Revisiting the Geometry of the Sala de Dos Hermanas

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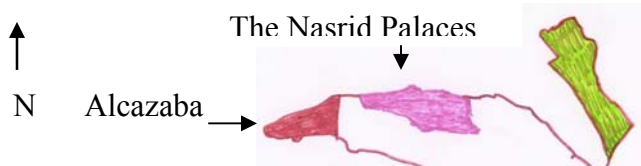E-mail: arob@conncoll.edu

Abstract

The Court of the Lions complex at the Alhambra, with primary focus on the Sala de Dos Hermanas, provides the basis for a Euclidean geometry module for an ethnomath course to be taught spring 2008 at Connecticut College. The module will undertake to: 1. incorporate art history of the Nasrid Kingdom, 2. use rare book collections, modern texts and software, 3. introduce Islamic architecture, 4. survey various mathematical topics found at the monument, 5. examine the poetry of the hall and its court to discuss various themes found at the Alhambra, 6. offer students ideas about making conjectures and 7. apply the subject material to many disciplines. What follows is an overview of the above seven thematic elements.

Why the Hall of Two Sisters?

Some common features of Islamic architecture over time include square rooms; multiple vaults within a larger vault; circular vaults over square rooms; sequences of arches, bays, and columns; arches within rectangular frames including the mihrāb[1] form; stalactite-type hangings (muquarnes); latticed windows; and the use of primary and secondary grids. All of the features are to be found in the Sala de Dos Hermanas (Hall of Two Sisters), a magnificent vaulted square room, where illusion plays a significant role. The hall attracted recent attention with its appearance on a poster for the ICM (International Congress of Mathematicians) meeting of August 2006 in Madrid and the February 2006 cover of the Notices of AMS (American Mathematical Society). Thus the Sala de Dos Hermanas became a candidate for course inclusion. The poetic lines inscribed on its walls, the poem's many references to architectural features, and general "Alhambra" themes present in the hall made it a particularly appropriate choice.

1. History: The Alhambra and the Nasrid Kingdom

The Alhambra sits on a hill in Granada on the outer edge of the Sierra Mountains in southern Spain. To the northeast is located the Generalife with its walks, gardens, fountains, orchards, residences and retaining walls. Vegetation, water, light in both day and night, and defense are all remarkably significant at the Alhambra. At the western edge of the compound sits the fortified town of Alcazaba. The Nasrid Palaces dominate the north central portion of the Alhambra site, the perimeter of which is delineated by defensive walls and towers. This medieval city of palaces and fortresses "could accommodate forty thousand people" [8]: 21.

Figure 1: Sketch of the Alhambra based on [14]

While Roman and Visigothic remains indicate early settlement in the Alcazaba region, the Nasrid kingdom dominated the region including Granada from 1232 to 1492 [6]: 32-3. The four periods of Nasrid art dating from the Emirates of Muhammad I to Yūsuf III will be introduced to the class by an art historian.

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[1] A mihrāb is a concave or shallow opening. The mihrāb of any mosque points to the Ka'ba, the spiritual and physical center of Islam in Mecca. Muslims turn in this direction in prayer five times a day [13]: 209.

Students will learn that "the main Nasrid Palaces, with their ornamentation of unrivalled brilliance and complexity, acquired their present appearance largely during the long reigns of Yūsuf I (1333-54) and his successor Muhammad V [1354-9, 1362-91], when Granada entered what is usually known as its Golden Age" [9]: 28. Interestingly, of the thirteen Nasrid sultans only five died a natural death.

2. Rare Books to Software: Owen Jones and Modern Updates

Owen Jones (1791-1871) was an English architect and designer, who with James Goury, a French architect and Arabic scholar, traveled to Egypt, Turkey, and Spain. There they undertook a detailed study of the Alhambra. Goury and Jones carried out "detailed on-the-spot drawings, but also made paper impressions and casts, and took scrapings of the original colors beneath the overprint" [9]: 28. Their work, an artistic, cultural and scientific study of the Alhambra preserves knowledge of the site that might otherwise have been lost. After Goury died of cholera in 1834, Jones continued their work publishing *Plans, Elevations, Sections and Details of the Alhambra* (2 volumes; London, 1842-5) [11]. Later, Jones' theories were expressed in *The Grammar of Ornament* (London, 1856) [12]. He also became involved with the production of wallpaper, carpets, silk and paper and is generally credited with modern ideas concerning marketing and graphic design. Jones' books provide many of the illustrations for this paper, presentation, and teaching module. Recently, many modern editions of the latter text have been published, including at least two available on cd-rom.

Antonio Fernández-Puertas, Professor of Hispanic Muslim Art History at the University of Granada and past Director of the Museo Nacional de Arte Hispanomusulmán at the Alhambra, is the author of *The Alhambra, I, From the Ninth Century to Yusuf I (1354)* [5]. In it, he compiled an exhaustive study of the Alhambra and reordered the plates of Jones' *Plans, Elevations, Sections and Details of the Alhambra* chronologically. He has also put forth a simple yet elegant theory of a progressive proportional system based on rectangles with relative diagonal lengths of $\sqrt{2}$, $\sqrt{3}$ and 2 for many of the plans and elevations at the Alhambra [5]: 18. Note that the irrational numbers (surds) such as $\sqrt{2}$, approximately equal to 1.414, were not known to mathematicians of the day. Rather, they used approximations such as 7/5 = 1.4 [8]: 117. Thus, comparative dimensions of rectangular regions were generated by approximations to square roots of 2, 3, and 5 [8]. The yet-to-be-published (as of April 2007), *The Alhambra Vol. 2 From Muhammad V (1354) to the Present Day to 1995* and *The Alhambra Vol.3 From 1391 to the Present Day to* 1995, will offer an additional scholarly perspective of the Alhambra.

3. Islamic Architecture

Students will learn that **"**Islamic Architecture is an art not so much of form as of decorative themes that occur both in architecture and in the applied arts, independently of material, scale and technique" [10]: 161. This is an architecture that can be difficult to interpret. "The interpretation … can only be successful and meaningful if it is seen against the backdrop of Islam[2] as a cultural, religious, political phenomenon…" [7]: 14. The function of a building can not usually be determined by its exterior appearance; its rooms might serve various purposes. Documentation of Islamic buildings in the time period of the construction of the Alhambra is poor. Fernández-Puertas reports that he had to use Arabic Chronicles, epigraphic text from the walls, and late Christian records to locate the original Nasrid names of the palaces [5]: 5. While architects could ascend to the upper/ruling classes, architecture was not generally a concern of the literate class. Arabic books solely devoted to architecture are not found before the thirteenth century [13]: 133. "The architectural treatise [of the pre-Christian era] of Vitruvius was probably not translated into Arabic in the early centuries, as was the case with so many other Latin and Greek authors" [13]: 133. *The Calculator's Key,* written by al-Kāshī in 1427 is an instance where mathematics was applied directly to carpentry [2]: 49.

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[2] "Islam, frequently translated as 'submission' [to God] is a contractual relationship between man and God, whereby man acknowledges the overlordship of God ..." [4]: 16.

It also contained tables for setting out arches [13]: 133. See Section 4 A for an example of al-Kāshī's work. Islamic architects were highly skilled in design and calculating ability. "Recent research suggests that 265 manual occupations existed as separate crafts in the Cairo of Fātimid [Fātimid: 969-1171] and Ayyūbid [Ayyūbid: 1171-1250] eras. Some idea of the extent of specialization may be gleaned from the division of woodworking into five different crafts: sawyers,… carpenters, …the maker of wooden door locks, … turners, … and the maker of chests" [13]: 133. This list did not include the decorators or the carvers. It is thought that the craftsmen could read the script of the Quran passages. Dalu Jones, in an article on "Surface, Pattern and Light," states "Primary and secondary grids in Islamic decorative surfaces demand to be 'read'—literally in the case of calligraphy…and the viewer, by reading their texts, also participates in a continuous recomposition of the elements of the decoration" [10]: 164. At times walls were constructed of clay or stucco. Such walls were thicker at the bottom than the top and sometimes had a ground floor of stone. In fact, the name, Alhambra, was derived from the color of its clay walls [5]: xix.

4. Survey of Mathematical Topics found at the Alhambra

The Alhambra is rich in geometric features including the use of proportions, primary and secondary grids, cyclic/dihedral rosettes, one/two dimensional ornamental bands/wallpaper, and prism combinations of muqarnes (mocárabes). A few examples that will be used in the course follow a brief introduction.

Background Information: Muqarnes are stalactite-like ornamentation used for vaults, arches, and cornices, etc. Figure 4 indicates a muqarnes style used in the Mirador de la Daraxa, a northern extension of the Hall of Two Sisters. Muqarnes are made by joining together tiny prisms of plaster or wood to give a honey-comb effect. They are not structural (load-bearing) despite their appearance. The use of muqarnes is a purely Islamic construction but not original to the Alhambra [6]: 175. Sometimes distinction is made between muqarnes with arc cuts from different centers and the western Muslim decoration known as mocárabes [5]: 448. The ceiling of the Hall of Two Sisters is composed of more than five thousand of these exquisite individual prisms of seven basic shapes (Figure 3). It is a suspended cone-shaped vault supported by unseen carved timber lintels over a square room (Figure 6). The ceiling's plan is an octagon (Figure 19). Light from a lower set of eight pairs of windows, set in an octagon, and the trompe d'oeil effects create an other-worldly effect of great height and depth. See Figure 7. Authors writing about the vaulted ceiling state that the intent of the domed ceiling was to represent the heavens and focus on the theme of intense honor and glory due to God [6], [8]. The hall's original Nasrid name is Qubba Mayor—indicating that it was the main vaulted lantern room of the palace [5]: 61. A lantern has a "ceiling or dome [that] is extended upwards with a turret containing windows to let in natural light" [14]: 188.
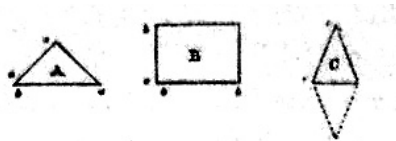

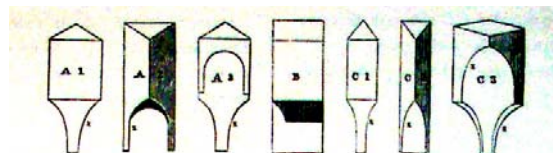**Figure 2:** *Jones' sketch of basic shapes*　　**Figure 3:** *Muqarnes/mocarabes sketched by Jones in [11]*
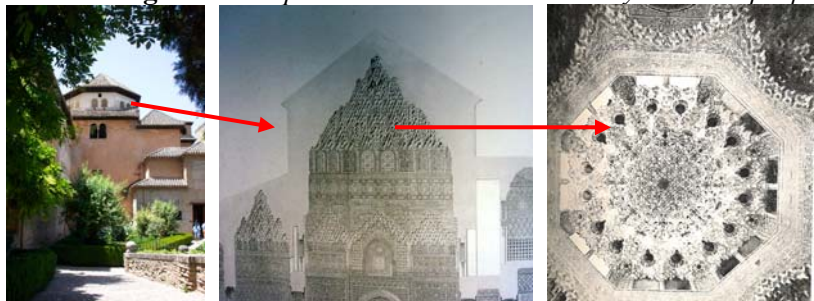

Figure 4: *Photo courtesy of D. Blevins*　　**Figure 5, 6 and 7:** *Hall of Two Sisters from photo, [11] and [3]*
Mirador de la Daraxa

A. Course Example from Medieval Islam : Finding the Curvature of a Muqarnes
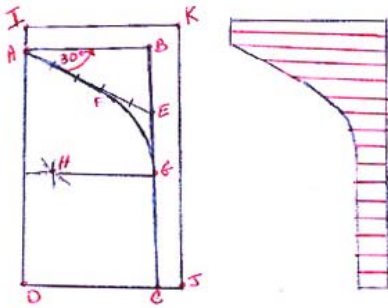

Figure 8: *Curvature of Muqarnes based on algorithm/ sketch of [15]*

Al- Kāshī, a Muslim scientist and mathematician from the latter half of the 14th century, showed that the curvature of a muqarnes could be determined and its surface area calculated in his *Calculator's Key*. Note the length of rectangle ABCD is twice its width. AE is constructed with a 30° angle to the horizontal and divided into 5 equal parts. F is then located 3/5 of the distance from A to E. G is located so that EF = EG = 2/5AE. From both F and G arcs are constructed with a radius equal to FG. The two arcs intersect at H inside rectangle ABCD. Arc FG is then constructed from H (1/6 of the circumference). Line segments DA and DC are extended to points I and J. Line segments KJ (parallel to BC) and IK (parallel to AB) are then drawn …

B. Course Example-- A Plan of Basic Shapes: Rectangles, Right Triangles, and Hexagon
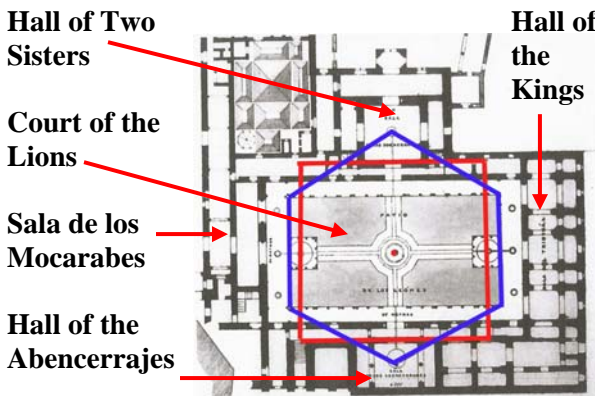

The Court of Lions is rectangular in shape (26.5 x 15.7 m) [6]: 77 with the **Sala de los Mocarabes** on its west**,** the **Hall of Two Sisters** to the north with a further northern extension of the **Mirador de la Daraxa**, the **Hall of the Kings** to the east and the **Hall of the Abencerrajes** to the south. Recently, Irwin, writing about the Alhambra, states that it was lavishly furnished with carpets, cushions and drapes [8]: 32. Further, he adds that because the Moors' vantage point at rest was close to the floor, their windows were placed low on the wall.

Figure 9: *Plan of Court of the Lions Complex from [11] with hexagon*

The triangle formed by the sides of the Court of Lions and its diagonal is described by a 30°, 60°, 90° triangle [5]: 56 and as such has relative sides of $1, \sqrt{3}$ and 2. If one approximates $\sqrt{3}$ by 17/10, the calculation of the smaller angle is arctan (1/ (17/10)) = 30.5°. This agrees closely with the actual measurements of the court: arctan (15.7m/26.5m) = 30.6°.

C. Course Example-- Symmetry of Friezes and Wallpaper

Background Information on Ornamentation at the Alhambra: Design motifs found within the interior walls, frequently in plaster, consist of religious or poetic inscriptions in the form of calligraphy, ataurique (decoration with floral or vegetal motifs), and geometric designs. Ataurique, using palm leaves, palmettes, and pinecones as motifs, may be in a natural or stylized form. In Muhammad V's time (1354-1391), ataurique began to be interwoven with text and used as background within panels [9]: 76. "Lazo or ribbons [are] formed by combinations of flatbands which knot and intertwine with great ingenuity" (Figure 10) [1]: 27. A lazo, based on the design of 6, 8 or 12 pointed stars, can be geometrically complex. It can be a tessellation that can repeat indefinitely and is made of tiles, wood or plaster. Cyclic and dihedral rosettes, and one and two dimensional ornamental bands/wallpaper, are frequently found within a dado, the lower portion of a wall when decorated. Many of the original surfaces at the Alhambra, except the tiling, have been white-washed.
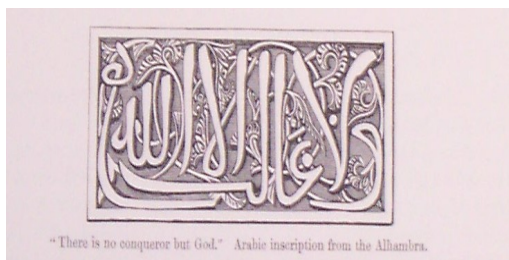
Figure 10: *A Common Inscription from [12]*   **Figures 11 and 12:** *Photos: AR—See below.*

The above figures demonstrate some of the various forms of design found in the Alhambra. The inscription of Figure 10 translates as: "There is no conqueror but God" [12]. The theme of devotion and submission to Allah is seen throughout the Alhambra. Almost all of the surfaces are covered with such detailed ornamentation, that many have remarked, that its effect is to negate its existence entirely. This and the repetitiously subtle nature of the design elements and repeated inscriptions perhaps were intended to put one in a meditative state to honor God. Irwin writes "It [the Alhambra] is a text-laden building, an inhabitable book" [6]: 88.

It is important for students to note that ornamentation was used to cloak the walls with delicate filigree patterns, fill in the backdrop of panels and cartouches where poems and text were inscribed, and provide unifying themes connecting structural pieces and other decorative elements. Different textures within entire walls of ornamentation were commonplace. The purpose of ornamentation was to conceal the core structures rather than reveal them [10]: 144. No one motif dominated but was repeated on different scales.

C. Course Example--Illustrating the Mathematics of Friezes and Wallpaper at the Hall of Two Sisters

Students will learn to use Crystallographers' notation to classify the common tree frieze of Figures 11-13 and other strip patterns. A strip pattern is a pattern that has an indefinite number of repetitions in one direction. This coding scheme has four elements and allows for the classification of seven symmetries. Similarly, the wallpaper patterns of Figures 13-15 will be examined in terms of the notation published by the International Crystallographic Union. Wallpaper patterns are patterns in the plane that have an indefinite number of repetitions in more than one direction. The full notation again consists of four elements and designates seventeen unique patterns/symmetries. The creation of various tilings using ruler and compass will also be examined.

Figure 13   **Figure 14**   **Figure 15**

Figures 13, 14 and 15: *Dados of mosaic tilings from the Hall of Two Sisters from [3] and [11]*

5. Poetry of the Hall of Two Sisters/Court of the Lions Reveals Themes at the Alhambra

Students will survey the poetry found at the Alhambra noting that the fourteenth-century poet and vizier, Ibn Zamrak, composed the poem inscribed on the Hall of Two Sisters' walls. It is clear that the chamber with its ceiling was to be considered the crowned jewel of the Alhambra. The poem begins: "A garden am I,

by beauty adorned: you will know me by what I am if you look at my delights. ~For Muhammad, my sultan, elevates me to the most noble, that ever will be or has been. ~Sublime piece of work. Fortune wishes me to surpass all other monuments! ~What diversion is here for the eye! Here the noble [person] renews his appetite for beauty" [14]: 127-9. The poem continues with prose highlighting cupolas that shine, hidden corners, the moon, the stars, the dawn, God and celestial vaults. It continues with a commentary on fame, clear light, iridescent reflection and blooming gardens [14] 127-9.

A. Alhambra Theme--Intense glory (and submission) to Allah and the use of natural confines, gardens, water, and light to demonstrate harmony with his creations: The Alhambra and the Generalife symbolically represent paradise. Perfection of the heavens was a metaphor to honor Allah. Arab artisans were interested in infinity and saw abstractions as being "close to Allah/God" [8]: 109, 114. Everything was a sign of God. Everything was connected to worshiping and serving God.

B. Alhambra Theme--Defense and Victory: The plan of the Alhambra shows the lack of any central passageway and indicates an overall pattern of successive buildings and palaces. Defense must have been a strong consideration at the Alhambra given its location in Sierra Nevada Mountains and that its perimeter was surrounded by defensive walls and many towers.

The Court of the Lions is thought to have been built in the last half of the fourteenth century by a single architect working closely with the appointed poet of the Alhambra, Ibn Zamrak, and the artisans [4]:110. Grabar believes "The main impetus for building the Court of the Lions and for refurbishing the rest of the Alhambra consisted in Mohammad V's desire to commemorate his successes, particularly, his victory at Algeciras [July 1369]" [6]: 134. Based on the inscription, the fountain and the lions are symbols of the emperor's power and victory: "It *[the fountain] resembles in this the hand of the caliph when it happens that it sheds forth supports towards the lions of the Holy War. ...O thou heir of the Helpers [of Muhammad, the Prophet] and thus not through distant kin, a heritage of glory, enabling you to raise even the well-rooted [mountains], ~God's blessing upon thee and mayest thou be blessed eternally to reiterate celebrations and to wear down thine enemies"* [6]: 127. Fernández-Puertas believes that each sovereign inherited the building projects of his predecessor and chose at will to continue or replace them based on his own needs, the requirements of his kingdom and the style of the period. Because the fountain was constructed at the center of the complex, and the court uses figurative images (the lions and an outstretched hand), he argues that the palace was secular and "intended for the sovereign's daily use of his public life" [5]: 106.

C. Alhambra Theme--Secrecy: The Alhambra is difficult to interpret. There are openings that lead to passageways and others that are similar in appearance but are merely decorative. Grabar, the author of *The Alhambra,* [3] states that the complex is secretive and that there is a discrepancy of how a room appears from its interior to its exterior. Much of the Alhambra, within the Nasrid palaces, is additive architecture. (See Figure 16.) Does this "architecture by accretion" go beyond the usual reasons of defense and kinship? For instance, the most massive towers of the thirty original towers (now twenty-two [8]: 22) are constructed on the northern side which, paradoxically, has its own natural defense.
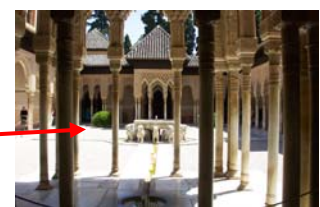

Figure 16: *Plan: Court of Lions from [11];* **Figure 17:** *Photo/AR*   **Figure 18:** *Photo/AR*

D. Alhambra Theme--Illusion: Fernández-Puertas refers to the Court of the Lions complex by its original name, Palacio del Riyād (Garden Palace) [5]: 52. The name is indeed suitable as its court had a sunken garden with below-ground scattered flowers and plants. It created an illusion of a carpet when viewed from above. At its center was the fountain surrounded by twelve lions. It main axes of symmetry were channels

for water that linked its interior halls to the courtyard. In some ways, the separateness of inside and outside did not exist; an illusion of a continuous space was created. Water, as a "solid" compositional element, brought nature and a sense of openness inside. In the words inscribed on the fountain's basin: *"A running stream evokes the illusion of being a solid substance and one wonders which one in truth is fluid"* [6]: 124. The multitude of pillars suggests the imagery of a forest of trees (Figure 18), particularly when viewed from a low vantage point. A succession of archways that were observed from the court contributed to a view of a three dimensional space as continuing—perhaps as a metaphor for eternity. The overall effect was one of harmony.

The ornamentation on the walls gave an illusion of depth and weightlessness. Large objects point to the same motifs minutely expressed as abstractions. The principles of reflection and refraction are used repeatedly with water, glossy tiles, and stained glass windows. Windows were carefully placed to give different optical effects at different times of the day and also at night. Further, the ornamentation and structural elements of arches within rectangles provide a unifying theme; sometimes this design deceptively presents no opening. Mocárabes continue the mihrāb form but with gleaming surfaces that both refract and reflect the light creating similar images that are not real but illusionary.

6. Making Conjectures

Students will be asked if conjectures A, B and C are reasonable. Discussion will follow.

A. Conjecture--Rotating Squares: Does Jones' plan of the ceiling of the Hall of Two Sisters reveal rotating squares—squares that are created by rotating the original set of squares by 45º?
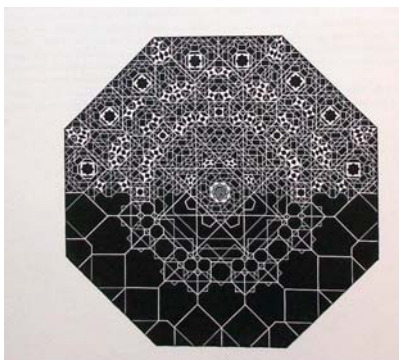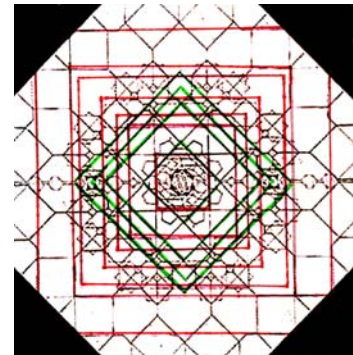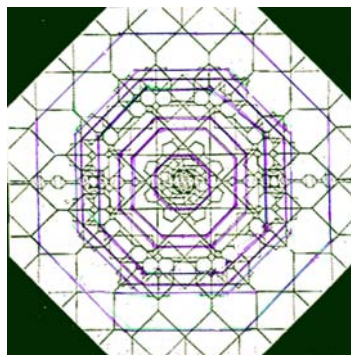

Figure 19: *Jones' Plan from [11]*   **Figures 20 and 21:** Inverted Plans [both based on Figure 19]

Can software help? Here the bottom half of Jones' plan was reflected and black/white then inverted using PhotoShop. Eight concentric octagons including the perimeter are then sketched based on line segments found in the inverted plan (Figure 20). The octagons can be obtained by rotating the 8 (red) squares, including the perimeter, by 45° (Figure 21). Again, only line segments found in the inverted plan were used. Once the eight red squares are produced, another eight squares are formed by rotating the red ones by 45º. Only four of the rotated squares (green) are sketched in Figure 21.

B. Conjecture--The Golden Mean: There is no evidence that the golden mean (that point on a line segment where the ratio of the whole to length of the longer piece is the same as the ratio of the lengths of the longer piece to the shorter one) was used in the construction of the Court of the Lions [5]: 72. However, the proportions used throughout the Alhambra produced harmonious results. In a very detailed analysis from the 1950s George Marçais, an architectural historian, sketched a two dimensional section of the portico of the Court of the Lions. He ignored 90 degree turns (corners). The scheme showed the archways drawn contiguously in a linear fashion. He then claimed that northeastern archway of the Court of Lions demonstrated various overlapping axes of symmetry with "proportions between parts in the elevation of the

portico organized according to … principles of the Golden Mean" [6]: 182. Could the golden mean inadvertently be present? Is it reasonable to investigate his claim?

C. Conjecture--Dynamic Squares: Does the Hall of Two Sisters' plan contain dynamic squares where each diagonal becomes the length of the next larger square/rectangle? The question was motivated by a comment found in Grabar's *Alhambra* [6]: 165. Does the floor plan relate to the architectural features of the elevation in a proportional way? Do the individual heights of dados, doorways, arches, windows follow a system of proportion?
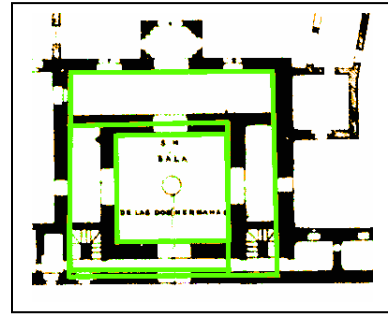

Figure 22: *Jones' Plan with overlay from [11]*

7. Applications across the Disciplines

In the ethnomath course, group projects will consist of an abstract, paper, and class presentation. The students will then be challenged to complete additional projects that relates to their majors. Initial suggestions might include: **A**. a composition of a piece of music incorporating some of the proportions encountered at the Alhambra, **B.** a summary and critique of Washington Irving's editions of his *Alhambra*, **C**. a conceptual model of the water/irrigation systems used at the Alhambra, **D.** overlay maps of the four periods of Nasrid art history, **E.** a design based on a motif, plan, etc. found at the monument, **F.** a computer-aided-design rendering of muqarnes, **G.** tessellations produced in Escher style, **H.** a dance incorporating some of the geometric elements or other themes encountered at the Alhambra, …

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