Mathematical Models for Binarization and Ternarization of Musical Rhythms

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Abstract

The process of mapping a ternary rhythm of, say 12 pulses, to a binary rhythm of 16 pulses, such that musicologically salient properties are preserved is termed binarization. By analogy the converse process of mapping a binary rhythm to a ternary rhythm is referred to as *ternarization*. New algorithms based on geometric proximity rules are proposed and investigated for the binarization and ternarization of musical rhythms with the goal of understanding the historical evolution of traditional rhythms through inter-cultural contacts.

1 Introduction

Examples of the evolution of musical phenomena may be observed in those musical traditions created by the mixture of other existing traditions, such as in jazz and Latin American music. Benzon [2] analyzes the development of ever more differentiated control over rhythmic patterns in the jazz music of the twentieth century, and argues that rhythmic elaboration in traditional jazz was followed by melodic progress in swing, and finally harmonic control in bop. In an award-winning book, Pérez Fernández [7] describes how African ternary rhythms that travelled to the Americas may have mutated to duple-metered forms as the more traditional music developed into more commercial popular music, a process he labelled *binarization*. For several critical discussions of the theory put forward in this book see [17], [11], and [4]. Manuel [12] describes a similar binarization transformation that occurred in Spain and Cuba, in which ternary 3/4 and 6/8 rhythms such as the flamenco *Guajira* mutated into the binary rhythm *Guajira-Son*. A more general discussion of the evolution of Cuban rhythms may be found in [1].

Toussaint [21] reviews methods that differ from the aforementioned approaches, and mimic those used in bioinformatics, where an organism is represented by its DNA molecule which, in turn, is modelled as a sequence of symbols [16]. However, the use of phylogenetic trees in the analysis of families of rhythms is just beginning [19], [20], [6].

Here we are concerned with mathematical models for two particular kinds of rhythmic transformations, namely, binarization and ternarization. In order to compare our models we use the transformations contained in the work of Pérez Fernández [7]. The algorithms proposed and investigated here are useful for a variety of applications including their use as tools for composition, for exploring the evolution of rhythms, and for studying the mechanisms of rhythmic transformations.

2 Mechanisms of Rhythm Mutation

A typical mutation operation found in the music literature explored by David Lewin [10], which he calls a *flip* in the context of the pitch domain (scales, chords, and pitch-class-sets), interchanges two adjacent elements in the cyclic sequence. Transferring this idea into the rhythmic domain for example, we could transform the

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clave Son timeline given by [x . . x . . x . . . x . x . . .] into the *clave Rumba* timeline by performing a flip on the $7th$ and $8th$ pulses to obtain [x . . x . . . x . . x . x . . .].

In [20] the flip operation is called a *swap*, and the *swap distance* between two rhythms with the same number of onsets is defined as the minimum number of swaps needed to convert one rhythm into the other. In order to be able to compare two rhythms with different numbers of onsets the swap distance was generalized in [6].

In computer science the first algorithms to compare two sequences in terms of the minimum number of a set of predefined operations necessary to convert one sequence to the other were designed for problems in coding theory by Vladimir Levenshtein [9]. His distance measure, which now goes by the name of *edit* distance, allows three operations: insertions, deletions, and substitutions (also called reversals).

A more general approach to the design of measures of string similarity is via the concept of an *assignment*, well developed in the operations research field. An assignment problem deals with the question of how to assign n items to m other items so as to minimize the overall cost [3]. If the two sets of n and m items are the corresponding two sets of onsets of two rhythms to be compared, and the cost of assigning an onset x of one rhythm to an onset y of the other rhythm, is the minimum number of swaps needed to move x to the position of y, then the cost of the minimum-cost assignment is equal to the swap distance discussed in the preceding [6].

3 The Data

Pérez Fernández follows the seminal work of the musicologist Nketia [14, 15] for some of his terminology. Nketia relates musical phrases to timespan, which is of fixed duration. The timespan, typically identified with a 12/8 bar when transcribing African music, is further divided into regulative beats, whose function is to serve as a reference for dancers. These regulative beats divide the timespan into two equal parts. By refining the timespan down to its smallest unit we find the basic pulse. The timespan is measured in terms of the number of basic pulses. Pérez Fernández then introduces the metric foot in between the regulative beat and the basic pulse as an intermediate level of rhythmic grouping. Metric feet consist of groupings of two or more basic pulses according to either their duration or accentuation patterns. Here we consider metric feet only with regards to duration. The main metric feet considered in this paper appear in Figure 1 .

Rhythmic Foot	Rhythm	Duration Pattern	Rhythmic Foot	Rhythm	Duration Pattern
Trochee	[x . x]	L-S	Tribrach	[x x x]	S-S-S
Iamb	[x x .]	S-L	Choriamb	[x . x x x .]	L-S-S-L
Molossus	[x . x . x .]	L-L-L	Zamba foot	[x x x x x .]	Tribrach+Iamb

Figure 1: The relevant metric feet. (L=long, S=short)

We now describe the data used for testing our models. We first introduce a set of ternary rhythms having 6-pulse timespans ([7], pages 82, 83, 91 and 101). They are combinations of two metric feet. Figure 2 shows those rhythms and their binarizations. Names in the rightmost columns correspond to one of the many possible names for the binarized rhythm.

Description of Rhythm	Ternary Rhythm	Binarized version	Description of the Binarized Version
Tribrach + Trochee	[x x x x . x]	[x x . x x . x .]	Argentinean milonga
Zamba Foot	[x x x x x .]	[x x . x x . x .]	Argentinean milonga
Choriamb	[x . x x x .]	[x . . x x . x .]	Habanera

Figure 2: Binarized rhythms having timespans of 6 basic units.

For the second set of rhythms, Pérez Fernández gathers rhythmic patterns with timespans of 12 pulses, and their corresponding binarized versions ([7], page 102 and following); see Figure 3. The rhythms consist of the 6/8 clave son (also called the clave *fume-fume*), [20], some of its variations, and the ubiquitous bembé, the binarization of which was already studied by Chernoff [5].

Description of Rhythm	Notation of the Rhythm	Binarized version	Description of the Binarized Version
6/8 clave Son	[x . x . x . . x . x . .]	[x . . x . . x . . . x . x . . .]	clave Son
6/8 clave Son	[x . x . x . . x . x . .]	[x . . x . . x . . x . . x . . .]	Variation Son - 1
6/8 clave Son variation 1	[x . x . x . . x . x . x]	[x . . x . . x . . . x . x . x .]	Variation Son -2
6/8 clave Son variation 2	[x . x . x . . x . x x .]	[x . . x . . x . . . x . x . x .]	Variation Son -2
Bembé	[x . x . x x . x . x . x]	[x . . x . . x x . . x . x . . x]	Binarized bembé

Figure 3: Binarized rhythms with timespans of 12 basic units.

Names of the binarized rhythms in the last column are provided for reference in the following sections.

For a third set of rhythms, Pérez Fernández displays what he calls resources of rhythmic variation ([7], pages 73-74 and 112-122). These rhythms are formed by variations of the molossus [x . x . x .], the first part of the clave son. Figure 4 shows these rhythmic variations with their associated binarized counterparts. Note that some variations have more than one binarized version.

Description of Rhythm	Notation of the Rhythm	Binarized version	Name of the Rhythm
Variation 1(c)	[x x x . x .]	[x . x x . . x .]	Binarized var. 1(c)-1
Variation 1(c)	[x x x . x .]	[x x . x . . x .]	Binarized var. 1(c)-2
Variation 1(d)	[. x x . x .]	[. . x x . . x .]	Binarized var. 1(d)-1
Variation 1(d)	[. x x . x .]	[. x . x . . x .]	Binarized var. 1(d)-2
Variation 2(c)	[x x x . x x]	[x . x x . . x x]	Binarized var. 2(c)
Variation 4(c)	[x x x x x x]	[x x . x . x x x]	Binarized var. 4(c)
Variation 4(a)	[x . x x x x]	[x . . x x x x .]	Binarized var. 4(a)
Variation 5(a)	[x . x . . .]	[x . . x . . . .]	Binarized var. 5(a)
Variation 6(c)	[x x x . . x]	[x x . x . . . x]	Binarized var. 6(c)

Figure 4: Binarized rhythms derived from rhythmic variations.

Again, names of the binarized rhythms are mnemonics used for reference to them in the following sections.

4 Mapping Rules

The process of binarization proposed by Pérez Fernández uses the metric foot as the starting point. Indeed, the binarization of a ternary rhythm is broken down in terms of its metric feet. Afterwards, each foot is binarized according to a set of binarization rules, also called *mapping rules*. Finally, the binarized feet are put back together so that they constitute the new rhythm. For instance, let us consider the binarization of the zamba foot [x x x x x .]. It is formed by the concatenation of a tribrach [x x x] and an iamb [x x .]. For this case, the mapping rules are [x x x] ⟶ [x x . x] and [x x .] ⟶ [x . x .]. Finally, gluing these patterns together yields the binarized rhythm [x x . x x . x .]. All the mapping rules used by Pérez Fernández are shown in Figure 5.

Metric Feet	Binarized Patterns	Snapping Rules	Metric Feet	Binarized Patterns	Snapping Rules
[x x x]	[x x . x]	NN	[x x .]	[x x . .]	NN
	[x . x x]	CN		[x . x .]	FN
	[x x x .]	CCN			
[x . x]	[x . . x]	NN			
	[x . x .]	FN			

Figure 5: The transformations used by Pérez Fernández, and their proposed geometric interpretations.

The rules described in the preceding are expressible in terms of *snapping rules*. Some transformations of a ternary metric foot (or rhythm) into a binary pattern can be interpreted geometrically as a snapping problem on a circle. Consider a three-hour clock with a four-hour clock superimposed on it, as depicted in Figure 6. The problem is reduced to finding a rule to snap onsets in the ternary clock to onsets in the binary clock. Since both clocks have a common onset at “noon” (the north pole), this onset is mapped to itself. For the remaining ternary onsets, several rules may be defined. One that arises naturally is snapping to the nearest onset. By doing so, the durational relationships among the onsets are perturbed as little as possible, and intuitively, one would expect that the perceptual structures of the two rhythms should remain similar. For instance, this rules takes a tribrach [x x x] to [x x . x]; it is called the *nearest neighbour* rule (NN). Other rules to be used in our study are the following: *furthest neighbour* rule (FN), where each onset is snapped to its furthest neighbour; *clockwise neighbour* rule (CN), which moves an onset to the next neighbour in a clockwise direction; and *counter-clockwise neighbour* rule (CCN), which is analogous to the clockwise rule, but travels in counter-clockwise direction. In Figure 5 the two rightmost columns identify the mapping rules used by Pérez Fernández in terms of the four snapping rules just introduced.

The reader may wonder what the rationale is for using the counter-intuitive FN rule. Two points are worth mentioning here. First, one would expect mapping rules that make musicological sense to use high-level musicologically relevant knowledge to select which onsets in one rhythm should be mapped to which onsets in the other. This is a difficult problem left for future research. In this study we have chosen to start our investigation with the simplest context-free rules possible, purely mathematical rules if you will, to determine how useful they can be. Therefore, from a combinatorial and logical point of view it makes sense to include the FN rule in our study. Second, and surprisingly, we observed that the musicological rules used by Pérez Fernández at the metric foot level were, in several cases, matched perfectly only by the FN snapping rule. Thus we were motivated to compare this rule with the others in order to better understand the entire snapping process.

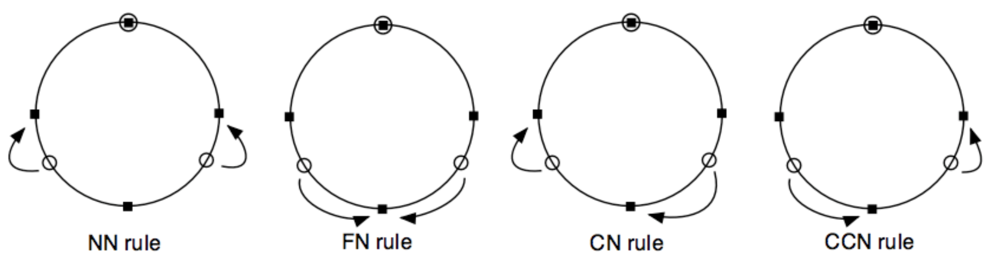

Figure 6: The snapping rules used.

Note that the nearest neighbour and furthest neighbour rules may snap two onsets onto one and the same onset. Consider the tribrach [x x x] with the furthest neighbour rule; one obtains the pattern [x . x .]. On the other hand, when these snapping rules are applied to an entire rhythm, an onset may be mapped to two different onsets, since it may have two nearest or furthest neighbours. For example, in the ternarization of [x x . x x . x .], we obtain two possible rhythms, namely, [x x x x x .] and [x x x x . x] . This creates the problem of breaking ties; we deal with this problem in the following.

5 Design of the Experiments

Since this paper is concerned with rhythmic transformations in general, we compute the mappings in both directions, that is, from ternary rhythms to binary rhythms, and from binary rhythms to ternary rhythms. As a matter of fact, we would like to have at our disposal a set of purported ternarized rhythms, just as we have for binarization. In the absence of such a set, we use Pérez Fernández’s set of binarized rhythms; refer to the appropriate columns in Figures 2, 3 and 4.

The first experiment consists of the binarization of the ternary rhythms contained in Pérez Fernández’s books [7] (Figures 2, 3 and 4) using the four snapping rules defined in the preceding section: NN, FN, CN, and CCN. These four rules yield procedures for both binarization and ternarization, since they are applicable in both directions. Furthermore, in this study we apply them to the whole rhythmic pattern rather than at the metric foot level.

The second set of experiments deals with centers of rhythm families. Such centers were first used by Toussaint [19, 20] for analysing binary and ternary clave rhythms, and proved to be good initial approximations for the center-points of phylogenetic trees. Toussaint also computed the complete phylogenetic graphs of families of rhythms. However, due to lack of space, we do not carry out such an analysis here.

As pointed out in Section 4, when the NN and FN snapping rules are used, ties may arise when an onset has two equidistant nearest or furthest neighbouring pulses. Among the many ways to break ties, we have chosen a method based on rhythmic contours because of their importance in music perception [13, 18].

5.1 Rhythmic Contours

Rhythmic contours have been used for the analysis of non-beat-based rhythms, for the description of general stylistic features of music, for the design of algorithms for automatic classification of musical genres, and for the study of the perceptual discrimination of rhythms. The rhythmic contour is defined as the pattern of successive relative changes of durations in a rhythm. Some authors represent the rhythmic contour as a sequence of integers reflecting these changes; others simply describe the changes in a qualitative manner, observing whether a duration becomes longer, shorter or remains the same. As an example, consider the rhythmic contour of the milonga [xx.xx.x.]. First, we determine its ordered set of durations 12122. The pattern of durations using integers is $\{1,-1,1,0,-1\}$, and if we are only concerned with the direction of these changes, we can write $\{+-+0-\}$. We use the latter definition of rhythmic contour. The length of the rhythmic contour depends only on the number of onsets in the rhythm. To break ties, we compare the rhythmic contours of the snapped rhythms with those of the original rhythms. Comparison of two rhythmic contours can be made by using the Hamming distance. The Hamming distance counts the number of places in which the rhythmic contours do not match. This distance, however, does not take into account where the mismatches occur. Finally, the contour that has the smallest Hamming distance to the contour of the original rhythm is chosen to break the tie. Some cases arise where one contour is shorter than the other, and hence both contours cannot be compared using the Hamming distance. Such cases appear when two onsets are snapped onto the same pulse. As a result, the total number of onsets in the snapped rhythm is smaller than that of the original rhythm. In these cases, a different measure has to be used for the comparison.

5.2 Centers of Rhythm Families

The second set of experiments comprises the computation of several types of centers. Given a family of rhythms with time-spans of n-pulses, we define a center as a rhythm that optimizes some distance function either within the family or in the entire space of rhythms of time-span n. Here we consider centers that convey the idea of similarity. In order to do so, we select as optimization criteria the minimization of the maximum distance (min-max), and the minimization of the sum (min-sum) to all rhythms in either the family or the entire space. For distance (similarity) functions, we select two common distances, the Hamming distance and the directed swap distance [6]. Thus, we have eight possible types of centers, given by the two possible distances, the two possible optimization criteria, the two possible sets of rhythms, and whether optimization is carried out within the given family or the entire space.

The swap distance between two rhythms of equal time-span is the minimum number of interchanges of adjacent elements required to convert one rhythm to the other. An interchange of two adjacent elements, either rests or onsets, is called a swap. If the condition of requiring that both rhythms have equal timespans is relaxed, then a more general distance, the directed swap distance, can be defined as follows. Let rhythm A have more onsets than rhythm B. Then, the directed swap distance is the minimum number of swaps required to convert A to B according to the following constraints: (1) Each onset in A must go to some onset in B; (2) Each and every onset in B must receive at least one onset from A; (3) No onset may travel across the boundary between the first and the last position in the rhythm.

6 Experimental Results

The families of the rhythms analysed were grouped according to the lengths of their timespans. Given the scope of this paper, we cannot describe all the details of all the experiments carried out for the four snapping rules. Due to the problem of breaking ties between rhythms, tables displaying the results are several pages long in some cases. We show the most relevant results and briefly comment on those remaining. For the centers of rhythm families we follow a similar approach. Centers computed on a given family of rhythms are discussed at length, whereas centers computed on the entire space of rhythms are not analysed in full; in some of these cases nearly one hundred instances of centers were obtained.

6.1 Snapped Rhythms

We begin with binarization using the NN rule. Figures 7 and 8 display the results of the calculations. The rhythms in boldface match the binarized rhythms in the books by Pérez Fernández. As can be seen, the NN rule yields almost no match. Furthermore, its binarizations are of little interest in the sense that they keep little perceptual resemblance to their ternary counterparts; see, for example, the binarization of the bembé and compare it to [x . . x . . x x . . x . x . . x].

Ternary rhythm	Name	Binarized rhythm
x x x x x . x . x . x .	Zamba+Molossus	x . x . x . x . x . x . . . x .
x . x . x . . x . x . .	6/8 clave Son	x . x . . . x . . . x . x . . .
x . x . x . . x . x . x	6/8 clave Son -var. 1	x . x . . . x . . . x . x . x .
x . x . x . . x . x x .	6/8 clave Son -var. 2	x . x . . . x . . . x . x . x .
x . x . x x . x . x . x	Bembé	x . . x . x . x . x . . x . . x

Figure 7: Binarization of 12-pulse rhythms using the NN rule.

Ternary rhythm	Name	Binarized rhythm	Ternary rhythm	Name	Binarized rhythm
x x x x . x	Tribrach +Trochee	x x . x x . . x	x x x . x x	Var. 2(c)	x x . x . x . x
x x x x x .	Zamba foot	x x . x x x . .	x x x x x x	Var. 4(c)	x x . x x x . x
x . x x x .	Choriamb	x x . x x x . .	x x x . . x	Var. 6(c)	**x x . x . . . x**
x x x . x .	Var. 1(c)-1	x x . x . x . .	x . x . . .	Var. 5(a)	**x . . x . . . .**
. x x . x .	Var. 1(d)-1	. x . x . x . .	x . x x x x	Var. 4(a)	x . . x x x . x

Figure 8: Binarization of 6-pulse rhythms using the NN rule.

The table in Figure 9 shows the ternarization of the 8-pulse rhythms obtained with the NN rule. The meaning of the columns from left to right is the following: original rhythm; its name; the snapped rhythm; its name in case it is in our list of rhythms; the number of ties encountered; the list of rhythms with minimal Hamming distance in the tie breaking rule, or the list of all rhythms given by ties that have overlapping onsets; the contours of the ties and the contour of the original rhythm; the list of all rhythms generated by ties with overlapping onsets; the contours of the ties and the contour of the original rhythm. Several matches for the original binary rhythms are found. In this table the tie breaking procedure can be observed in detail. For instance, variation 6(c) was ternarized in a unique manner since no ties arose. However, variation 1(c)-1 produced two ties, [x x x . x .] and [x x x . . x]. In the latter case we compare their rhythmic contours to break the tie. The rhythmic contours are $\{0+0-\}$ and $\{0+-0\}$, respectively. For the ternary rhythm the rhythmic contour turns out to be $\{-+-0\}$; therefore, the rhythmic contour of [x x x . x .] is more similar, and [x x x . x .] is output as the ternarized rhythm. In the case of variation 1(d)-1, the rhythmic contour cannot break the tie between the two snapped rhythms; note that both contours are made up of the same symbols. Hence, the tie remains unresolved.

Bin. rhy	Name	Tern.	Name ternary	Ties	Min.Hamm.	Rhyt. Contour	Ons. Ov.	Rhyt. Contour Ov.
xx.xx.x.	Milonga	n/a	n/a	1	xxxxx. xxxx.x	000+- , +-+0- 00+-0 , +-+0-	n/a	n/a
x..xx.x.	Habanera	x.xx.x		1	n/a	n/a	n/a	n/a
x.xx..x.	Var1c1	xxx..x	Tern. Var6c	2	xxx.x. xxx..x	0+0- , -+-0 0+-0 , -+-0	x.x.x. x.x..x	000 , -+-0 +-+ , -+-0
xx.x..x.	Var1c2	n/a	n/a	1	xxx.x. xxx..x	0+0- , ++-- 0+-0 , ++--	n/a	n/a
..xx..x.	Var1d1	n/a	n/a	2	.xx.x. .xx..x	+0- , +-- +-0 , +--	..x.x. ..x..x	00 , +-- -+ , +--
.x.x..x.	Var1d2	.xx..x		1	n/a	n/a	n/a	n/a
x.xx..xx	Var2c	xxx.xx	Tern. Var2c	2	xxx.xx	0+-00 , -+-0+	xxx..x x.x.xx x.x..x	0+-0 , -+-0+ 0-0+ , -+-0+ +-+ , -+-0+
xx.x.xxx	Var4c	n/a	n/a	1	n/a	n/a	xxx.xx xxx.xx	0+-00 , +0-000 0+-00 , +0-000
xx.x...x	Var6c	xxx..x	Tern. Var6c	0	n/a	n/a	n/a	n/a
x..x....	Var5a	x.x...	Tern. Var5a	0	n/a	n/a	n/a	n/a
x..xxxx.	Var4a	x.xxxx	Tern. Var4a	1	x.xxxx	-000+ , -00++	x.xxx.	-0+0 , -00++

Figure 9: Ternarization of 8-pulse rhythms using the NN rule.

Figures 10 and 11 display the binarizations given by the CN rule. For both 12-pulse and 6-pulse rhythms many matches are found. For example, the bembé is transformed to its commonly accepted binarized form [x . . x . . x x . . x . x . . x]. As mentioned in the preceding, this rule does not produce ties.

Ternary rhythm	Name	Binarized rhythm
x x x x x . x . x . x .	Zamba+Molossus	x . x x x . x . x . . x . . x .
x . x . x . . x . x . .	6/8 clave Son	**x . . x . . x . . . x . x . . .**
x . x . x . . x . x . x	6/8 clave Son -var. 1	x . . x . . x . . . x . x . . x
x . x . x . . x . x x .	6/8 clave Son -var. 2	**x . . x . . x . . . x . x . x .**
x . x . x x . x . x . x	Bembé	**x . . x . . x x . . x . x . . x**

Figure 10: Binarization of 12-pulse rhythms using the CN rule.

The FN rule produced no matches in either the binarization or the ternarization. Interestingly enough, there were no ties with binarization, but there were many with ternarization, in some cases as many as three. Furthermore, in numerous cases the ties are unresolved (in the case of 16-pulse rhythms no output was produced; all ties were unresolved). For the binarization, rhythms produced with the FN rule are somewhat monotonous, in many cases consisting of rhythms with many consecutive eighth notes, and again they do not reflect the perceptual structure of their counterparts. With respect to the CCN rule, the overall results are better than those obtained for the CN rule. Ternarization worked very well; for example, the ternarization of [x . . x . . x x . . x . x . . x] is the bembé. No match was obtained in the binarization, but the rhythms produced are still interesting inasmuch as they maintain a perceptual resemblance with their counterparts. However, there is a caveat here; CN and CCN rules do not give good results when transforming rhythms such as [. x x . x .], since both rules produce rhythms with an onset on the first pulse. Obviously, this changes the essence of the rhythm since it takes an upbeat to a downbeat.

Ternary rhythm	Name	Binarized rhythm	Ternary rhythm	Name	Binarized rhythm
x x x x . x	Tribrach+ Trochee	x . x x x . . x	x x x . x x	Var. 2(c)	x . x x . . . x
x x x x x .	Zamba foot	x . x x x . x .	x x x x x x	Var. 4(c)	x . x x x . x x
x . x x x .	Choriamb	**x . . x x . x .**	x x x . . x	Var. 6(c)	x . x x . . . x
x x x . x .	Var. 1(c)-1	**x . x x . . x .**	x . x . . .	Var. 5(a)	**x . . x . . . .**
. x x . x .	Var. 1(d)-1	**. . x x . . x .**	x . x x x x	Var. 4(a)	x . . x x . x x

Figure 11: Binarization of 6-pulse rhythms using the CN rule.

6.2 Centers of Rhythm Families

For the computation of centers of families of rhythms we used two distances, the Hamming distance and the directed swap distance; and two types of centers, the min-sum and the min-max functions. Both binary and ternary rhythms for all possible timespans of rhythms were included.

For binary rhythms of 8-pulses the results are summarized in Figure 12. We note that the rhythm [x x . x . . x .] is the center for all the distances and functions. The center for the directed swap distance with the min-max function contains four rhythms. Therefore, rhythm [x x . x . . x .] may be considered as the one most similar to the others.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	23	[x x . x . . x .]	Bin. var. 1(c)-2
Hamming	Min-Max	3	[x x . x . . x .]	Bin. var. 1(c)-2
Directed swap	Min-Sum	24	[x x . x . . x .]	Bin. var. 1(c)-2
Directed swap	Min-Max	4	[x . . x x . x .]	Habanera
			[x . x x . . x .]	Bin. var. 1(c)-1
			[x x . x . . x .]	Bin. var. 1(c)-2
			[. x . x . . x .]	Bin. var. 1(d)-2

Figure 12: Results for centers of the family of 8-pulse rhythms.

For the binary rhythms of 16 pulses we obtain the table shown in Figure 13. The clave son and its variation [x . . x . . x . . . x . x . x .] appear as centers in all cases. This is not surprising, since the set of binary rhythms considered are rhythms based mainly on the clave son.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	13	[x . . x . . x . . . x . x . . .]	Bin. clave Son
			[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Hamming	Min-Max	6	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Directed swap	Min-Sum	12	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1
Directed swap	Min-Max	4	[x . . x . . x . . . x . x . x .]	Bin. clave Son-var. 1

Figure 13: Results for centers of the family of 16-pulse rhythms.

Figure 14 displays the results for the ternary rhythms of 6 pulses. As with the binary case, there is a rhythm that appears in all the centers, namely, variation 1(c)-1 [x x x . x .]. Again, this indicates that this rhythm is the one most similar to the others.

Finally, the centers for ternary rhythms of length 12 are shown in Figure 15. The situation is very similar to that of binary rhythms. The 6/8 clave son and its variation [x . x . x . . x . x x .] determine the entire set of centers, the latter appearing in three centers out of four.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	19	[x x x . x .]	Var. 1(c)-1
Hamming	Min-Max	3	[x x x x x .] [x x x . x .] [x x x . x x]	Zamba foot Var. 1(c)-1 Var. 2(c)
Directed swap	Min-Sum	18	[x x x . x .]	Var. 1(c)-1
Directed swap	Min-Max	3	[x . x x x .] [x x x . x .] [. x x . x .]	Choriamb Var. 1(c)-1 Var. 1(d)

Figure 14: Results for centers of the family of 6-pulse rhythms.

Distance	Function	Value	Rhythm	Name
Hamming	Min-Sum	11	[x . x . x . . x . x . .]	6/8 clave Son
Hamming	Min-Max	6	[x . x . x . . x . x x .]	6/8 clave Son var. 2
Directed swap	Min-Sum	9	[x . x . x . . x . x x .]	6/8 clave Son var. 2
Directed swap	Min-Max	4	[x . x . x . . x . x x .]	6/8 clave Son var. 2

Figure 15: Results for centers of the family of 12-pulse rhythms.

7 Concluding Remarks

In addition to some general conclusions, numerous specific conclusions may be drawn from the results of these experiments, concerning the data, the snapping rules, and the centers.

Data: It would be desirable to have more documented examples of binarized rhythms. The 12-pulse rhythms considered here are rather limited since they are almost all based on the 6/8 clave son. Figure 3 contains only five rhythms, four of which consist of the 6/8 clave son and its variants. Since we are using its binarizations as our set of binary rhythms, the situation repeats itself for the ternarization results.

Snapping rules: The snapping rules based on nearest (NN) and furthest (FN) neighbours appear not to work very well. In particular, the behaviour of the NN rule is surprizing. One would expect this rule to respect the perceptual structure of the original rhythm, but this is not the case, at least when it is applied to the entire rhythmic pattern. More experiments applying both rules at the metric foot level should be carried out. Rules based on snapping in a preferred direction, such as CN and CCN, work better than NN and FN. Curiously, CN works much better for binarization than for ternarization. On the other hand CCN performs better for ternarization.

Centers: As a consequence of the small size of the sets of 12- and 16-pulse rhythms, the results for the families of rhythms are poor in general. Centers computed on the families of 6- and 8-pulse rhythms are more meaningful. It would appear that perhaps a certain critical number of rhythms is necessary for the centers to make musical sense in the context of binarization and ternarization. Interestingly enough, on the whole, the computation of the centers yields large families of rhythms that are musicologically interesting on their own. Thus centers provide a nice tool for generating new rhythms that are similar to a given group, and can be used as composition tools as well as automatic rhythmic modulation rules. The entire collection is not listed here for lack of space, but it can be found on the web [8].

General conclusions: The results of these experiments are encouraging and suggest several avenues for further research towards our goals, which are the automatic generation of new rhythms as a composition tool, the possible testing of evolutionary theories of rhythm mutation via migration, the understanding of perceptual rhythm similarity judgements, and the development of a general computational theory of rhythm. In this preliminary study we only considered snapping rules on the entire rhythmic patterns. The next step is to repeat these experiments at the finer level of the metric feet contained in rhythmic patterns. The families of rhythms considered in this study are rather small since they consist of the documented examples of binarizations found in the literature. It would be very useful for comparison purposes to repeat these experiments using all known binary and ternary rhythms used in world music to discover which other binary-ternary pairs are identified by the snapping rules investigated here. Finally, mapping rules that use higher level musicological knowledge should be designed and compared to the context-free snapping rules used here, to determine how relevant such high level knowledge might be.

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