscribed/ circumscribed polyhedron	Pattern class
Tetrahedron	p2	hexagonal	1/2	(self dual)	-
	c2mm	hexagonal	1/2		-
	p6	hexagonal	1/2		-
	p6mm	hexagonal	1/2		-
Octahedron	p3	hexagonal	1/2	cube	p4
	p31m	hexagonal	1/2		p4mm
	p3m1	hexagonal	1/2, 1/6		p4gm
	p6	hexagonal	1/2, 1/6		-
	p6mm	hexagonal	1/2, 1/6		-
Icosahedron	p6	hexagonal	1/2	dodecahedron (via cube)	p4
	p6mm	hexagonal	1/2		p4gm
Cube	p4	square	1	octahedron	p3
	p4mm	square	1		p31m
	p4gm	square	1		p3m1
	-	-	-		p6
	-	-	-		p6mm
Dodecahedron	-	-	-	cube	p4
	-	-	-		p4gm
	-	-	-	icosahedron	p6
	-	-	-		p6mm

Avenues for Further Research

An obvious extension of this enquiry would encompass the Archimedean polyhedra. Such an investigation should uncover different rules for the application of tilings to the polyhedral structures due to the presence of more than one type of regular polygonal face in each solid. The truncated polyhedra may be patterned by manipulation of their Platonic counterparts in a similar manner to that mentioned above. The cuboctahedron, for example, could be patterned through the manipulation of either a patterned cube or a patterned octahedron. By inscribing a cuboctahedron within either a regularly patterned cube or octahedron, the pattern can be projected inwards onto the surface of the cuboctahedron. Figure 9 illustrates a *p4mm*-derived pattern applied to regularly repeat across the faces of a cube. The patterned cuboctahedron that results from the projection of a *p4mm*-derived pattern inwards from the surface of the cube is shown in Figure 10, with a net for the resultant patterned cuboctahedron.

Schattschneider and Walker [4] present an alternative method for tiling the cuboctahedron based on the Archimedean tessellation 3.3.4.3.4. This method is not totally satisfactory, however, as the plane pattern does not correspond at certain points. The net for the cuboctahedron may be cut from this pattern, omitting the non-corresponding tiles. Knowledge of plane tessellations and their relationship to the three-dimensional solids, may be of significance when considering the tiling of more complex Archimedean polyhedra.

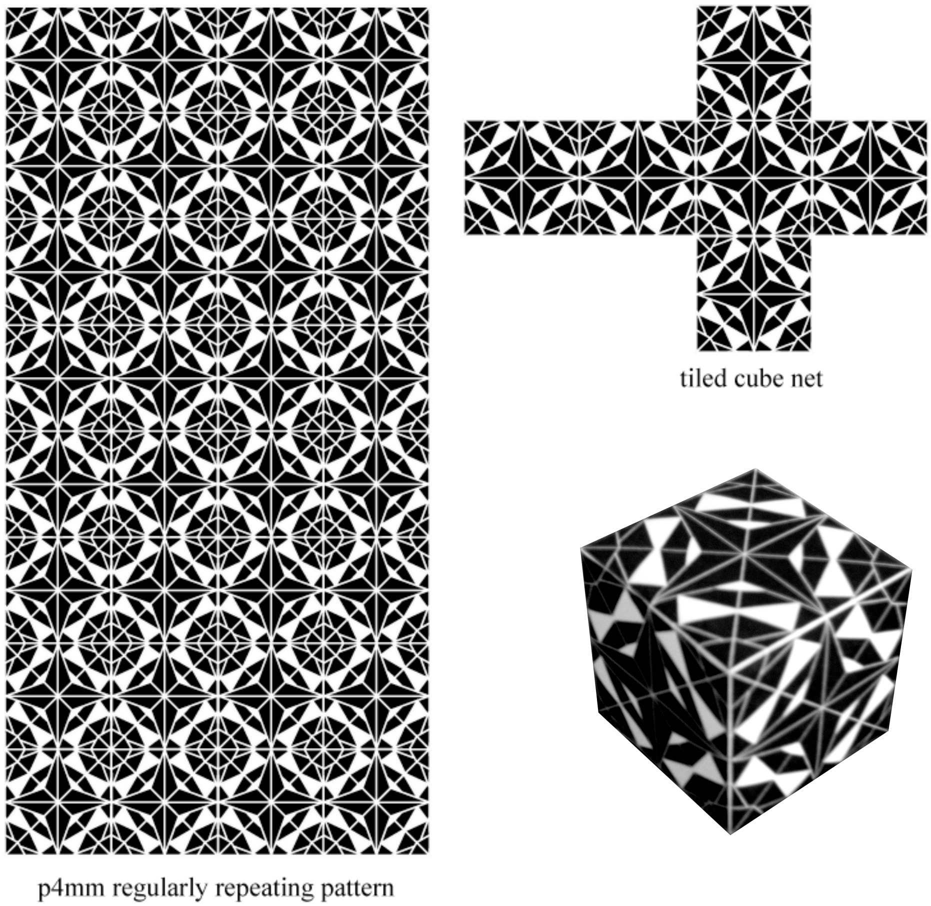

Figure 9: *Illustration of a design for the cube regularly tiled with a class p4mm-derived pattern*

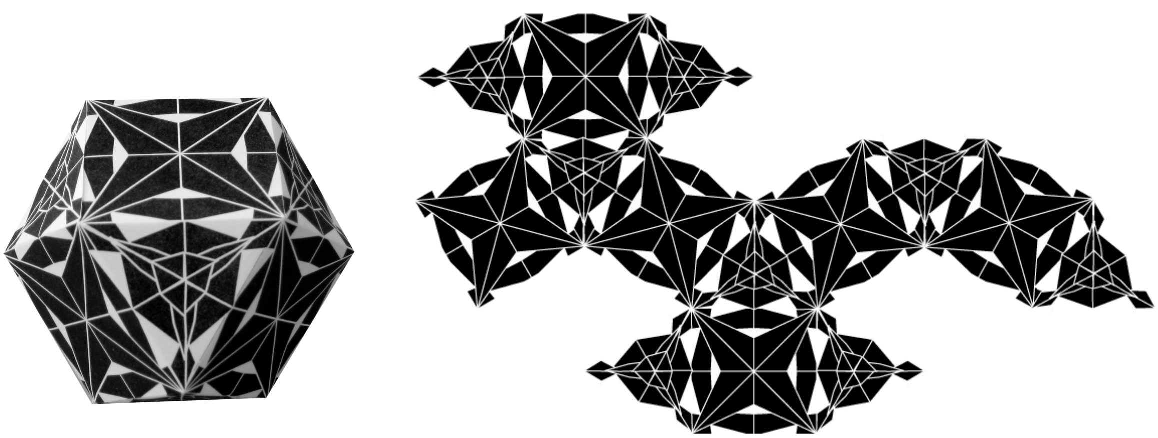

Figure 10: *Illustration of the cuboctahedron and corresponding net, tiled with a class p4mm-derived pattern following projection from the cube*

In Conclusion

An understanding of the symmetry characteristics of patterns and polyhedra can act as a basis for the development of a means by which patterns may be applied to the surface of polyhedra in a systematic and complete way. The initial investigation focused on the application of areas within a pattern's unit cell to act as a tile when applied to the faces of a polyhedron. This study identified which of the seventeen pattern classes were capable of regular repetition around the faces of Platonic solids, applying only the restriction that the unit cell must repeat in exactly the same way that it does in the plane pattern. The application of pattern to the dodecahedron required the development of a different method due to the impossibility of a regular five-sided figure tiling the plane. This paper presents and discusses a method by which pattern can be applied to repeat across the faces of a dodecahedron through projection from a related regularly tiled polyhedron. This method can also be applied to other solids dependant on their inter-relationships. It has been shown that the Platonic duals can be patterned through projection from a patterned inscribed or circumscribed solid. This method can also be extended to the truncated Archimedean solids, which can be patterned through projection from their primary Platonic solids. Preliminary outcomes of the investigation are encouraging and the remarkable mathematical solids developed from the methods described within this paper have been exhibited as a major collection of work [5].

This paper reports on the outcome of conceptual developments completed to date and further attention should be focused on irregular pentagonal tessellations and other such tilings. Knowledge of these pentagonal tessellations could be of great importance to the process, as was shown in the patterning of the dodecahedron using the Cairo tessellation. Platonic duals are able to tile each other through the method of projection. Further investigation is required into other relationships between the solids and how these may be taken into account in future tiling exercises.

References

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[4] D. Schattschneider and W. Walker. *M.C. Escher Kaleidocycles.* Norfolk, Tarquin Publications. 3rd ed., pp.26-27. 1982.

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