Sculptures which Stellarize Non-Planar Hexagons

Douglas G. Burkholder
School of Computing Science & Mathematics
Lenoir-Rhyne College
625 7th Avenue NE
Hickory, North Carolina, 28603
E-mail: BurkholderD@LRC.edu

Abstract

In this paper we describe procedures for turning any random non-planar hexagon into five radically different sculptures that are mathematically interesting and esthetically pleasing. In each situation, we start by sketching a planar hexagon with relatively nice symmetries – noting that a regular hexagon is far too restrictive. We then describe a method for turning this sketch into a sequence of linear steps which, when applied to any non-planar hexagon will construct an affine image of our sketch. Since the resulting hexagon is affine, this process is mathematically interesting because it planarizes our non-planar hexagon. By adding motion to the hexagon that we sketch, we can add the sense of motion to the sculpture. This added motion provides unexpected results which can turn our sculptures into art.

1. Background: Symmetric Linear Constructions

If the five vertices of any irregular non-planar pentagon slide down their respective median line at speeds proportional to the length of their respective line segment, then at a certain moment in time, these five points have folded themselves up and flattened out to form a stellar planar affine regular pentagon. As these points continue sliding, they unfold, and flatten out once again to form a non-stellar planar affine regular pentagon. With the exception of a few degenerate cases, affine regular polygons appear regular when viewed from a certain direction. A sculpture, shown to the right, visually shows this transformation of random non-planar polygons into planar affine regular polygons. This sculpture puts into motion a theorem of Jesse Douglas [5] from 1960.

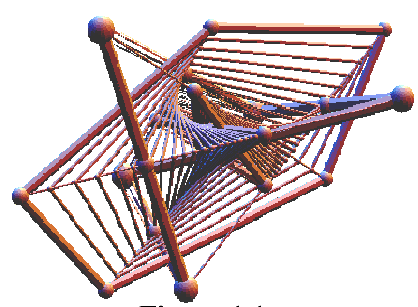

Figure 1.1

This example begs the question: *What other constructions planarize random polygons and which of these can be used to produce artistically interesting sculptures?*

Before considering the artistic aspect of the above question, we must consider the mathematical aspect. To clarify the question: Starting from a random non-planar *n*-gon, describe a sequence of steps consisting of constructing line segments between pairs of known points followed by locating a new point on this line at a specified ratio of this length. At the completion of these steps, label the last point constructed q_1. Likewise, describe a sequence of linear steps for constructing the remaining vertices of the polygon. Connect the vertices in order to form a new *n*-gon. We shall call this a *linear construction*. Thus, our first goal is to determine linear constructions which produce planar polygons when applied to any random *n*-gon, for a fixed known value of *n*. The most familiar linear construction is the midpoint rule for quadrilaterals which planarizes quadrilaterals by producing parallelograms. In [5], Jesse Douglas locates specific points on the median lines of pentagons which planarize pentagons. He also observed that his process always produces a pentagon which is as regular as possible in the sense that it is affine regular. A polygon is affine regular if it is the image of a regular, either stellar or convex, polygon under a linear transformation. Notice that parallelograms are affine regular.

We shall call a linear construction *symmetric* if the construction generates each vertex by repeating the same sequence of steps from different vantage points. Notice that the two examples listed above are indeed symmetric. In [1], the author proved that a symmetric linear construction planarizes if and only if it produces affine images of regular polygons. Moreover, as we shall see below, there is a procedure for designing these constructions for polygons of any size. For example, the distances in Figure 1.2 can be used to construct one portion of Jesse Douglas's construction. Thus, for symmetric linear constructions, the mathematical aspect has been completely answered.

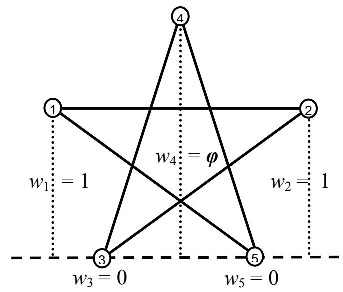

Figure 1.2

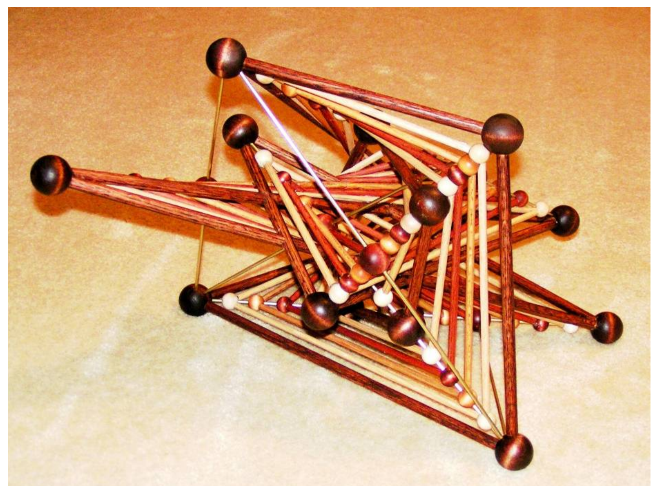

Figure 1.3

This leaves the artistic aspect: How do we apply the mathematical constructions in an interesting way to design pleasing sculptures? In [2], [3], and [4], the author has used symmetric linear constructions to describe and produce a variety of (computer generated) sculptures. Figure 1.3 shows a wood and brass sculpture created by the author using this technique. This sculpture was shown at the Bridges Conference Art Exhibit in London in 2006. In this paper, we wish to focus on hexagons. Applying the theorem stated above to a random hexagon produces the sculpture shown in Figure 1.4. Notice that the sculptures in Figures 1.1 and 1.3 not only incorporate two different planar pentagons, but they also both incorporate stellar affine regular pentagons. Since regular hexagons only come in one form, we can not obtain either of these two characteristics when applying *symmetric* linear constructs to hexagons.

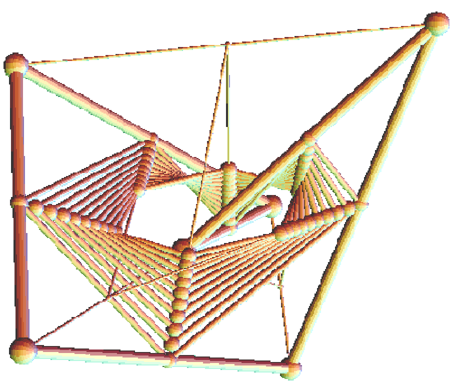

Figure 1.4

As we shall see in this paper, removing the symmetry restriction greatly expands the possibilities both from the mathematical aspect and from the artistic aspect.

2. Designing Linear Constructions which Planarize Hexagons

It turns out to be surprisingly straightforward to create a mathematical formula which can be used to construct a sequence of steps which planarizes a polygon. Before considering the non-symmetric case, we shall consider the symmetric case. By the theorem stated above, this must create an affine regular hexagon. Start by drawing a regular hexagon and label the points in order. Then, as shown in Figure 2.1, draw a dashed line anywhere and measure, using any choice of units, the distance between each point and the line. For simplicity, we shall place our line along an edge. These distances describe the relative weights necessary for constructing the planar affine regular hexagon. Normalize these weights {1,2,2,1,0,0} so that they sum to 1 and use these as a weighted average of the vertices p_1, p_2, p_3, p_4, p_5, and p_6 of the random hexagon. Thus, the first point must be located at $(1/6)p_1+(1/3)p_2+(1/3)p_3+(1/6)p_4$. Notice that this equals $\frac{2}{3}\left(\frac{p_2+p_3}{2}\right)+\frac{1}{3}\left(\frac{p_1+p_4}{2}\right)$.

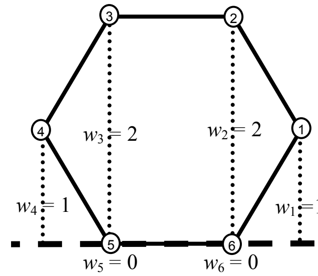

Figure 2.1

Recall that $2/3A + 1/3B$ locates a point on the line segment AB which is 1/3 of the distance from A towards B. This is closer to A than to B because A has a heavier weight. The hexagon weights can be interpreted as: "*Construct new points 1/3 of the way from the midpoint of each side to the midpoint of the corresponding diagonal*". We can see this in Figure 2.2. In this sculpture the vertices of hexagons slide from the midpoint of each side to their critical locations.

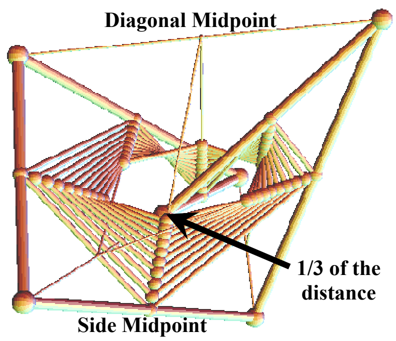

Figure 2.2

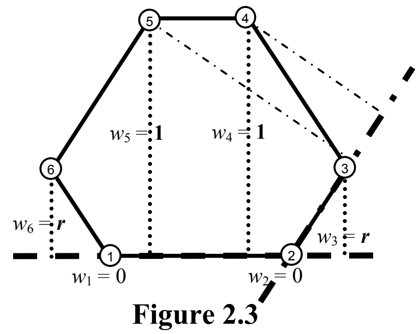

Figure 2.3

We now modify this for *non-symmetric* linear constructions. Start by drawing any desired polygon and label the vertices in order. We shall first consider the hexagon shown in Figure 2.3. Draw the line passing through vertices 1 and 2 and measure the distances to get $\{0,0,r,1,1,r\}$. Next, draw the line passing through vertices 2 and 3 and measure the distances to get $\{1-r,0,0,1-r,1,1\}$. Continue this process for all six sides. We now construct a matrix using these distances as the column vectors as shown in Figure 2.4. To planarize, it is critical that every row of this matrix must have the same non-zero sum. After dividing by this sum, the rows become the weights for our construction. Specifically, if p_1, p_2, p_3, p_4, p_5, and p_6 are the vertices of our original polygon, then we get the locations q_1, q_2, q_3, q_4, q_5, and q_6 of the new polarized polygon by multiplying our matrix times the column vector consisting of p_1, p_2, p_3, p_4, p_5, and p_6 as shown in Figure 2.5. Notice, that if $r = 1/2$, then we have the symmetric construction described above. For $r = 1/3$, the first row can be written as $q_1=\frac{2}{3}\left(\frac{p_3+p_4}{2}\right)+\frac{1}{3}\left(\frac{2}{3}p_2+\frac{1}{3}p_5\right)$ and the second row can be written as $q_2=\frac{2}{3}\left(\frac{p_4+p_5}{2}\right)+\frac{1}{3}\left(\frac{1}{3}p_3+\frac{2}{3}p_6\right)$. Notice that, ignoring horizontal translation of each row in the matrix, the odd rows contain the sequence $\{1-r,1,1,r\}$ and the even rows contain the sequence $\{r,1,1,1-r\}$. This is not surprising since the sequences of distances alternate between the odd sides and then even sides in Figure 2.3.

$$\begin{pmatrix} 0 & 1-r & 1 & 1 & r & 0 \\ 0 & 0 & r & 1 & 1 & 1-r \\ r & 0 & 0 & 1-r & 1 & 1 \\ 1 & 1-r & 0 & 0 & r & 1 \\ 1 & 1 & r & 0 & 0 & 1-r \\ r & 1 & 1 & 1-r & 0 & 0 \end{pmatrix}$$

Figure 2.4

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{pmatrix} = \begin{pmatrix} 0 & 1-r & 1 & 1 & r & 0 \\ 0 & 0 & r & 1 & 1 & 1-r \\ r & 0 & 0 & 1-r & 1 & 1 \\ 1 & 1-r & 0 & 0 & r & 1 \\ 1 & 1 & r & 0 & 0 & 1-r \\ r & 1 & 1 & 1-r & 0 & 0 \end{pmatrix} \div (3) \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{pmatrix}$$

Figure 2.5

3. Designing a "*Wacky Triangular Basket with a Pedestal*" Sculpture

To make an artistically interesting sculpture we shall use the ideas above to construct a sequence of planar hexagons. The following sculpture will construct a sequence of hexagons using the method above for values of r between -2 and 3. It turns out that for all of the sculptures considered in this paper, instead of constructing each planar polygon independently for each value of r, it suffices to construct planar hexagons for only two values of r and then to connect these planar hexagons with "sliding lines" between the corresponding vertices. The vertices of the remaining planar hexagons are located on these sliding lines. To keep the initial two hexagons as simple as possible, we shall use $r = 0$ and $r = 1$. For $r = 1$, the matrix from the previous section instructs us to average three successive vertices. By breaking the triplet into pairs we have $q_1 = \frac{p_3 + p_4 + p_5}{3} = \frac{2}{3}\left(\frac{p_3 + p_4}{2}\right) + \frac{1}{3}p_5$. This says that for the odd sides we need to connect the midpoint of a side to the point a third of the way towards the next vertex. For the even sides we also average a triple, but this time we connect the midpoint of the side to the point a third of the way towards the *previous* vertex. Since this constructs the same point $q_1 = q_2$ and we end up with a triangle. The original hexagon in Figure 3.1 is the outside bold zigzagging hexagon. Applying this procedure creates the six thin lines and the triangle on the bottom half of the figure. The case r=0 switches evens and odds and creates the six thin lines and the triangle on the top half of the figure. Notice that the first procedure merges q_1 and q_2 whereas the second procedure merges q_1 and q_6.

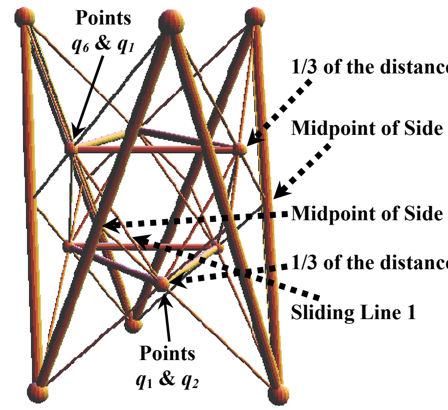

Figure 3.1

Next we connect the top q_1 to the bottom q_1 to create "sliding line 1" as shown. Likewise we create five more sliding lines. As we side, at r=½ we have constructed the hexagon half way along these sliding lines, as shown in Figure 3.2. This is the affine regular hexagon discussed at the beginning of section two. For $r = 1/3$ we slide a third of the way to produce an affine regular image of the non-regular hexagon shown in Figure 2.3. For our sculpture, we shall stretch each sliding line to five times its length and center the extended line on the original line segments, as shown in Figure 3.3. This figure also shows the shape of the hexagon as it slides upward two steps and downward two steps. While initially not obvious, we can check that these six hexagons correspond to the values r = -2, -1, 0, 1, 2, and 3. Applying this process for 36 values of r between -2 and 3 to the random hexagon shown in Figure 3.4 we obtain the computer generated sculpture in Figure 3.5 titled "*Wacky Triangular Basket with a Pedestal.*" Because this process is designed to produce affine images of the hexagon in Figure 2.3 for values of r between -2 and 3, this particular construction will produce an image similar to that shown in Figure 3.5 regardless of the initial random hexagon chosen in Figure 3.4. On the other hand, there was nothing about the process to expect the resulting planar hexagons to end up being parallel. In the remaining sculptures we shall see situations where the planar hexagons do not end up parallel.

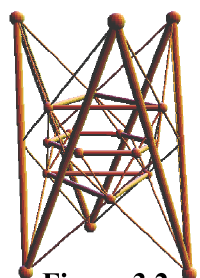

Figure 3.2

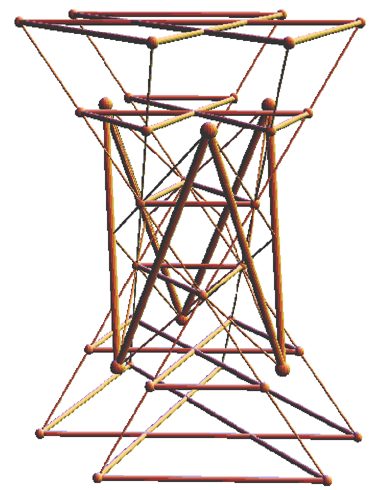

Figure 3.3

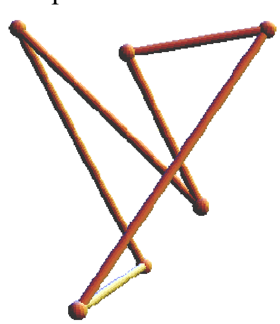

Figure 3.4

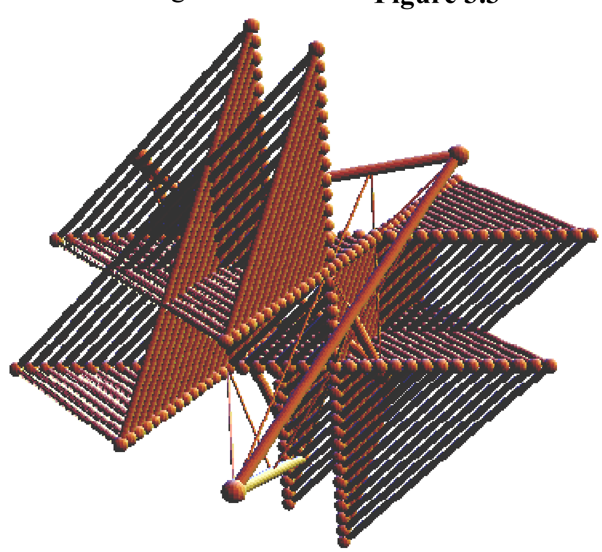

Figure 3.5:
"*Wacky Triangular Basket with a Pedestal*"

4. Designing "*Sailing*"

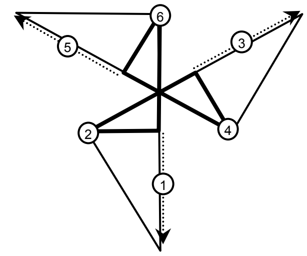

Figure 4.1

Our next sculpture is based upon pinwheel hexagons. Figure 4.1 shows both a small dark pinwheel hexagon and a large light pinwheel hexagon. It also shows how we can slide the vertices 1, 3, and 5 continuously between these two pinwheels while leaving the even vertices fixed. Observe that the light and the dark pinwheels "turn" in opposite directions. That is, once we adjust for size, they are mirror images across a horizontal line of reflection. As before, we start with a random hexagon and then construct affine images of the two pinwheel hexagons shown in Figure 4.1. We attach "sliding lines" between corresponding vertices of these two pinwheel hexagons. And then we construct a sequence of hexagons between these to give us a sense of motion as these transform from one pinwheel to the other.

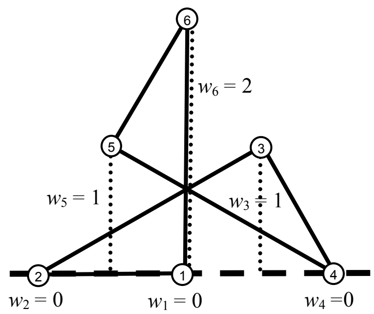

Figure 4.2

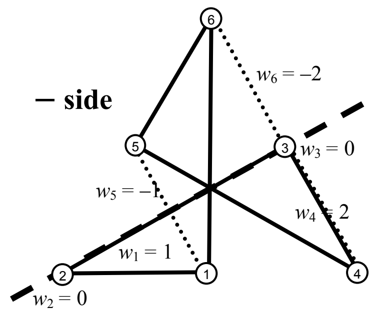

Figure 4.3

The first pinwheel, the smaller version from Figure 4.1, is shown in Figure 4.2. The first column of our matrix consist of the distances {0,0,1,0,1,2} from each vertex to the dashed line through vertices 1 and 2. In Figure 4.3 we see that the line through vertices 2 and 3 cuts through the polygon and so we must denote one side positive and the other side negative. Although the final construction will be different depending upon this choice, they will both have the desired affine image of our pinwheel. By denoting distances above the line as negative column 2 becomes {1,0,0,2,-1,-2}. Because of the even and odd symmetry of our pinwheel, the remaining columns will be similar. These distances form the matrix shown in Figure 4.4. As a final step, each row must sum to the same non-zero value which we must then divide out.

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 2 & 0 & 0 & 2 & -2 \\ 1 & -1 & 1 & 0 & 0 & 1 \\ 2 & -2 & 0 & 2 & 0 & 0 \end{pmatrix} \cdot 1/2$$

Figure 4.4

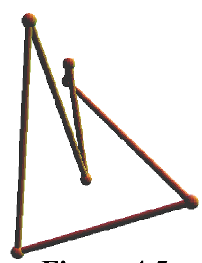

Figure 4.5

Let us now apply this matrix to the random hexagon in 3-space shown in Figure 4.5. From our matrix, row six tells us that vertex q_6 of our pinwheel hexagon must be located at $p_1 - p_2 + p_4$. By rearranging, this is equivalent to $2\left(\frac{p_1 + p_4}{2}\right) - 1p_2$. Thus, as shown in Figure 4.6, we first construct the midpoint m_{14} between points p_1 and p_4. Letting d denote the distance from p_2 to m_{14}, we next construct a line segment of length $2d$ and place this as shown in Figure 4.6. Thus, point q_6 is located at distance d from m_{14} located in the opposite direction from p_2. The remaining evens are constructed in a similar fashion.

From our matrix, our first row tells us that vertex q_1 of our pinwheel hexagon must be located at $\frac{1}{2}p_2 + \frac{1}{2}p_3 - \frac{1}{2}p_4 + \frac{1}{2}p_5$ which can be written as

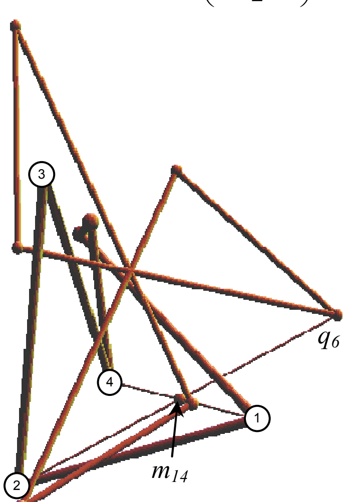

Figure 4.6

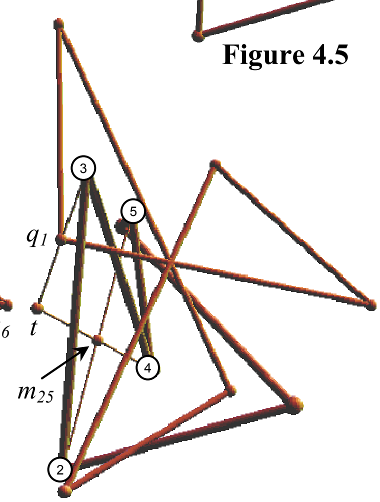

Figure 4.7

$q_1 = \frac{1}{2}\left(2\left(\frac{p_2 + p_5}{2}\right) - 1p_4\right) + \frac{1}{2}p_3$. Figure 4.7 shows this construction where $t = 2\left(\frac{p_2 + p_5}{2}\right) - 1p_4$. Notice the consistency between Figure 4.2 and Figures 4.6 and 4.7. That is, q_6, and hence all the evens, are located on the "tips" of the pinwheel, and q_1, and the odds, are located on the "elbows" of the pinwheel. The complete set of lines used to construct this first pinwheel is shown in Figure 4.8.

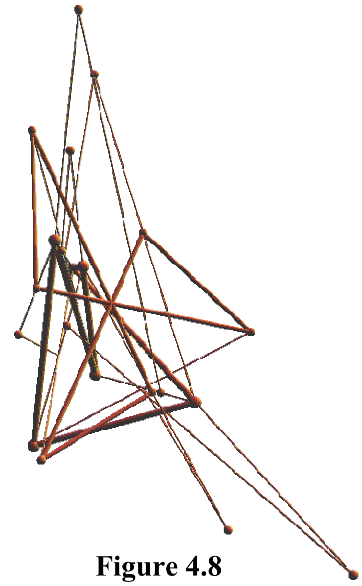

Figure 4.8

Returning back to Figure 4.1 we now wish to construct a pinwheel similar to the larger pinwheel. This time, just to be different, we double the units for the odd columns, as shown in Figure 4.9, but do not double the units for the even columns, as shown in Figure 4.10. (If we doubled both, then this doubling would be divided out when we average each row vector.) Comparing the distances from Figure 4.2, {0,0,1,0,1,2}, to the distances in Figure 4.10 {0,0,4,2,0,2} we notice

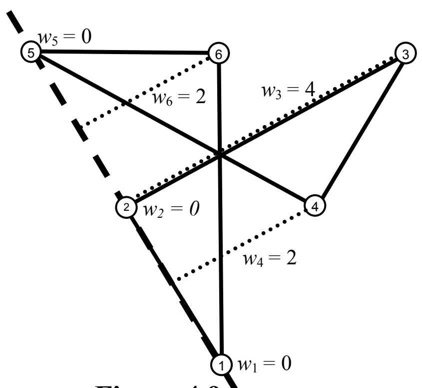

Figure 4.9

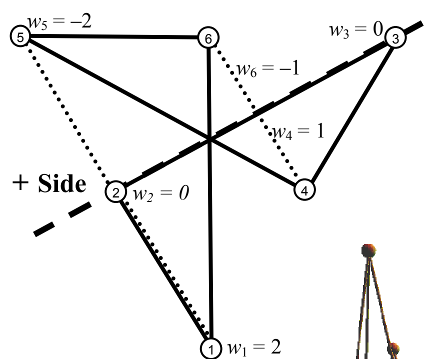

Figure 4.10

$$\begin{pmatrix} 0 & 2 & 0 & -2 & 4 & 0 \\ 0 & 0 & 2 & -1 & 2 & 1 \\ 4 & 0 & 0 & 2 & 0 & -2 \\ 2 & 1 & 0 & 0 & 2 & -1 \\ 0 & -2 & 4 & 0 & 0 & 2 \\ 2 & -1 & 2 & 1 & 0 & 0 \end{pmatrix} \cdot 1/4$$

Figure 4.11

that the difference in orientation of the two pinwheels is reflected in the column vector. The matrix generated by Figures 4.9 and 4.10 is shown in Figure 4.11. When we apply this matrix to the same random hexagon as shown above, we obtain the pinwheel shown in Figure 4.12. This figure also shows the construction lines used to construct q_3 and q_4. We leave the details to the reader with only a hint that the far bottom right ball is located by moving from vertex 6 towards vertex 1 and then going beyond an equal distance.

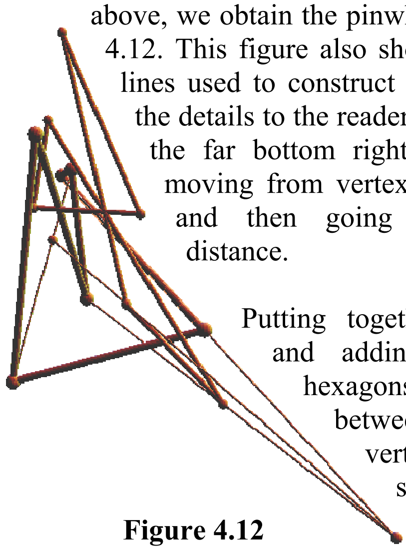

Figure 4.12

Putting together both pinwheels and adding a collection of hexagons for values of r between the corresponding vertices produces the sculpture titled "Sailing" shown in Figure 4.13.

Figure 4.13: "*Sailing*"

5. A Triple of Stellar Hexagon Sculptures

Our final three sculptures are based upon the stellar hexagon formed by the dark lines in Figure 5.1. As before, we add variety by having this hexagon transform shape by having the odd vertices slide in and out. We shall let t denote the distance between the edge of the dotted triangle and the odd vertices and we shall set the height of the dotted triangle equal to 1. Two very simple values for t are t=1 and t=0 as shown in Figures 5.2 and 5.3, respectively. We generate matrices similar to the above examples. To construct the odd vertices of the hexagon shown in Figure 5.2 we find the midpoint of the sides starting with odd vertices. For the even vertices we find the average of four successive vertices, which we can construct by finding the midpoint between two pairs of midpoints. Figure 5.4 shows a random hexagon with these two "hexagons" constructed. Notice that, as triangles, Figure 5.2 has odd vertices and even midpoints, while Figure 5.3 has even vertices and odd midpoints. Thus, when we connect corresponding vertices of the two hexagons in Figure 5.4, midpoints of the triangles are connected with vertices of the other triangle. Figure 5.4 also shows the sliding lines from Triangle A to Triangle B. Notice that all sliding lines start at Triangle A and go to and *beyond* Triangle B. Figure 5.5 shows the hexagon generated for the value, of t = *3,* which generates an affine image of a stellar hexagon of the type shown in Figure 5.1. Notice that this image is at the end of the sliding lines and that the edges moved in slightly from Triangle B whereas the vertices moved out considerably from this same Triangle. This explains why the sliding lines shown in Figure 5.4 are not the same relative length for the odds as for the evens. Figure 5.6 shows a sculpture designed by sliding Triangle A towards Triangle B and beyond to t = *3.*

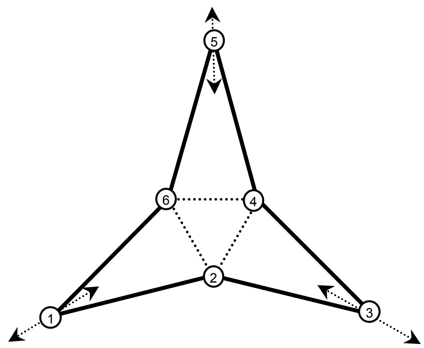

Figure 5.1

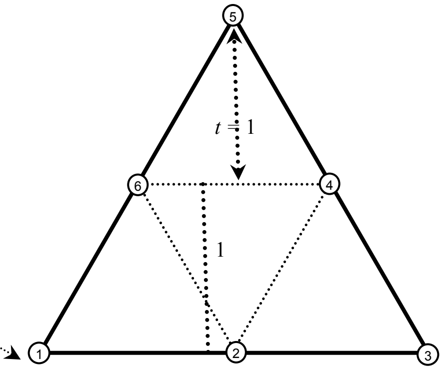

Figure 5.2

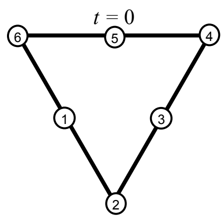

Figure 5.3

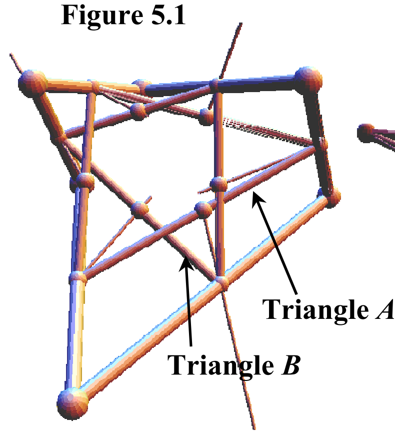

Figure 5.4

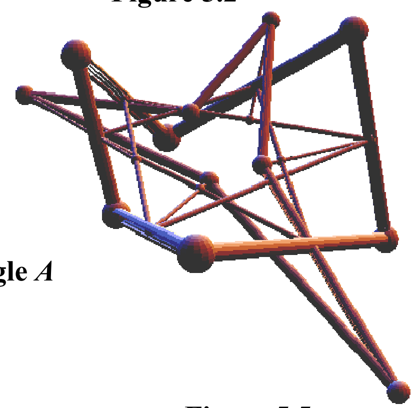

Figure 5.5

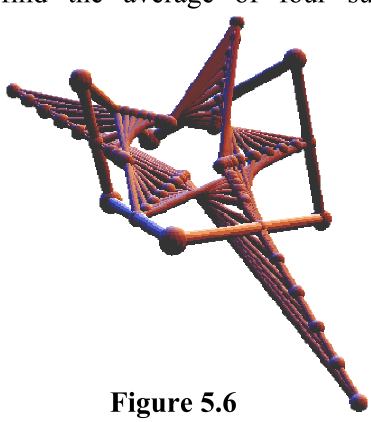

Figure 5.6

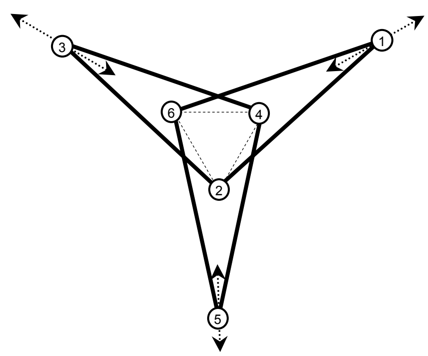

Figure 5.7: *t* = *-3.*

We now consider the interesting case when $t < -1$. This moves the top vertex, point 5, in Figure 5.1 down below the central triangle forming the stellar hexagon shown in Figure 5.7. Applying the

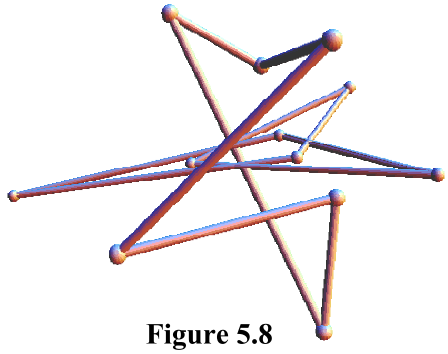

Figure 5.8

procedures, as described above, to a random hexagon produces the affine image shown in Figure 5.8 and applying t for values between $t = -1$ and $t = -3$ generates the sculpture shown in Figure 5.9. Figure 5.10 shows a sculpture for values between $t = -\frac{1}{2}$ and $t = -3$. While this seems to show that our “sliding lines” turn a corner, they in fact stay straight. Rather, by looking at Figure 5.1 we note that for $t = -1$, the hexagon folds up into a triangle traveled twice. This process causes the pairs of sliding lines in Figure 5.10 to cross.

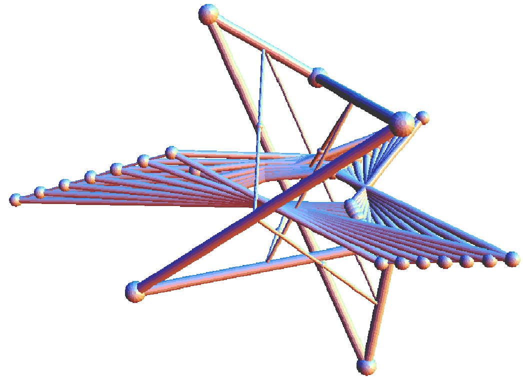

Figure 5.9

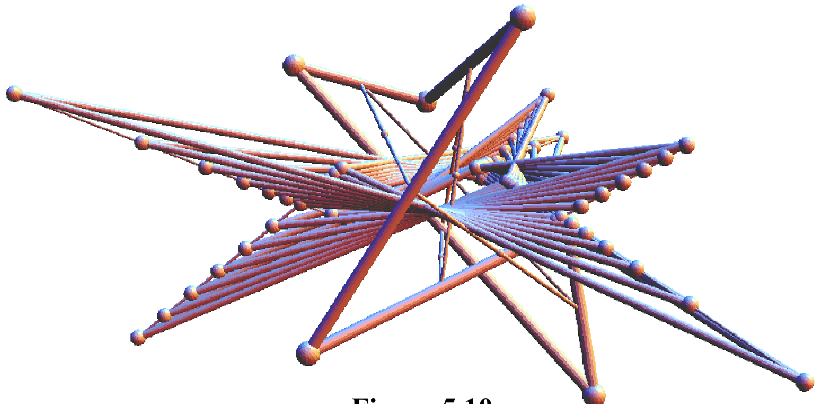

Figure 5.10

Summary

The freedom released by expanding from symmetric linear constructions to non-symmetrical linear constructions vastly expands the potential for interesting sculptures. Rather than merely picking among the rather limited choices of regular polygons, we now have the almost unlimited possibility from non-regular polygons. By creating a transformation from one polygon to another, applying the processes above of measuring distances, creating a matrix and constructing a sequence of linear steps, many surprising and wonderful sculptures can arise.

References

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