Hyperbolic Semi-Regular Tilings and their Symmetry Properties

Ma. Louise Antonette N. De Las Peñas, mlp@mathsci.math.admu.edu.ph
Glenn R. Laigo, glaigo@yahoo.com
Eden Delight B. Provido, edenprovido@yahoo.com
Math Department, Ateneo de Manila University
Loyola Heights, Quezon City, Philippines

Abstract

In this paper, symmetry groups of certain classes of semi-regular tilings on the hyperbolic plane are discussed.

Introduction

Symmetry groups play an important role in tiling theory. Certain classes of tilings are studied and characterized based on the symmetries they contain. In color symmetry theory, a colored symmetrical tiling or pattern is constructed and analyzed based on the symmetry group of its corresponding uncolored tiling or pattern [4]. A *color symmetry* of a tiling or pattern is a symmetry of the uncolored tiling or pattern that maps all parts of the tiling or pattern having the same color onto parts of a single color – that is, the symmetry permutes the colors. If every symmetry of a tiling or pattern is a color symmetry, the tiling or pattern is said to have *perfect color symmetry*. For example, in the colored hyperbolic semi-regular tiling shown in Figure 1(a), the 4-fold rotations about the centers of the 8-gons fixes black and interchanges the dark and light gray colors. Moreover, the reflections with axes the lines passing through the midpoints of two opposite edges shared by an 8-gon and a 6-gon fixes all colors. As a matter of fact, all symmetries of the uncolored tiling permute the colors of the given coloring, thus the tiling has perfect color symmetry.

Semi-regular tilings and patterns have aesthetic appeal even more so if they have non-trivial color symmetry. A very striking feature, for instance, of Escher's patterns is their color symmetry. In fact, Escher did pioneering work in color symmetry before the theory was developed by mathematicians and crystallographers. In Figures 1(b) and (c) are shown Dunham's computer renditions of Esher's patterns [9, 10] which exhibit perfect color symmetry. In Figure 1(b), Escher's *Circle Limit III* is superimposed on the semi-regular $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling. The symmetry group of this tiling, as well as that of the uncolored *Circle Limit III* pattern, include 3-fold rotations with centers located at the vertices of the tiling and centers of the 3-gons; and 4-fold rotations with centers located at the centers of the 4-gons. All the symmetries in this group permute the colors of the pattern. In Figure 1(c), we present Dunham's modification of the *Circle Limit III* [9, 10] using the semi-regular $5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3$ tiling. The symmetry group of the uncolored pattern, as well as the symmetry group of the $5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3$ tiling, include 3-fold rotations with centers located at the vertices of the tiling and centers of the 3-gons; and 5-fold rotations with centers located at the centers of the 5-gons. All the symmetries of the uncolored pattern, and also the tiling, permute the colors of the fishes.

It is interesting to note that certain aesthetically pleasing colored patterns and designs arise from semi-regular and regular tilings; the color symmetries of the patterns and designs emerging from the symmetries of the corresponding semi-regular and regular tilings. Another example is the *Hyperbolic Spiderweb* shown in Figure 1(d), Tony Bomford's first hyperbolic rug [10, 11]. This rug was inspired by M.C. Escher's *Circle Limit IV* shown in Figure 1(e), based on the regular 6^4 tiling in which four 6-gons meet at a vertex. The *Hyperbolic Spiderweb* displays non-perfect color symmetry. The reflection whose axis is a horizontal line passing through the center of the central 6-gon does not permute the colors. The

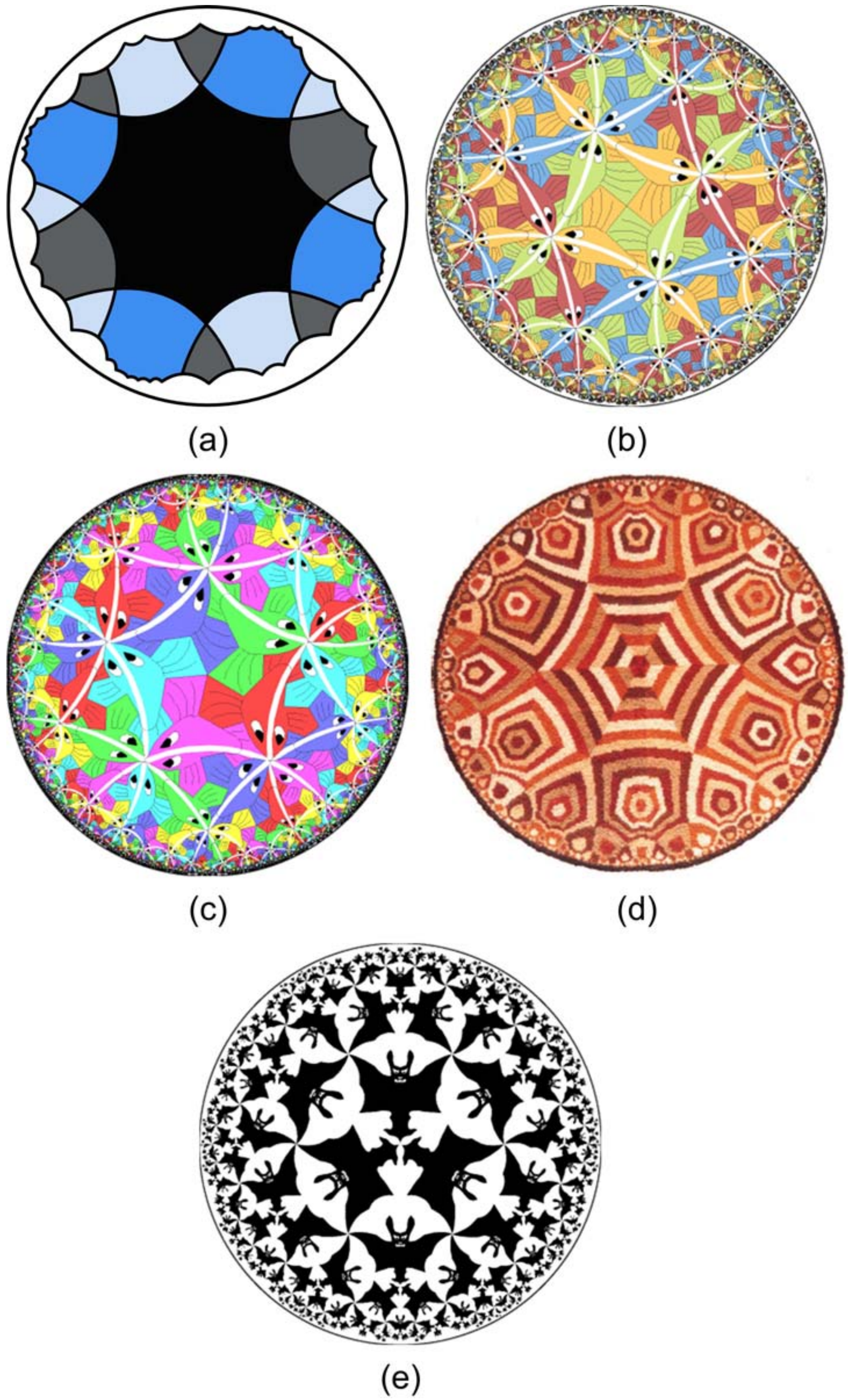

Figure 1. (a) *a colored semi-regular* $4 \cdot 6 \cdot 8 \cdot 10$ *tiling*; (b) *a rendition of Escher's Circle Limit III*; (c) *a modification of Escher's Circle Limit III*; (d) *Tony Bomford's Hyperbolic Spiderweb*; (e) *a rendition of Escher's Circle Limit IV*.

reflection fixes the color black of the innermost 6-gon, and at the same time interchanges black and brown in the seventh 6-gonal ring from the center. (There are eight 6-gonal rings of colors that make up the central 6-gon.)

In this paper, we discuss symmetry properties of certain classes of semi-regular tilings on the hyperbolic plane. By a *semi-regular tiling* on the hyperbolic plane, we consider a $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling that is edge to edge, having regular polygons as its tiles; a p_1-gon, a p_2-gon, …, and a p_q-gon surrounding each vertex in cyclic order, where $(1/p_1) + (1/p_2) + \ldots + (1/p_q) < (q - 2)/2$, and satisfying the additional property that the symmetries of the tiling act transitively on its vertices. If $p_1 = p_2 = \ldots = p_q = p$, we denote the semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_s$ tiling to be the regular p^q tiling, sometimes denoted by its Schlafli symbol $\{p, q\}$.

The semi-regular tilings that will be studied in this work will be exhibited on the Poincaré circle model. This model is conformal, where the hyperbolic measure of an angle is just its Euclidean measure. Moreover, it has the additional property that it is represented in a bounded region of the Euclidean plane. This is useful since we desire to show an entire pattern. The points of this model are interior points of the bounding circle. The hyperbolic lines are circular arcs orthogonal to the bounding circle, including diameters. For example, the backbones of the fish in Figures 1(b) and (c) lie on hyperbolic lines. Moreover, the fishes (as well as the angels and devils in Figure 1(e)) are of the same hyperbolic size, showing that equal hyperbolic distances are represented by decreasing Euclidean distances as one approaches the bounding circle.

A Method to Determine the Symmetry Groups of Semi-regular Tilings

In this work, we will describe the symmetry group of a given semi-regular tiling using Conway's orbifold notation [1, 2]. The Conway notation for crystallographic groups enumerates the type of non-translational symmetries occurring in the group. The symbol * indicates a mirror reflection while x indicates a glide-reflection and a number n indicates a rotation of order n (that is, a rotation by $2\pi/n$). In addition, if a number n comes after the * symbol, the center of the n-fold rotation lies on mirror lines.

To describe the symmetry group of a given semi-regular tiling, the elements of the symmetry group will be indicated on a fundamental region of the tiling. By a *fundamental region* of the semi-regular tiling we mean a smallest region of the plane that can cover the entire plane by copies of itself using the symmetries of the tiling. Moreover, the images of a fundamental region under the symmetries do not overlap except at the boundary points. The existence of symmetries on a fundamental region will be indicated in this manner: a center of a rotation of order n, $n > 2$ will be labeled in the tiling with a small regular polygon of n sides. A 2-fold rotation will be labeled by a small circle. The axes of reflections will be illustrated by dark heavy lines.

The key element in the determination of the symmetry group of a semi-regular tiling lies in the construction of the given semi-regular tiling. In [15], Mitchell constructed semi-regular tilings using the *incenter process*. The center of a polygon's inscribed circle is called the *incenter*; located at the point of intersection of the polygon's angle bisectors. The basic idea in the construction of a semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling using the incenter process is to first create an *auxiliary tiling*, a tiling by q-gons. Then, in the tiling by q-gons, the incenters of the q-gons are connected to obtain the semi-regular tiling.

Some Results on the Symmetry Groups of Semi-Regular Tilings

We begin the discussion by considering the $4 \cdot 6 \cdot 8 \cdot 10$ tiling shown in Figure 1(a) and finding its symmetry group. The $4 \cdot 6 \cdot 8 \cdot 10$ tiling is constructed using the incenter process from a tiling by quadrilaterals with interior angles $\pi/2$, $\pi/3$, $\pi/4$ and $\pi/5$. Consider the centers of the inscribed circles of each quadrilateral. The centers are the incenters of the quadrilaterals. Connecting the incenters of the quadrilaterals will give the $4 \cdot 6 \cdot 8 \cdot 10$ tiling (Figure 2(a)). Note that a particular quadrilateral in the tiling by quadrilaterals has 4, 6, 8, and 10 quadrilaterals meeting at each of its vertices. Consider quadrilateral *ABCD* shown in Figure 2(b). The eight quadrilaterals meeting at *C* has eight incenters. When connected, these quadrilaterals give an 8-gon about *C*. Also, there are six quadrilaterals that meet at *B*. The incenters of these quadrilaterals give rise to a 6-gon about *B*. Similarly, the four and ten quadrilaterals meeting at *A* and *D*, respectively, give rise to a 4-gon and a 10-gon about *A* and *D*. There will be 4-, 6-, 8-, 10-gons arising from the vertices of any quadrilateral.

It can be observed that a fundamental region of the tiling by quadrilaterals is also a fundamental region of the $4 \cdot 6 \cdot 8 \cdot 10$ tiling, as displayed in Figure 2(c). Figure 2(d) shows the symmetries of the $4 \cdot 6 \cdot 8 \cdot 10$ tiling, which include reflections with axes passing through the sides of a fundamental region and 5-, 4-, 3- and 2-fold rotations with centers lying on the vertices of a fundamental region. Thus, the $4 \cdot 6 \cdot 8 \cdot 10$ tiling, using Conway's notation, can be described as having symmetry group *5432.

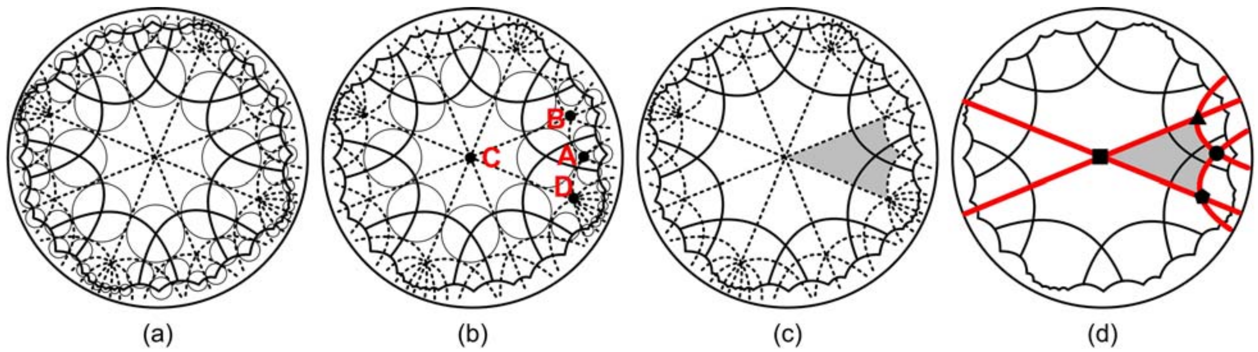

Figure 2. *The* $4 \cdot 6 \cdot 8 \cdot 10$ *tiling* (a) *obtained from a tiling by quadrilaterals (dashed) using the incenter process*; (b) *on a tiling by quadrilaterals (dashed) with a particular quadrilateral ABCD*; (c) *on a tiling by quadrilaterals (dashed) with a fundamental region and* (d) *some of its symmetries on a fundamental region.*

The process just described of obtaining the symmetry group of the $4 \cdot 6 \cdot 8 \cdot 10$ tiling can be applied to obtain the symmetry group of the semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling, where $p_1, p_2, \ldots, p_q$ are all distinct and even. We now have the following result:

Theorem 1. The semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling, where $p_1, p_2, \ldots, p_q$ are all distinct and even, has symmetry group $*(p_1/2)(p_2/2)\ldots(p_q/2)$.

Proof. Consider a tiling by q-gons, where the q-gons have interior angles $2\pi/p_1$, $2\pi/p_2$, …, and $2\pi/p_q$ such that $1/p_1 + 1/p_2 + \ldots + 1/p_q < (q-2)/2$. This tiling is obtained by reflecting a q-gon across its sides [3]. A q-gon serves as a fundamental region of the tiling. Each q-gon has an incenter, which is the center of the circle inscribed in the q-gon. Incenters of adjacent q-gons are joined to produce a semi-regular tiling. Since the angles of the q-gons in the tiling are $2\pi/p_1$, $2\pi/p_2$, …, and $2\pi/p_q$, then a semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling is obtained.

Since $p_1, p_2, \ldots, p_q$ are distinct, a q-gon has trivial symmetry. Thus, a fundamental region of the q-gon tiling, which is a q-gon, is also a fundamental region of the semi-regular tiling. The reflections with axes passing through the sides of a q-gon are symmetries of the semi-regular tiling. These reflections yield the $(p_1/2)$-, $(p_2/2)$-, …, and $(p_q/2)$-fold rotations with centers at the vertices of a q-gon. Hence, the symmetry group of the semi-regular $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ tiling is $*(p_1/2)(p_2/2)\ldots(p_q/2)$. ■

The next example pertains to a tiling belonging to the class of $r \cdot s \cdot r \cdot s \cdot \ldots \cdot r \cdot s$ tilings consisting of $(q/2)$ r-gons and $(q/2)$ s-gons.

The $5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3$ tiling shown previously in Figure 1(c), is derived from a tiling by 6-gons, where a given 6-gon has alternating interior angles $2\pi/5$ and $2\pi/3$. The incenters of the adjacent 6-gons are joined to obtain the $5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3$ tiling (Figure 3(a)). A fundamental region of a tiling by 6-gons is a tile one-sixth of a 6-gon, in particular, a triangle with interior angles $\pi/5$ and $\pi/3$; this is also a fundamental region of the 5 · 3 · 5 · 3 · 5 · 3 tiling (Figure 3(b)). There are reflections with axes passing through the sides of a fundamental region and 5- and two 3-fold rotations with centers on the vertices of a fundamental region (Figure 3(c)). Thus, the 5 · 3 · 5 · 3 · 5 · 3 tiling has symmetry group *533.

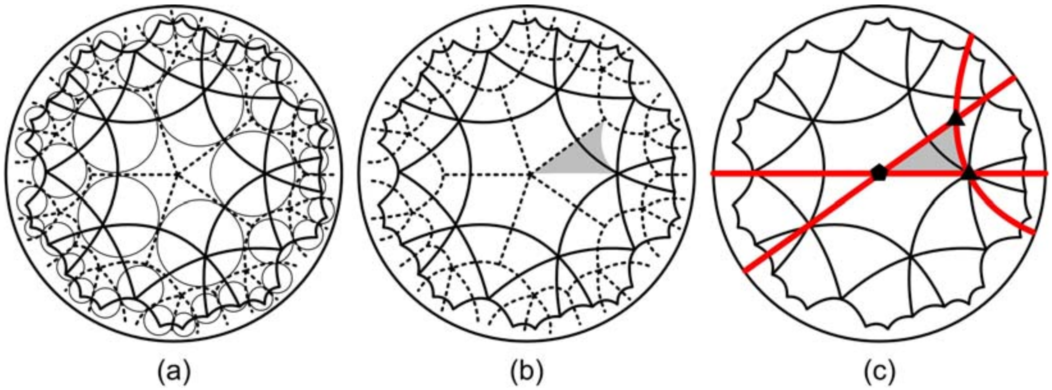

Figure 3. *The* 5 · 3 · 5 · 3 · 5 · 3 *tiling* (a) *obtained from a tiling by* 6*-gons (dashed) using the incenter process*; (b) *on a tiling by* 6*-gons (dashed) with a fundamental region and* (c) *some of its symmetries on a fundamental region.*

The next theorem gives the result on the symmetry group for any *r* · *s* · *r* · *s* · … · *r* · *s* tiling.

Theorem 2. The alternating semi-regular *r* · *s* · *r* · *s* · … · *r* · *s* tiling, consisting of (*q*/2) *r*-gons and (*q*/2) *s*-gons, has symmetry group **rs*(*q*/2).

Proof. Following the essence of the proof in the previous theorem, the *r* · *s* · *r* · *s* · … · *r* · *s* tiling is obtained by applying the incenter process to a tiling by *q*-gons, where a given *q*-gon has alternating interior angles 2π/*r* and 2π/*s*. The incenters of adjacent *q*-gons are joined to produce a semi-regular tiling. Since each *q*-gon in the tiling has alternating interior angles 2π/*r* and 2π/*s*, a semi-regular *r* · *s* · *r* · *s* · … · *r* · *s* tiling is obtained.

A fundamental region of the tiling by *q*-gons, as well as the *r* · *s* · *r* · *s* · … · *r* · *s* tiling, is a triangle with interior angles π/*r*, π/*s* and 2π/*q*. The reflections with axes passing through the sides of a fundamental region are symmetries of the semi-regular tiling. These reflections give rise to the *r*-, *s*- and *q*/2-fold rotations with centers at the vertices of a fundamental region. Thus, the symmetry group of the *r* · *s* · *r* · *s* · … · *r* · *s* tiling is **rs*(*q*/2). ■

The last theorem discusses the symmetry group of the 3 · *r* · 3 · *r* · 3 · *s*/2 tiling, where $s \geq 6$ and *s* is even. This tiling is a semi-regular tiling with auxiliary tiling constructed from a regular r^s tiling. The symmetry group of this type of tiling depends on the relationship between *r* and *s*.

We first look at two examples before giving the result. The first example discusses the 3 · 4 · 3 · 4 · 3 · 3 tiling; this is the case when $s \neq 2r$. The second example discusses the 3 · 4 · 3 · 4 · 3 · 4 tiling; this is the case when $s = 2r$.

For the first example, let us discuss the construction of the 3 · 4 · 3 · 4 · 3 · 3 tiling first. Consider a 4^6 tiling shown in Figure 4(a). On the tiling, we show the center *O* and a vertex *Y* of a 4-gon, a point *X* on the edge of the 4-gon containing *Y* and *X*'. *X*' is obtained by rotating *X* by an angle π/2 about *O*. Now *OX*, *OX*' and *X*'*Y* become the motif to generate the tiling by 6-gons shown in Figure 4(b). This tiling by 6-gons is obtained by using as symmetries the rotations of 2π/4 = π/2 about the center of each 4-gon and reflections across the edges of the 4-gons. Moreover, each 6-gon has interior angles 2π/3, 2π/4, 2π/3, 2π/4, 2π/3 and 2π/3. The incenter process will give us the desired tiling shown in Figure 4(c).

Now, Figure 4(d) shows the 3 · 4 · 3 · 4 · 3 · 3 tiling superimposed on a tiling by 6-gons. A fundamental region for the tiling is also shown. The symmetries of the 3 · 4 · 3 · 4 · 3 · 3 tiling are 4-fold rotations about the centers of the original 4-gons in the 4^6 tiling and reflections whose axes lie across the edges of these 4-gons. The reflections yield a 3-fold rotation about the vertices of the 4-gons. The center of a 4-fold rotation does not lie on any axis of reflection (Figure 4(e)). Thus, the symmetry group of the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 3$ tiling is 4*3. We would like to mention that the *Hyperbolic Spiderweb* in Figure 1(d) has been constructed from the dual of the 4^6 tiling, which is the 6^4 tiling. Moreover, the symmetry group of the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 3$ tiling is the same color fixing group of Escher's *Circle Limit IV*, shown in Figure 1(e).

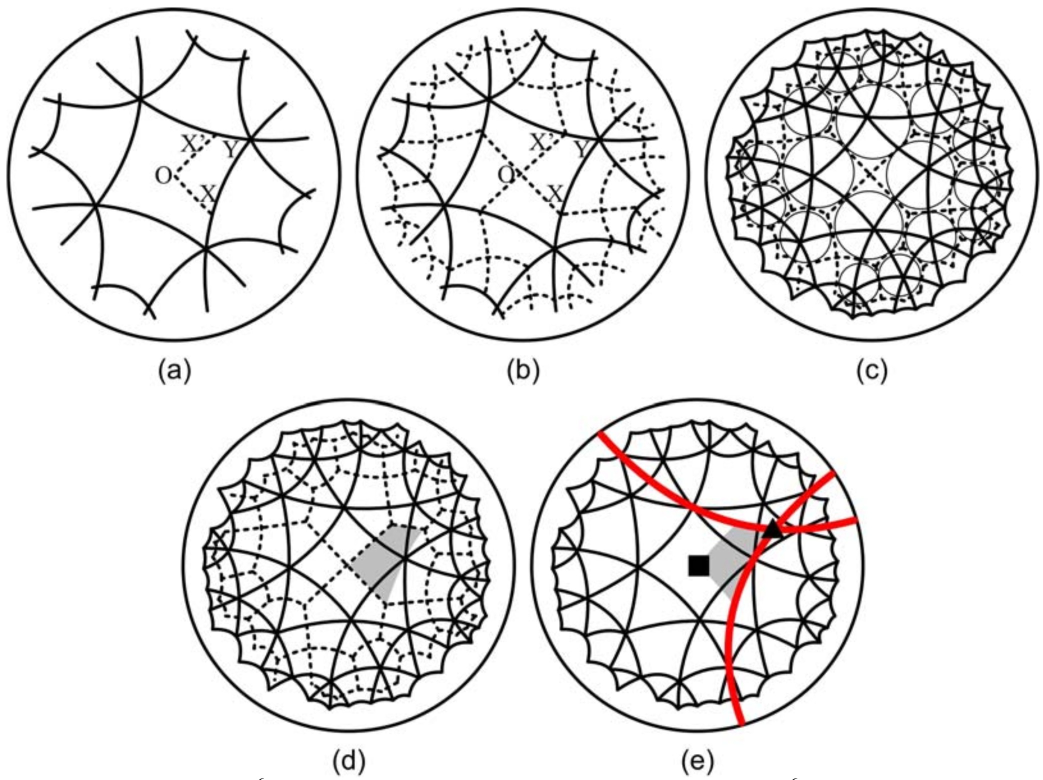

Figure 4. (a) *The motif on a* 4^6 *tiling;* (b) *tiling by* 6*-gons (dashed) on the* 4^6 *tiling (solid). The* $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 3$ *tiling* (c) *obtained from a tiling by* 6*-gons (dashed) using the incenter process*; (d) *on a tiling by* 6*-gons (dashed) with a fundamental region and* (e) *some of its symmetries on a fundamental region.*

The second example we will look at illustrates the case when $s = 2r$. Consider the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling shown in Figure 1(b). This tiling is obtained from the regular 4^8 tiling as follows: Take the center O and a vertex Y of a 4-gon, a point X on the edge of the 4-gon containing Y and X'. X' is obtained by rotating X by an angle $\pi/2$ about O (Figure 5(a)). Now OX, OX' and X'Y constitute the motif to generate the tiling by 6-gons (Figure 5(b)). This tiling by 6-gons is obtained by using as symmetries the $\pi/2$ rotations about the center of each 4-gon and reflections across the edges of the 4-gons. Moreover, each 6-gon has interior angles $2\pi/3$, $\pi/2$, $2\pi/3$, $\pi/2$, $2\pi/3$ and $\pi/2$. The incenter process will give us the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling (Figure 5(c)).

Figure 5(d) shows a $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling superimposed on a tiling by 6-gons. Figure 5(d) also shows a fundamental region of the tiling, which is a triangle. One vertex of the triangle is a center of a 3-fold rotation (center of a 6-gon). Another vertex is a center of a 4-fold rotation (center of a 4-gon). There is another 3-fold rotation at the third vertex of the triangle where the sides of the 6-gons intersect. In Figure 5(e), we show a fundamental region of the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling. The centers of 4-fold and 3-fold rotations are shown on axes of reflections. Thus, the symmetry group of the $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ tiling is *433. Note that this result agrees with Theorem 2.

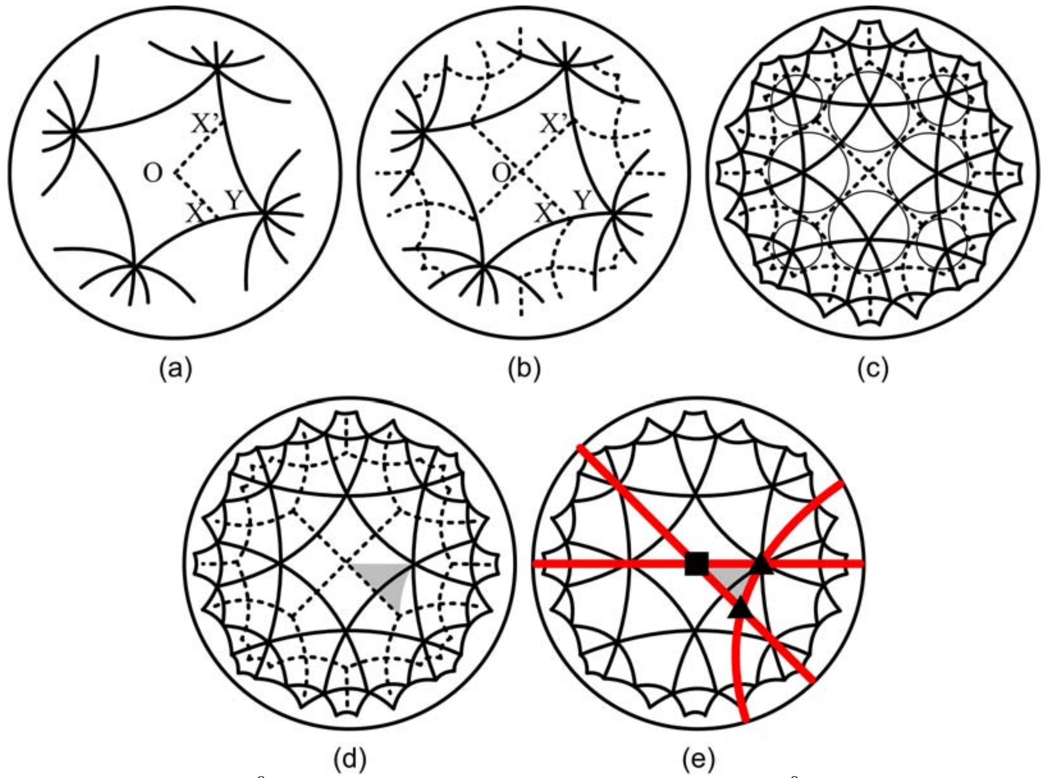

Figure 5. (a) *The motif on a* 4^8 *tiling;* (b) *tiling by* 6-*gons (dashed) on the* 4^8 *tiling (solid). The* $3 \cdot 4 \cdot 3 \cdot 4 \cdot 3 \cdot 4$ *tiling* (c) *obtained from a tiling by* 6-*gons (dashed) using the incenter process*; (d) *on a tiling by* 6-*gons (dashed) with a fundamental region and* (e) *some of its symmetries on a fundamental region.*

We now give the result for the $3 \cdot r \cdot 3 \cdot r \cdot 3 \cdot s/2$ tiling.

Theorem 3. The semi-regular $3 \cdot r \cdot 3 \cdot r \cdot 3 \cdot s/2$ tiling, where $s \geq 6$ and s even, has symmetry group $r{*}s/2$ if $s \neq 2r$; otherwise, it has symmetry group $*r33$.

Proof. Starting with an r^s tiling, take the center O and a vertex Y of an r-gon. Then select a point X on the edge of the r-gon containing Y such that m<$OXY = 2\pi/3$. Rotate X by an angle of $2\pi/r$ about O to get X'. Using OX, OX' and $X'Y$ as the motif, generate a tiling by 6-gons using as symmetries the r-fold rotations about the centers of each r-gon and the reflections across its edges. The incenter process can easily be adapted to this situation. Since the interior angles of the 6-gons are $2\pi/3$, $2\pi/r$, $2\pi/3$, $2\pi/r$, $2\pi/3$ and $4\pi/s$, we obtain a $3 \cdot r \cdot 3 \cdot r \cdot 3 \cdot s/2$ tiling, for $s \geq 6$ and s even.

The symmetry group of the tiling by 6-gons and the $3 \cdot r \cdot 3 \cdot r \cdot 3 \cdot s/2$ tiling is $r{*}s/2$ if $s \neq 2r$. The tiling by 6-gons was generated using as symmetries the r-fold rotations about the centers of the r-gons and the r^s tiling, and the reflections across the edges of those r-gons. The reflections from the r^s tiling, with axes the lines emanating from the centers of the r-gons to the vertices and midpoints of the edges are not symmetries of the tiling by 6-gons as a result of the introduction of a motif which does not have these as symmetries. The reflections across the edges of the r-gons give rise to the $s/2$-fold rotation about the vertices of the r-gons.

If $s = 2r$, the reflections across the edges of the r-gons give rise to the r-fold rotations about the vertices of the r-gons. Consequently, reflections will exist with axes lying along the diagonals of the 6-gons containing the vertices which are centers of $2\pi/r$ and $2\pi/3$ rotations. The reflections will yield $2\pi/3$ rotations with center situated at the center of the 6-gons. A fundamental region of the tiling by 6-gons is a triangle which is one-sixth of a 6-gon. One vertex of a triangle is a center of a $2\pi/3$ rotation (center of a 6-gon); another vertex is a center of a $2\pi/r$ rotation (a vertex or a center of an r-gon). The sides of the 6-

gons form a $2\pi/3$ rotation at the third vertex of the triangle. Moreover, the vertices of the triangle are situated at the sides of the r-gons, which lie on axes of reflections of the tiling. Hence, the symmetry group of the $3 \cdot r \cdot 3 \cdot r \cdot 3 \cdot s/2$ tiling is $*r33$. ■

Conclusions and Future Work

In this short note, we discussed symmetry groups of some classes of semi-regular tilings on the hyperbolic plane. The method used to describe the symmetry groups may be extended to determine the symmetry groups of other general classes of hyperbolic semi-regular tilings. Among such tilings are the $p^k \cdot r \cdot p^k \cdot s$ tiling, the $p^k \cdot r$ tiling, the $p_1 \cdot p_2 \cdot \ldots \cdot p_k \cdot r$ tiling, and the $p_1 \cdot p_2 \cdot \ldots \cdot p_k \cdot r \cdot p_k \cdot \ldots \cdot p_2 \cdot p_1 \cdot s$ tiling.

This work will facilitate future, related research on the colorings of semi-regular tilings in the hyperbolic plane, using the knowledge of the symmetry group structure discussed here. Colorings arising from general classes of semi-regular tilings may be analyzed and characterized, including the study of properties of related symmetry groups and their subgroups. We hope results presented here will lead to many aesthetically pleasing colored tilings.

Remarks: The authors would like to thank Professor Douglas Dunham for the use of the images presented in Figure 1 [6-13]. The semi-regular tilings presented in this paper were generated using the *Mathematica* packages *L2Primitives* and *Tess* [14, 17]. For a more detailed discussion on the study of semi-regular tilings with the aid of technology, refer to [5].

References

[1] Balke, L. and D. H. Huson, *Two Dimensional Groups, Orbifolds and Tilings*, in Geometr Dedicata **60,** 89-106, (1996).

[2] Conway, J. H., *The Orbifold Notation for Surface Groups*, in Groups, Combinatorics and Geometry: London Mathematical Society Lecture Notes Series **165,** 438-447, (1992).

[3] Coxeter, H. S. M. and W. O. Moser, *Generators and Relations for Discrete Groups* 2nd ed, (Springer –Verlag, 1965).

[4] De Las Penas, M. L. A. N., R. Felix and G. Laigo, *Colorings of Hyperbolic Plane Crystallographic Patterns*, in Z. Kristallogr **221**, 665 – 672, (2006).

[5] De Las Penas, M. L. A. N., G. Laigo and E. D. Provido, *Studying Non-Euclidean Tilings and their Groups with the Aid of Technology*, in Proceedings of the Eleventh Asian Technology Conference in Mathematics ATCM 2006, Hongkong, W.C.Yang, et al., Eds, 317 – 326, (ATCM Inc, USA., 2006), http://www.atcminc.com/mPublications/EP/EPATCM06/enter.shtml.

[6] Dunham, Douglas, *Creating Hyperbolic Escher Patterns* in M.C. Escher: Art and Science, H.S.M. Coxeter, et al., Eds, (Elsevier Science Publishers B.V., North Holland, 1986).

[7] _____, *Hyperbolic Symmetry* in Comp. & Maths. with Appls. Vol. 12B, Nos. 1/2, 1986, 139-153.

[8] _____, *Transformations of Hyperbolic Escher Patterns*, in Visual Mathematics, Vol 1 No 1, (1991). http://www.mi.sanu.ac.yu/vismath/dunham/index.html.

[9] _____, *New Hyperbolic Escher Patterns*, in Visual Mathematics Math Art. http://www.mi.sanu.ac.yu/vismath/dunham1/index.html.

[10] _____, *Families of Escher Patterns*, in M.C. Escher's Legacy: A Centennial Celebration, Schattschneider and Emer, Eds, (Springer – Verlag, USA, 2002)

[11] _____, *Tony Bomford's Hyperbolic Hooked Rugs*, in Bridges: Mathematical Connections in Art, Music and Science, R. Sarhangi, et al., Eds, (USA, 2004).

[12] _____, *H.S.M. Coxeter and Tony Bomford's Colored Hyperbolic Rugs*, in Bridges: Mathematical Connections in Art, Music and Science, R. Sarhangi, et al., Eds, (USA, 2005).

[13] _____, *More "Circle Limit III" Patterns*, in Bridges: Mathematical Connections in Art, Music and Science, R. Sarhangi, et al., Eds, (USA, 2006).

[14] Knezevic, I., R. Sazdanovic and S. Yukminovic, *Basic Drawing in the Various Models of the Lobachevskian Plane*, for Mathematica 4.0, (2002), file: L2Primitives.m, http://www.library.wolfram.com/infocenter/MathSource/4540/.

[15] Mitchell, K., *Constructing Semi-Regular Tilings*, a paper presented at the Spring 1995 Meeting of the Seaway Section of the Mathematical Association of America at Hobart and William Smith Colleges, Geneva, NY, (1995). http://people.hws.edu/mitchell/tilings/Part1.html.

[16] Provido, E., *On the Symmetry Groups of Hyperbolic Semi-Regular Tilings and the Index 2 Subgroups of Some Hyperbolic Groups*, a Masters Thesis, Ateneo de Manila University, (2006).

[17] Sremcevic, M. and R. Sazdanovic, *Tessellations of the Euclidean, Elliptic and Hyperbolic Plane*, for Mathematica 4.0, (2002), file: Tess.m. http://www.library.wolfram.com/infocenter/MathSource/4540/.