# 2D and 3D Animation Using Rotations of a Jordan Curve

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## Abstract

The authors discuss briefly a method and the art behind it to create the possible maximum number $2^n$ of planar simple connected regions that can be distinguished with different binary string codes with length exactly $n$. This method uses the rotations of a single simple closed planar curve $n$ times over $360/n$ degrees. The method creates planar diagrams which are called (rotational) symmetric Venn diagrams. The authors describe that how they created a 2D and 3D animation video to show this rotations for the case $n=11$. They also illustrate how this can be projected into the surface of a sphere creating marvelous spherical images. (A video will be presented at the 2007 Conference of Bridges Donostia, Mathematics, Music, Art, Architecture, Culture, and some images from it in this paper.)

## Introduction

We discuss a method to create the maximum possible number $2^n$ of planar or spherical simply-connected regions that can be distinguished with different binary string codes of length exactly $n$. One of these methods is the following. Find and draw a simple closed curve in the plane (known as a Jordan curve) and a center of rotation in the interior but not on it. Rotate this curve first by $360/n$ degrees. Place the rotated curve on the top of the original curve. Some connected regions are created, see Figure 2. Some but few of the created regions will not be changed in the rest of the process. Mark those regions which will not be changed in the rest of the process. Then rotate the new curve again about the same center and by the same angle and place the double rotated curve on top of the two already drawn ones. (In fact this step can be considered such that second time the first curve is rotated about the given center over $2\cdot 360/n$ degrees.) Again some new regions are created. Some of them will not be changed by the rest of the process. Mark those new regions which will not be changed in the remaining part of the process. Repeat this $n-1$ times. (At the $n^{\text{th}}$ rotation you rotate back to the original selected curve.) After each rotation draw the new curve on top of the already drawn curves and select those newly created regions which will not be changed in the rest of the process, (see Figure 3). At the end these are the regions that you label with different binary codes or with different colors. In this process one has created $n$ copies of the Jordan curve; label them with numbers 1 through $n$. Now designate a length-$n$ binary string code to each of these created regions; we call them $n$-binary string codes. It is clear that the maximum number of these regions cannot exceed $2^n$. These string codes can be assigned to each of the region by writing a 1 in place $k$ if the region is inside curve $k$ and 0 if this region is outside of curve $k$, where $k=1,2,\ldots,n$. Since we have $n$ labeled curves we create an $n$ long binary string code for each of the regions. This process is illustrated in Figures 1, 2, and 3.

When you finish the creation of the planar diagram and its labeling, use the method of stereographic projection from the plane to the surface of a sphere to project the planar image to the surface of a sphere, (see Figure 4). By doing this you create a partition of the surface of a sphere into $2^n$ regions and its labeling with $2^n$ many length-$n$ binary string codes. By coloring its orbits (see the definition of orbits below) you get marvelous spherical images, see Figures 10, 11.

If we have to use all different binary string code with length-$n$ for labeling the regions, then the diagram obtained above is an example of a so-called symmetrical planar Venn diagram with *n* curves. In an *n*-Venn diagram each of the *n* sets is represented as the interior of a Jordan curve. Furthermore, for each choice of any *k* sets, the regions inside the chosen sets (and outside of others) must be connected. An *n*-Venn diagram is called a (rotationally) symmetric Venn diagram if all the curves are obtained by rotating one of the curves about a common center by multiples *360/n* degrees. (Diagrams in which some intersections fail to be connected are called independent families of curves; and diagrams in which some intersections fail to appear are called Euler diagrams. They first appeared in Euler's famous *Letters to a German Princess*.)

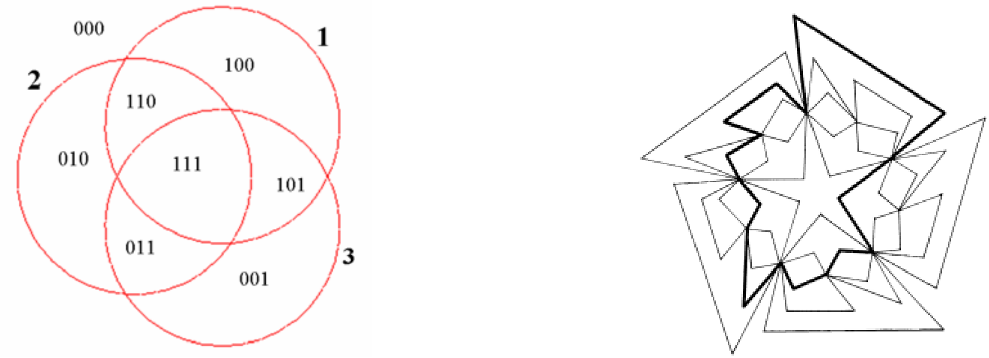

**Figure 1**: *Rotation of a circle three times over 120 degrees, and the labeling of its regions results in the well known symmetric 3-Venn diagram. A symmetric 5-Venn diagram.*

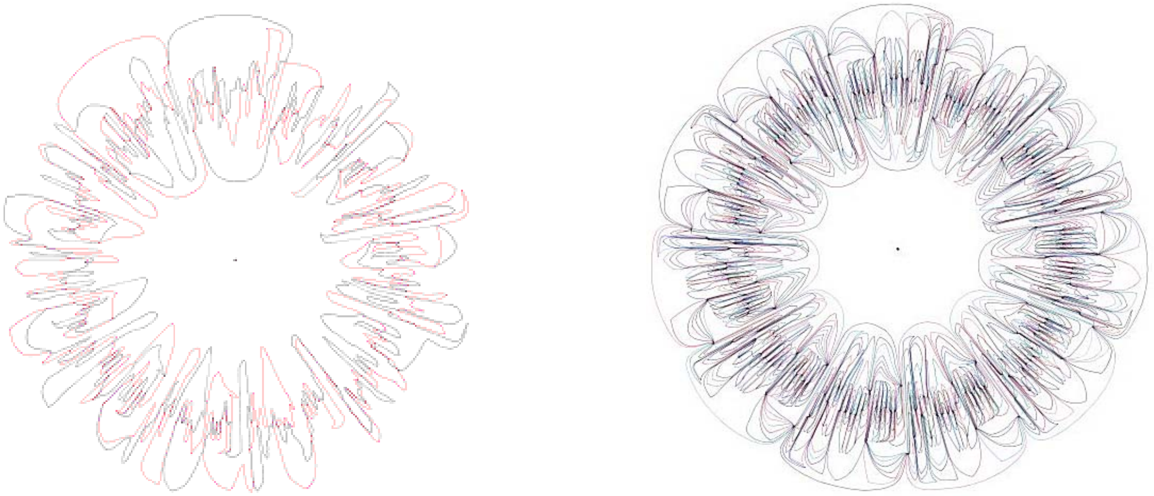

**Figure 2**: *Two curves and nine curves drawn in top of each other to create a symmetric 11-Venn diagram; the 11-Venn diagram is shown in the first image of Figure 6.*

For the history and for the definition of different type of Venn diagrams we refer the readers to papers [1, 2, 3, 4, 5, 7, 8, 9,] and [16]. For the details of the different type of constructions of symmetric Venn diagrams we refer the readers to papers [6, 10, 11, 2,] and [13].

In [12] it is said "It looks like just about anybody can draw *n* curves on the plane--- and it might not even be hard to draw *n* non-self-intersecting curves to divide the plane into $2^n$ connected regions. Then the challenge is a topological one, to get exactly one connected region for each possible *n*-digit code, so no in/out combination is missed, and none occurs in two regions. John Venn was the first, who showed that this could be done for every positive integer number *n* in 1880, [17]. He used a recursive process, starting from one curve, and then adding a new curve to the existing diagrams in each steps." Later in the same paper it is said "To create a symmetric *n*-Venn diagram looks even simpler; you draw a curve, pick a center, rotate the curve to obtain all the other curves, and check that you indeed get *$2^n$* intersections and they are connected. The difficulty is that you do not know such curves. If you try with a curve you think will work, you often will not get all the intersections, and when you do they are often disconnected or the same binary code appears in two different regions." For the complexity of the curves that are needed to create symmetric Venn diagrams see Figures 1 (second part), 4, and 7.

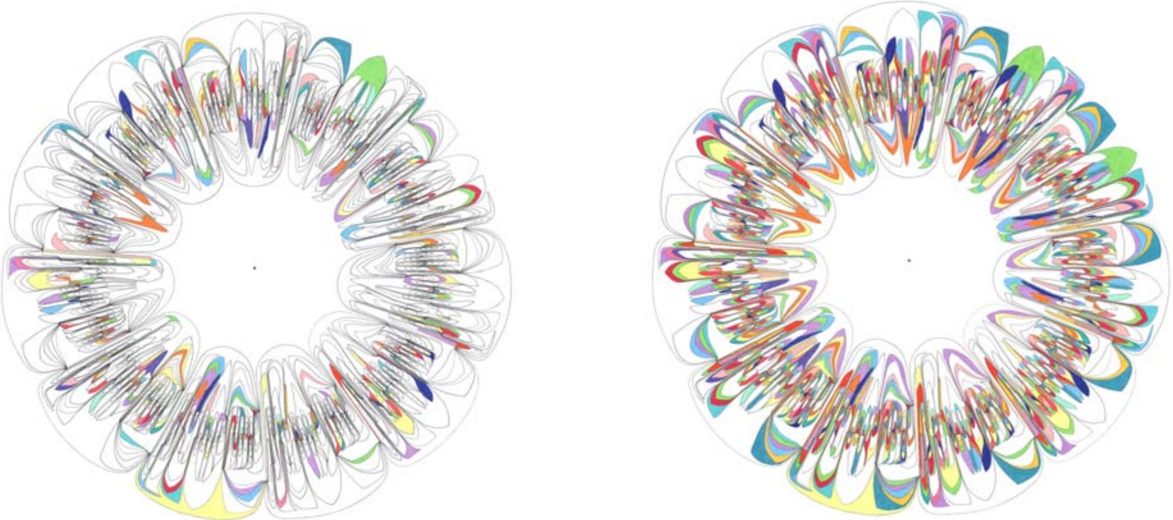

**Figure 3**: *The regions that are created exactly in the ninth rotation and in the first nine rotations in the process of drawing a symmetric 11-Venn diagram; this diagram is shows in the left of Figure 6.*

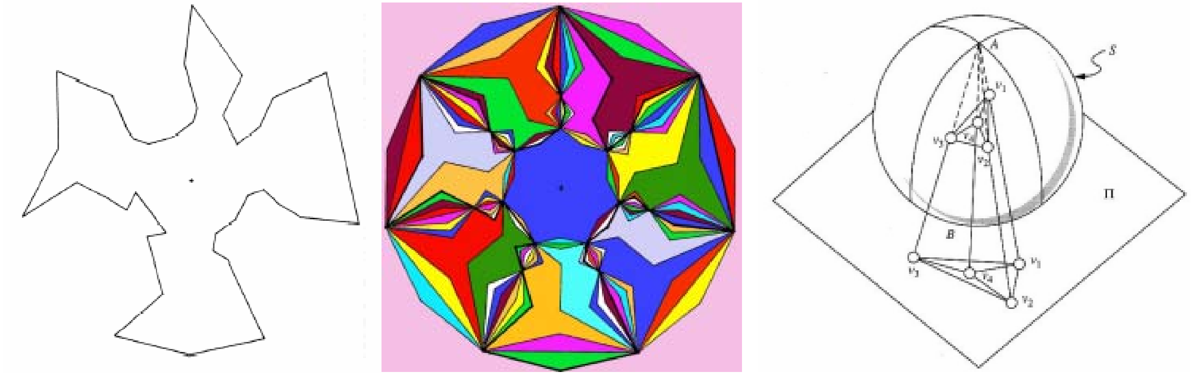

**Figure 4**: *This curve creates this symmetric 7-Venn diagram, it has 128 regions. The illustration of the method of a stereographic projection.*

### A mathematical method of finding a suitable curve for a symmetric p-Venn diagram

The Boolean lattice $B_\text{n}$ is the set of all subsets of *{1,2,…,n},* partially ordered by inclusion. It can also be thought of as the *n*-dimensional hypercube, with vertices of *n*-tuples of 0's and 1's, in which two *n*-tuples are connected if and only if they differ exactly in one coordinate. Now it is clear that there is a one-to-one correspondence between the regions of the Venn diagram and the elements of the Boolean lattice and the n-hypercube, respectively. A symmetric chain is a path in the hypercube whose starting point has as many 0's as its endpoint has 1's, and in each step turns a 0 into a 1. This is equivalent to enlarging a subset in the Boolean lattice by one item in each step. A saturated chain is a chain with starting point and end point having different many 0's and 1's but does not skip any step in the enlarging process. A path, symmetric, or saturated chain partition separates the lattice into distinct paths or chains. Chain partitions and special paths are useful objects in the theory and application of Boolean lattice.

Cipra writes: “In a paper titled “Doodles and Doilies, Non-Simple Symmetric Venn Diagrams,” presented in 1999 at a conference and published in 2002 in Discrete Mathematics, Hamburger introduced a new approach to the problem that enabled him to solve the $n=11$ case-with no computer search at all. Hamburger calls his rotationally symmetric diagrams “doilies” for the lacy crocheted pieces they resemble. His “doodles” are the key. Roughly speaking a doodle is a compact blueprint for a $2\pi/n$ wedge of the doily. Consisting of loops nested within loops, a doodle is derived from a study of symmetric chain decompositions of the Boolean lattice.” See [3] and Figure 5.

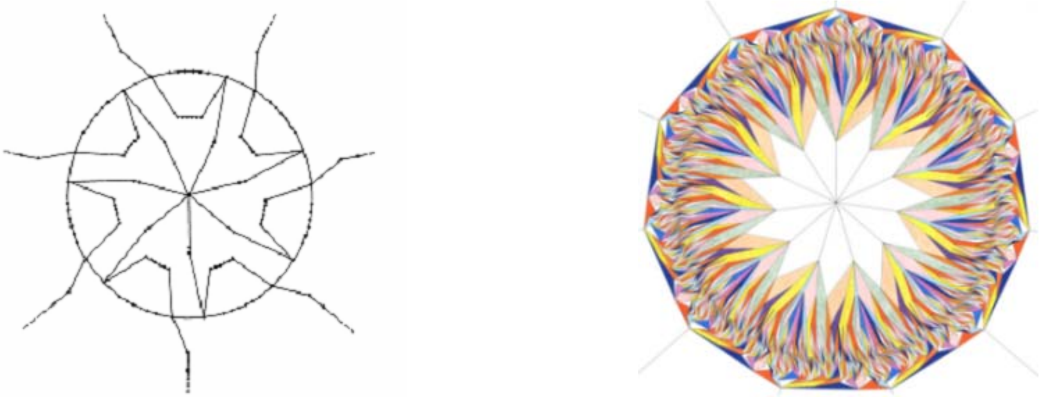

**Figure 5**: *The image of a symmetric and a saturated chain partition of the 7- and the 11-Boolean lattice. These chain decompositions are the blueprints for the diagrams in Figures 4 and the second image in Figure 6. The unconnected line segments must meet in the plane in infinity or at the South Pole on the sphere.*

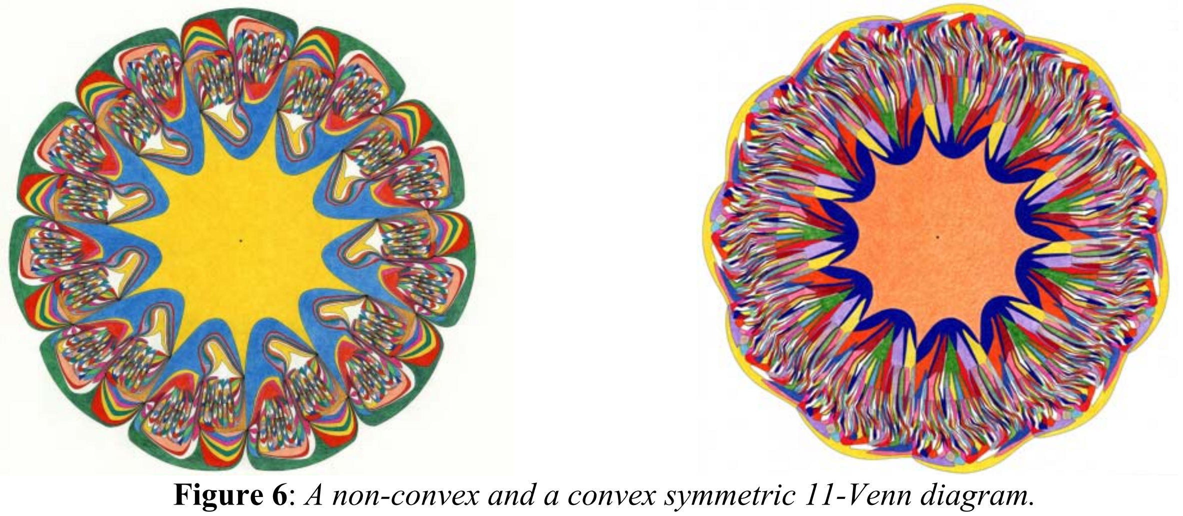

**Figure 6**: *A non-convex and a convex symmetric 11-Venn diagram.*

He continues later: “It’s not hard to turn a symmetric chain decomposition into a decomposition of the plane whose $2^n$ regions correspond to subsets of $\{1, 2,\dots, n\}$. There are two tricky parts. One is to pick symmetric chain decomposition that produces a Venn diagram, rather than a jumble of disconnected duplicate intersections. The other is to get a diagram that is rotationally symmetric. But the real trick is to accomplish the two simultaneously.” “After a warm-up with $n=7$, Hamburger found a decomposition of $B_{11}$ (actually a subposet thereof) into symmetric chains whose doodle produced a doily.” (See [3]).

Hamburger’s method was enhanced with some additional techniques in [6] to show that this can be proved for all prime number *p*. Henderson showed that in 1963, symmetric Venn diagram cannot exist for composite *n*, see [14] and also [18] for a correction to the proof. Also it is easy to see that under the very specific rotation under which the diagram is symmetric a planar region that has a given bit string will become a region for which the bit string changes by shifting the last bit of the string to the first place and moving all the other bits by one place ahead. For example, the bit string 0111001 under one rotation becomes 1011100. This is called a shift. For a given fixed bit string the set of all shifts is called an orbit. In fact in a symmetric Venn diagram not only the curves but the elements of any orbit are also in rotational symmetry.

Using a doodle the method to find a curve is the following, see [10]: find a loop that associated with a region with a code having first bit 1 and continue to follow those loops that are all associated with regions having first bit 1 in the first wedge; then follow in the first rotation of this compact wedge those loops that are associated with regions having first bit 1, and 1 in the third rotation of the wedge, and so on, throughout the diagram until you return to the starting wedge and the original loop. The result is a simple closed curve; this curve creates the diagram.

## The art behind this method

The mathematical argument cannot determine the curves completely, nor the overall images. Some small perturbations of the curves might not ruin the required geometric and topological properties, while some other very small perturbation will destroy them. Other small perturbations of the curves can preserve the major required properties, such as the number of regions and the structure of the bit strings, but can change some of the not-so-important geometrical properties, such as the number of intersection points of the curves. Especially in [13], (the web page of Hamburger), in the extended version of his paper [12], it is shown how little perturbation of a curve is necessary to create diagrams with more and more vertices for the cases *n=5* and *7*. The gradual change of a curve from one vertex set to the next possible vertex set is almost impossible to see. There the big difference between two curves that are needed to create a so-called monotone or a non-monotone diagrams with the same number of curves is also shown. (According to a theorem of Grünbaum, a diagram is monotone if and only if it can be drawn with all convex curves.)

The curves and the image of the diagram are specified only topologically by the mathematical argument, which does not say anything about the exact geometric measurements of the curves. This freedom is where the artist's creativity adds beauty to the mathematical aesthetics. The mathematical aesthetic is created by the multifold symmetries while the artistic aesthetic comes from the artist through this freedom. If the number of curves exceeds 7, it is difficult to color all regions (there are at least 128 regions) with a different color due to the fact that there are no such many different colors that one can buy commercially. Thus one way of coloring such a diagram is to color the regions of a given orbit with the same color. This reduces the number of colors, for instance, for an *11*-Venn diagram from 2048 to 186. (White is considered here as a color.) This reduction is not good enough since there are 186 orbits for a symmetric *11*-Venn diagram. Therefore a choice is to color some of the orbits with the same colors which are close to each other. The most stunning images are those where the symmetries are intentionally destroyed by the artist in order to dive deep into the mathematical principle. It is obvious that the removal of the exterior of one of the curves leaves half of the diagram. Knowing this fact, Hepp observed that this process creates wonderful abstract images; see Figure 7. (It also helps us follow one of the curves throughout the diagram since the contour of the exterior space is one of the curves.) Neither of these images is computer generated; they were created by Edit Hepp. In fact they were created with the possible simplest technique; they are all handmade using color and black pencils, a ruler, and a compass.

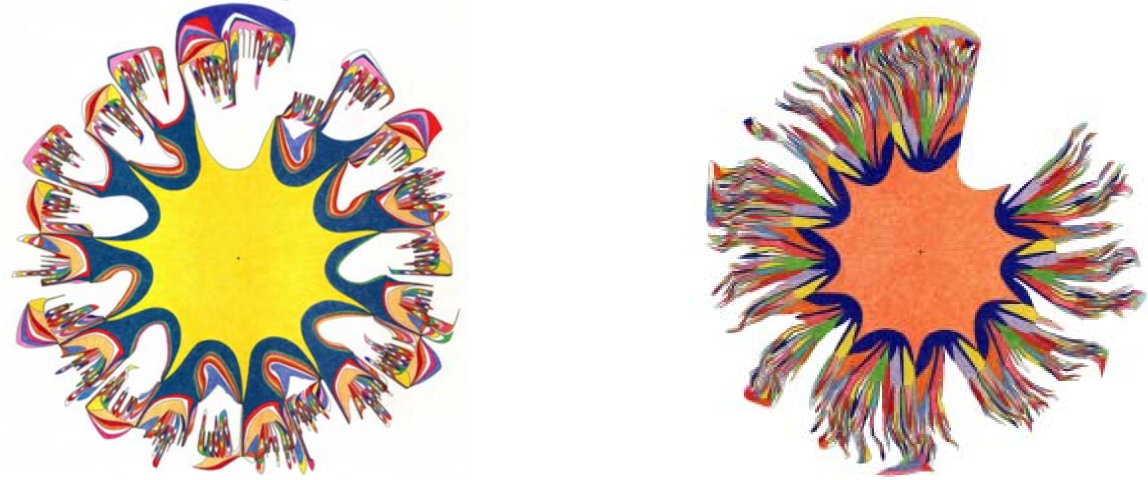

**Figure 7**: *The images of the interior of one of the curves in a non- convex and convex symmetric 11-Venn diagram.*

Recently, Stan Wagon generated one similar image by a computer with some manual intervention. This was featured recently as the cover page of Notices of the American Mathematical Society in 2006; see [18]. His tool was a Mathematica package written by him, and the symmetric Boolean decomposition of Killian et. al. [15]. He used their decomposition the same way as Hamburger's doodles method describes it. Wagon manually entered the data of Killian et. al.'s into the computer, and then used the computer to finish up the job.

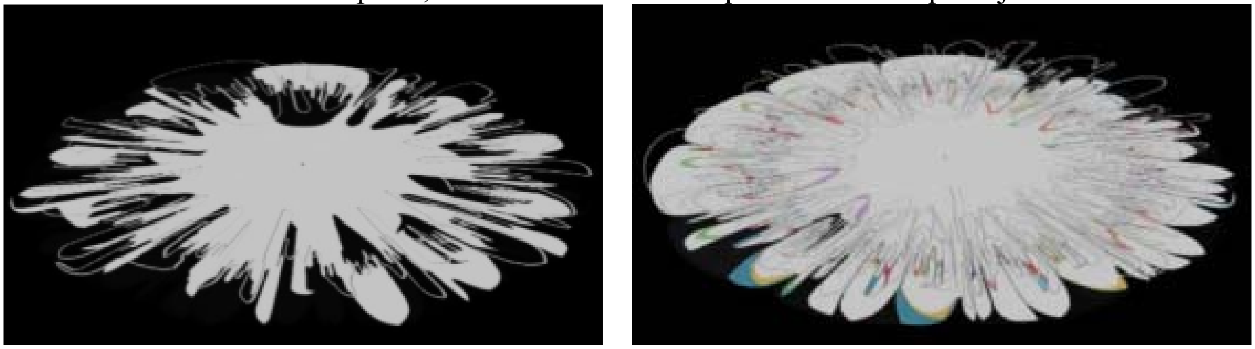

**Figure 8**: *In the animation the first and the sixth rotations. In both images the rotating curves are above the plane. The second image shows those regions which will not be changed by the rest of the process.*

The animation video was created by R. Wartell. The process was the following. Hepp manually drew all steps of the process by drawing the 30 different images. They are the following. The first image is the suitable curve that is used to create the process. The next ten images are the rotations of this curve and the drawings of the new curve on top of each of the other curves, (see Figure 2). The next ten images illustrate those regions which are not being changed during the remaining part of the process, but were created exactly in the last step of the process, (see the first part of Figure 3). Finally the next nine images illustrate those regions that have been constructed up to this point but will not be changed during the rest of the process, (see second part of Figure 3). (In the second step we need only one image, since the first cure itself cannot create regions.) The last one is the final product; (see it in the first image of Figure 6).

The 32 inch by 32 inch handmade colored images of Hepp were scanned into a computer. Hepp enriched the copies and transformed the tif to jpeg and eps formats as well.

Wartell used the computerized images for the animation.

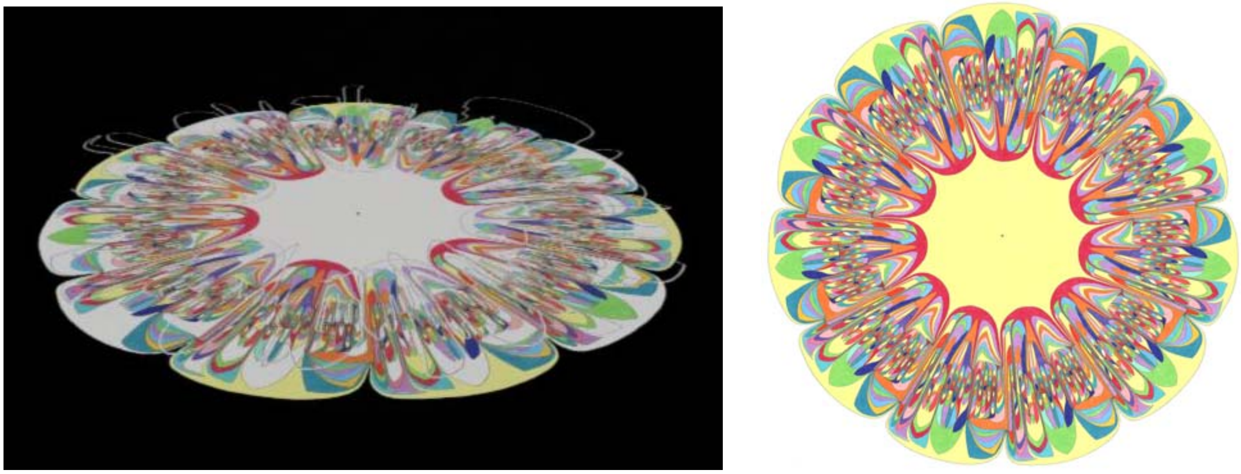

**Figure 9**: *The last step in the process before the diagram is finished; and the finished diagram.*

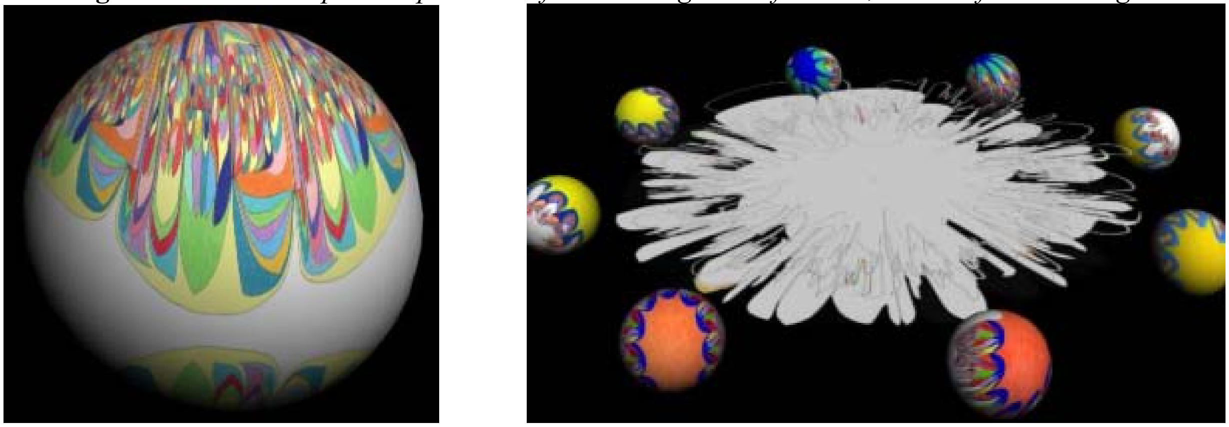

**Figure 10**: T*he stereographic projection of Figure 9 to the sphere; an image of the animation.*

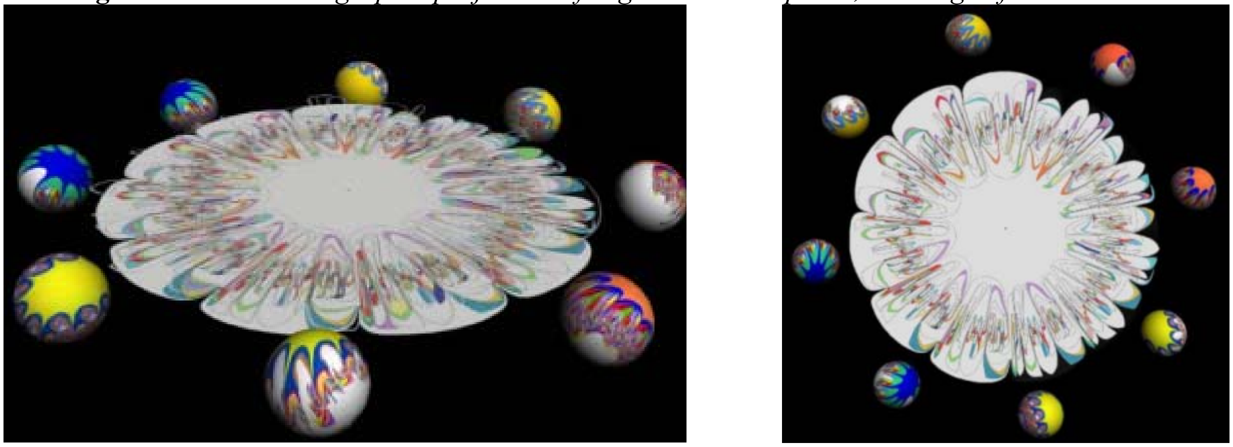

**Figure 11**: *Two images from the animation video.*

The program that he used to create the animation is called 3DS MAX. He used a free version of it for educational purposes. First, he created planes for every one of the iterations of the curve creation. He placed all of these planes in the same place and then he changed all of their visibilities to 0. Then, he started with a single curve, he showed it alone, then he rotated the camera to a lower angle to view the rotation of the curve. With the camera rotated, he pulled a single curve up and rotated it to the next position; (see Figures 8 and 9). He then rotated the camera back, and changed the visibility of the 1 curve alone to 0, and the 2 curves to 1. Then he changed the visibilities to show the plane that was all the newly finished sections, and then once again he changed the visibilities to show the plane which displayed all of the finished sections so far. Then he rotated the camera and repeated the process until he reached the finished diagram, (see Figures 3, 9 and 11). To create a sphere needs to specify how large a radius is. Then, to add texture to it, Wartell simply added a spherical UVW map to each sphere, effectively wrapping the texture around a sphere, (see Figure 10 and 11).

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