Poverty and Polyphony:
A Connection between Economics and Music

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Abstract

In polyphonic music, simultaneous melodic lines, or "voices," create a sequence of vertical harmonies or "chords." The melodic voices determine mappings, or *voice leadings*, from the notes of one chord to those of the next. While we have clear intuitions that some voice leadings are smaller than others, our intuitions are not robust enough to determine a precise "metric" of voice-leading size. Tymoczko (2006) proposed that any voice-leading metric should be consistent with two principles: (1) small voice leadings move their voices by short distances, and (2) small voice leadings move their voices along non-intersecting paths. We show that the partial order imposed on voice leadings by these constraints is equivalent to the submajorization partial order, originating in 1905 with the economist Lorenz. We further show how to use submajorization to compare distances between "chord types." Finally, we highlight surprising connections between the results discussed in this paper and problems in welfare economics.

1. Introduction

Western polyphonic music exhibits two dimensions of musical coherence: it articulates simultaneous melodies, each of which typically moves by short distances in pitch; and it articulates sequences of harmonies, which typically sound similar to one another. In Western musical notation, these two dimensions are represented spatially, with melodies notated horizontally and harmonies notated vertically. The challenge in composing is to design arrays of notes that are coherent in both ways at the same time.

So, for example, music students are often asked to realize a sequence of chords as a sequence of simultaneous melodic voices or lines. Two possible realizations of the chord progression C-F-C-G-C are depicted in figure 1. If the exercise is done well, as on the left, the notes assigned to individual voices form pleasing melodies. Furthermore, the individual voices will tend to avoid "voice crossings" in which a lower voice moves above a higher voice.[1] If voices constantly make great leaps in pitch, as on the right, we hear the music as choppy or disjointed, and it can be hard to distinguish the individual melodic lines. Such realizations also tend to be difficult for musicians to sing or play.

The practice of devising simultaneous melodies that form pleasing chord-successions is called counterpoint. A passage of contrapuntal music can be analyzed as a series of mappings, or voice leadings, between the pitches in adjacent chords. There is a sense in which voice leadings can ordered by size: the mapping between the first two chords on the left in figure 1 is "smaller" than the mapping between the first two chords on the right, because each voice moves a smaller distance. Unfortunately, our intuitions do not tell us how to compare two arbitrary voice leadings, let alone assign a specific number "measuring" their size. Should we compute the aggregate distance traveled by the voices? Should we weight large leaps more than small ones? Or should we use an entirely different metric?

[1] Anyone who has studied beginning music theory also remembers that there are several other constraints: for example, the prohibition against parallel fifths or octaves. However, these rules apply only to some styles of Western music. We are considering more general principles common to a wider range of styles.

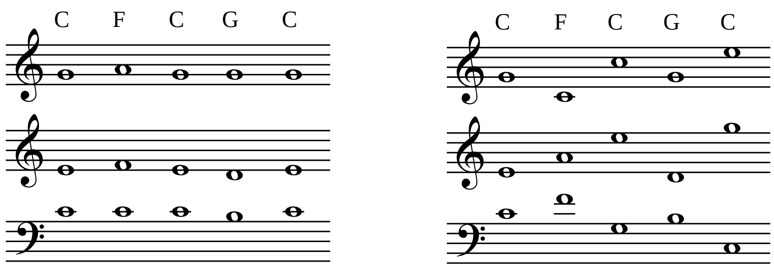

Figure 1: *Two voice leadings associated with the chord progression C-F-C-G-C.*

Music theorists have not devoted much attention to these questions; instead, they typically choose some particular metric without attempting to justify it. In this paper, we follow a different approach, stating conditions that any "reasonable" method of comparing voice leadings must satisfy. These conditions turn out to be quite restrictive; for example, we show that all reasonable metrics agree about the minimal voice leadings between two major or two minor triads and between two dominant or two half-diminished sevenths. In addition, we develop an algorithm for comparing classes of voice leadings between "chord types" that goes some way toward modeling our intuitions about their similarity.

Surprisingly, the topics we discuss are similar to topics arising in welfare economics. There are three reasons for this. First, music and economics both involve unordered collections of real numbers: just as a chord can be represented as an unordered collection of pitches, so can the distribution of wealth in a society can be represented as a multiset of individuals' assets. Second, in both music and economics these unordered collections can be more or less close to one another: it can take more or less economic "work" to transform one wealth distribution into another, just as it can take more or less musical "work" to move from one chord to another. And third, both music theorists and economists confront uncertainty about just how to measure the "distance" between unordered collections. In both fields, therefore, it is interesting to ask whether there are constraints shared by all reasonable metrics.

2. Basic terminology

Listeners comprehend music by abstracting from information—for example, treating the ordered pitch sequences (C4, E4, G4)[2] and (E2, G4, C5) to be instances of the same musical object, the C major chord. In this section we summarize recent work by Callender [2], Tymoczko [8], and Callender, Quinn, and Tymoczko [3] that uses quotient spaces to model this process of abstraction. Individual objects such as notes and chords can be represented as points in quotient spaces formed by identifying or "gluing together" points in $\mathbb{R}^n$. Sequences of musical objects, such as voice leadings and chord progressions, can be represented as line segments or pairs of points in these spaces.

Pitches are real numbers representing the logarithm of a note's fundamental frequency. Conforming to the MIDI standard, we set middle C equal to 60 and the size of the octave equal to 12. Thus pitch $=$ $69 + 12\log_2(\text{frequency}/440)$. (Note that pitch space is continuous, and not limited to the discrete pitches of Western equal temperament.) A musical *object* is an ordered tuple of pitches: (C4, E4, G4), or (60, 64, 67), is the object whose first element is middle C, and whose second and third elements are the E and G above that. *Progressions* are ordered tuples of objects. Thus (C4, E4, G4)→(C4, F4, A4) is the progression whose first object is (C4, E4, G4) and whose second is (C4, F4, A4).

Musicians classify objects and progressions using the five "$\mathbf{OPTIC}$ symmetries": octave displacement

[2] In "scientific pitch notation" pitches are indicated by combining a letter name with an integer indicating the pitch's octave. C4 is middle C, while C5 and C3 are an octave above and below C4, respectively. Octaves run from C to C, so that B3 is a semitone below C4.

(**O**), which identifies (C4, E5) with (C2, E6); permutation (**P**), which identifies (C4, E5, G4) with (E5, C4, G4), transposition or translation (**T**), which identifies (C4, E4) with (G4, B4); inversion or reflection (**I**), which identifies (C4, E4, G4) with (G4, E♭4, C4); and cardinality equivalence (**C**), which identifies (C4, C4, E4) with (C4, E4). These operations are described in table 1. Every combination of the four $\mathbf{OPTI}$ operations generates a quotient space of $\mathbb{R}^n$. The quotients by $\mathbf{P}$ and $\mathbf{I}$ are orbifolds, since they contain singularities. The $\mathbf{C}$ operation generates an infinite dimensional "Ran space" that is quite difficult to visualize.

	Symmetry	**Geometrical space**
None		$\mathbb{R}^n$
Octave	$x \sim_{\mathbf{O}} x + 12i, \quad i \in \mathbb{Z}^n$	$\mathbb{T}^n$ (n-torus)
Transposition	$x \sim_{\mathbf{I}} x + c(1, \ldots, 1), \quad c \in \mathbb{R}$	$\mathbb{R}^{n-1}$ or $\mathbb{T}^{n-1}$ (Orthogonal projection creates a simplicial coordinate system)
Permutation	$x \sim_{\mathbf{P}} \sigma(x), \quad \sigma \in \mathcal{S}_n$	add $/\mathcal{S}_n$
Inversion	$x \sim_{\mathbf{I}} -x$	Add $/\mathbb{Z}_2$ [or $/(\mathcal{S}_n \times \mathbb{Z}_2)$ if in conjunction with $\mathbf{P}$]
Cardinality	$(\ldots, x_i, x_{i+1}, \ldots) \sim_{\mathbf{C}} (\ldots, x_i, x_i, x_{i+1}, \ldots)$	Infinite dimensional "Ran space"

Table 1: *The five principal symmetries in Western music theory.*

There are two ways to apply the $\mathbf{OPTIC}$ operations to progressions. Let S refer to a collection of $\mathbf{OPTIC}$ operations. For musical objects $O_1, \ldots, O_k$, the progression $O_1 \to O_2 \to \ldots \to O_k$ is *globally* S-equivalent to $s(O_1) \to s(O_2) \to \ldots \to s(O_k)$ for any s in S. The progression $O_1 \to O_2 \to \ldots \to O_k$ is *locally* S-equivalent to $s_1(O_1) \to s_2(O_2) \to \ldots \to s_k(O_k)$ for s_i in S. Thus, global equivalence requires that a single operation s transform each object in the first progression into the corresponding object in the second; local equivalence requires only that *some* operation in S relate corresponding objects in the two progressions.

A large number of familiar musical terms can be described using this formalism. A *pitch class* is an equivalence class of pitches under octave displacement (**O**). Pitch classes are points on the circle $\mathbb{R}/12\mathbb{Z}$, and can be represented by numbers in the range $[0,\ 12)$. Integer pitch classes ($\mathbb{Z}/12\mathbb{Z}$) make up the chromatic scale of *twelve-tone equal temperament*, with 0 representing the pitch class C, 1 representing C♯, and so on. A *chord* is an equivalence class of objects under octave transposition and permutation—in other words, a multiset[3] of pitch classes. For example, the multiset {C, E, G}, or $\{0, 4, 7\}$, represents the C major chord. A *chord type* or "transpositional set class" is an equivalence class of objects under $\mathbf{OPT}$. Familiar terms like "major chord," "minor chord," and "major scale" refer to chord types.

A *chord progression*—the kind of thing one sees in a guitar fakebook—is a sequence of chords without any implied mapping between the chords' notes. Chord progressions are equivalence classes of progressions under local applications of $\mathbf{O}$ and $\mathbf{P}$. They are represented geometrically by ordered pairs of points in $\mathbb{T}^n/\mathcal{S}_n$, the n-torus modulo the symmetric group of order n. By contrast, a *voice leading* is an equivalence class of progressions under global $\mathbf{O}$ and $\mathbf{P}$, represented geometrically by the image, in $\mathbb{T}^n/\mathcal{S}_n$, of a line segment in $\mathbb{R}^n$. Voice leadings show how the notes of one chord "move" to those of another. Voice leadings are notated (C, E, G)$\xrightarrow{-1,0,1}$(B, E, G♯). This indicates that the C in the first chord moves down by semitone, the E stays fixed, and the G moves up by semitone. We omit the numbers above the arrow when they all lie in the range $(-6,\ 6]$.

[3]Representing chords as multisets (rather than sets), though somewhat unusual, *is* musically meaningful—for example, it is common for two voices in four-part harmony to double a pitch class.

3. Measuring voice leadings and the distribution constraint

Let $(x_1, x_2, \ldots, x_n) \xrightarrow{d_1, d_2, \ldots, d_n} (y_1, y_2, \ldots, y_n)$ be a voice leading. Its *displacement multiset* is the multiset of distances moved by each voice, $\{|d_1|, \ldots, |d_n|\}$. We are interested in measuring the voice leading's "size"—the amount of musical "work" done in moving from the first chord to the second. We assume that the size of the voice leading depends only on the voice leading's displacement multiset, and that it is nondecreasing in each of the multiset's elements: that is, the voice leading with displacement multiset $\{|d_1| + |c|, |d_2|, \ldots, |d_n|\}$ is at least as large as that with displacement multiset $\{|d_1|, |d_2|, \ldots, |d_n|\}$ (see figure 2, left).

What are the minimal constraints on measures of voice-leading size? To answer this question, DT [8] proposed the "distribution constraint." It says that the composer's aim of avoiding voice crossings should be compatible with the aim of minimizing voice-leading size, however we measure it.[4] See figure 2, right, for an illustration. The distribution constraint imposes a partial order on multisets of nonnegative real numbers that may be expressed in a number of different ways.

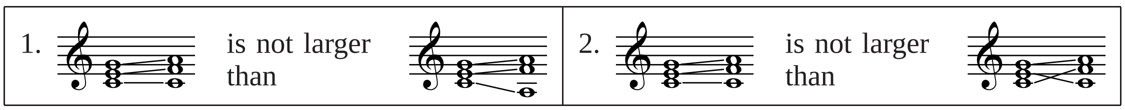

Figure 2: *The distribution constraint.*

In what follows, $x_{[i]}$ denotes the ith largest element of the multiset $\{x_1, \ldots, x_n\}$.

Theorem 1 *Let $\leq$ be a partial order on the space of n-element multisets of nonnegative real numbers that is nondecreasing in each of its elements. The following definitions of $\leq$ are equivalent:*

1. The Dalton transfer principle. *Let $\{x_1, \ldots, x_n\}$ be a multiset of nonnegative real numbers. For any nonnegative c and any pair of indices (i, j) where $x_i \leq x_j$,*

$$\{x_1, \ldots, x_i + c, \ldots, x_j, \ldots, x_n\} \leq \{x_1, \ldots, x_i, \ldots, x_j + c, \ldots, x_n\}. \quad (1)$$

2. The no crossings principle. *For any multisets of real numbers $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$,*

$$\{|b_{[1]} - a_{[1]}|, \ldots, |b_{[n]} - a_{[n]}|\} \leq \{|b_1 - a_1|, \ldots, |b_n - a_n|\}.$$

   In other words, there is a minimal length voice leading $A \rightarrow B$ between any two multisets of pitches such that if $a_i < a_j$ in the source, then $b_i \leq b_j$ in the target.

3. The weakened triangle inequality. *There is a minimal-length path between any two points in $\mathbb{R}^n/\mathcal{S}_n$ or $\mathbb{T}^n/\mathcal{S}_n$ that does not pass through a singularity.*

4. Submajorization. *Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be multisets of nonnegative real numbers. Then $X \leq Y$ if and only if*

$$\sum_{i=1}^{j} x_{[i]} \leq \sum_{i=1}^{j} y_{[i]} \ \text{ for } 1 \leq j \leq n. \quad (2)$$

[4] If the aim of avoiding voice-crossings were incompatible with the aim of minimizing voice leading size, we would expect to see two distinct sets of voice-leading preferences: when voices were close together, the goal of avoiding crossings would trump the goal of minimizing voice-leading size; when voices were far apart, crossings would pose no danger, and minimal voice leadings could be used freely. However, composers' voice leading preferences do not typically depend on how far apart musical voices are; therefore it is reasonable to postulate that the goal of avoiding crossings is compatible with the goal of minimizing voice-leading size.

DT [8] proved that the Dalton transfer principle and the no crossings principle are equivalent. The weakened triangle inequality is a restatement of the no crossings principle in the language of geometry. The final condition is better known as the definition of the submajorization partial order:

Definition 1 *Let X and Y be multisets of nonnegative real numbers. We say that Y* submajorizes *X (written $X \prec_{\mathrm{w}} Y$) if and only if inequality (2) holds.*

Submajorization is a weakened form of the majorization partial order, which requires equality in (2) when $j = n$. The equivalence of (1) and (2) when a partial order is increasing in each of its elements follows from lemmas 1 and 2 in Hardy, Littlewood, and Pólya [5]. Henceforth, we will use the symbol $\prec_{\mathrm{w}}$ to indicate the partial order of theorem 1.

Majorization originated with the economist Lorenz [6], who reasoned that taking wealth away from a rich person and giving it to a poorer person should not make a society less fair; this is the so-called *Dalton transfer principle*. It imposes a partial order on multisets of nonnegative numbers representing individual incomes or assets. Note that the condition that the partial order be nondecreasing in each of its elements does not apply, because the total wealth in the society remains constant. We explore economic applications more fully in section 6.

Many have proposed measures—that is, functions from the space of displacement multisets to the non-negative reals—for voice-leading size (see [8]). We say that $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is an "acceptable" voice-leading metric if and only if $f(X) \leq f(Y)$ whenever $X \prec_{\mathrm{w}} Y$; in general, any method of comparing (but not necessarily measuring) voice leadings that respects the submajorization ordering is "acceptable." Examples include L^p-norms for $p \geq 1$ and reverse lexicographic order.

3.1. The geometry of submajorization. We develop the geometry of submajorization in $\mathbb{R}^n$ with a view towards comparing distances in the geometrical spaces described in section 2. In what follows, let $\Sigma_j(X)$ denote the sum of the j largest elements of a multiset X (that is, $\Sigma_j(X) = \sum_{i=1}^{j} x_{[i]}$). In addition, for a vector $\mathbf{v} = \{v_1, \ldots, v_n\} \in \mathbb{R}^n$, let $\{|v_i|\} = \{|v_1|, \ldots, |v_n|\}$.

Definition 2 *The* submajorization ball *of a multiset X of nonnegative real numbers is the set of vectors $\mathbf{v}$ such that $\{|v_i|\} \prec_{\mathrm{w}} X$. The* Σ_j-ball *of X is the set of vectors $\mathbf{v}$ in $\mathbb{R}^n$ such that $\Sigma_j(\{|v_i|\}) \leq \Sigma_j(X)$.*

The submajorization ball is the intersection of Σ_j-balls for $1 \leq j \leq n$. In general, a Σ_j-ball is a set of simultaneous solutions to a system of linear inequalities of the form $e_1 v_1 + \ldots + e_n v_n \leq \Sigma_j(X)$, where each e_i is chosen from the set $\{0, 1, -1\}$, and exactly j of the e_i are nonzero. (Note that $\Sigma_1(X) = L^\infty(X)$ and $\Sigma_n(X) = L^1(X)$.) Each Σ_j-ball is a convex polytope that is invariant under the group of signed permutations. For example, in $\mathbb{R}^3$, the Σ_1-ball is a filled cube, the Σ_2-ball is a filled rhombic dodecahedron, and the Σ_3-ball is a filled octahedron. The intersection of the Σ_j-balls for $1 \leq j \leq n$—that is, the submajorization ball—is a convex polytope whose vertices are the signed permutations of the elements of X. The set of vectors $\mathbf{u}$ such that $\{|u_i|\}$ submajorizes X is the complement (in $\mathbb{R}^n$) of the union of the interiors of the Σ_j-balls for $1 \leq j \leq n$. If a vector $\mathbf{y}$ does not lie in either this complement or in the submajorization ball, $\{|y_i|\}$ and X are not comparable.

Since the orbifolds $\mathbb{T}^n/\mathcal{S}_n$ representing the spaces of chords locally resemble $\mathbb{R}^n$ except at their singularities, the submajorization ball gives us an idea of what it means for one chord to be "closer" than another to a given chord. Let A and B be chords in $\mathbb{T}^n/\mathcal{S}_n$ sufficiently far from any singularity of the orbifold, and let D be the displacement multiset of a minimal voice leading from B to A. The set of chords closer than B to A is congruent to the submajorization ball of D in $\mathbb{R}^n$. Figure 3 (left) shows the polytope—in this case an octagon—containing the intervals (two-voice chords) "closer" than DF♯ to CF situated in two-dimensional chord space $\mathbb{T}^n/\mathcal{S}_2$. Intervals lying in the exterior of the star are "farther" from CF than DF♯ is from CF. At right, the same relationship is depicted for the intervals DF and CF.

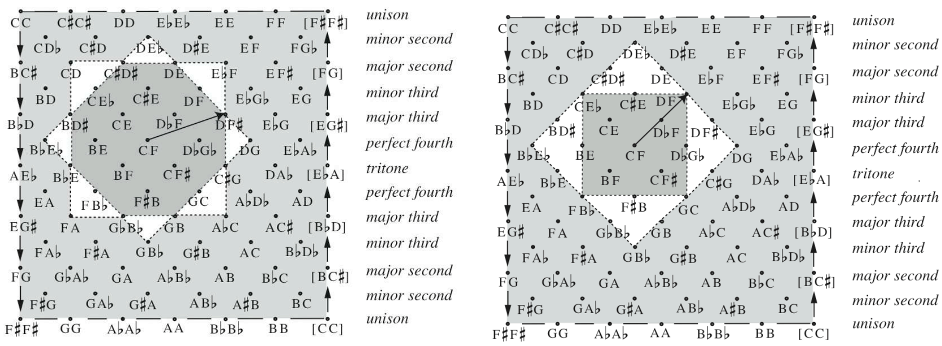

Figure 3: *Submajorization polytopes in $\mathbb{T}^n/\mathcal{S}_n$ (Möbius strip).*

4. Measuring distance between chord types

A *chord type* or "transpositional set class" is an equivalence class of n-tuples of pitches under octave displacement, permutation, and transposition. Chord types are equivalence classes of unordered sets of points on the pitch circle modulo rotation. In order to capture the voice-leading possibilities of chord types, we return to the orbifold models introduced in section 2. The space of chord types $\mathbb{T}^{n-1}/\mathcal{S}_n$, called *transposition-class space*, or $\mathbf{T}$-*space*, is the projection of the space of chords $\mathbb{T}^n/\mathcal{S}_n$ onto the zero-sum plane (see [2, 8, 3]). Points in this space represent chord types. Paths represent voice leadings modulo the individual transposition of either chord; they are the projections of directed line segments in $\mathbb{R}^n$ onto $\mathbf{T}$-space.

We have a sense that some chord types offer fairly similar voice-leading possibilities and that others offer very different choices: major chords like {C, E, G} seem very similar to augmented triads like {C, E, G♯}, and not so similar to clusters like {C, C♯, D}. Any measure of voice-leading size provides some measure of distance between chord types: let the distance between two chord types be the size of the minimal voice leading between chords belonging to those types. However, as in our discussion of submajorization, we would like to compare distances in $\mathbf{T}$-class space without committing ourselves to a particular metric. We use the partial order that the distribution constraint imposes on displacement multisets to define a "distance" between chords modulo transposition of either chord. This allows us to compare voice leading possibilities between chord types. As in our discussion of submajorization, we begin by defining a method of comparing distances (think voice leadings) in $\mathbb{R}^n/\mathbf{T} \simeq \mathbb{R}^{n-1}$, and then use the fact that this space locally resembles $\mathbf{T}$-class space.

Definition 3 *Let* $\mathbf{v}$ *be and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^n$*. We say that* $\mathbf{w}$ *is* $\mathbf{T}$-smaller *than* $\mathbf{v}$ *if, for any real number* x*, there exists a real number* y *such that*

$$\{|w_i + y|\} \prec_{\mathrm{w}} \{|v_i + x|\}. \tag{3}$$

In other words, $\mathbf{w}$ is $\mathbf{T}$-smaller than $\mathbf{v}$ if and only if the set $\{\mathbf{w}+y(1,\dots,1)|y \in \mathbb{R}^n\}$ has a representative that lies in the intersection of the submajorization balls of $\{|v_i + x|\}$ for $x \in \mathbb{R}$. Therefore, $\mathbf{w}$ is $\mathbf{T}$-smaller than $\mathbf{v}$ if the projection of $\mathbf{w}$ to the zero-sum plane lies in the projection of the intersection of the submajorization balls of $\{|v_i + x|\}$ onto the zero-sum plane.

Definition 3 does not tell us how to determine an ordering based on $\mathbf{T}$-smallness in finite time. Theorem 2 shows that it suffices to find y in inequality (3) for finitely many x, thus permitting a polynomial-time algorithm for $\mathbf{T}$-closeness.

Theorem 2 *Let* $\mathbf{v}$ *be a vector in* $\mathbb{R}^n$*, let* S *be the set of points* $S = \{(1/2)(v_i + v_j) : 1 \le i \le j \le n\}$*, and* *let $M \subset S$ be the set $M = \{(1/2)(v_{[i]} + v_{[n-i+1]}) : 1 \le i \le \lceil n/2 \rceil\}$. Let S' be the set of elements of S that are no less than $\min(M)$ and no greater than $\max(M)$. The vector $\mathbf{w}$ is $\mathbf{T}$-smaller than $\mathbf{v}$ if and only if for each s in S' there exists a y such that*

$$\{|w_i + y|\} \prec_{\mathrm{w}} \{|v_i - s|\}. \tag{4}$$

Corollary 1 *If $\mathbf{v}$ is symmetric about a value m, then $\{|v_i - m|\}$ is minimal.*

We now develop a notion of distance in transposition-class space. Let $\mathbf{T}_x(A)$ represent the transposition of chord A by x pitches, and let $[A] = \{\mathbf{T}_x(A) : x \in \mathbb{R}\}$ represent the chord type of A.

Definition 4 *Let A, B, and C be chords. We say that $[C]$ is* $\mathbf{T}$-closer *to $[A]$ than $[B]$ is to $[A]$ if, for every voice leading from A to some transposition $\mathbf{T}_x(B)$, there exists some $\mathbf{T}$-smaller voice leading from A to a transposition of C.*

Example. We claim that the chord type $[\{0, 4, 7\}]$ of major triads is $\mathbf{T}$-closer to the class of augmented triads $[\{0, 4, 8\}]$ than the cluster $[\{0, 1, 2\}]$ is to the class of augmented triads. The voice leading $(x, 1+x, 2+x)\to(0, 4, 8)$ has displacement multiset $\{|x|, |x-3|, |x-6|\}$ and the voice leading $(y, 4+y, 7+y)\to(0, 4, 8)$ has displacement multiset $\{|y|, |y|, |y-1|\}$. We must show that for every x there exists a y such that $\{|x|, |x-3|, |x-6|\}$ submajorizes $\{|y|, |y|, |y-1|\}$. In this case, setting y equal to zero suffices for all x.

A voice leading is *inversionally symmetric* if the multiset of directed distances traveled by each of its voices is symmetric about some value m. Corollary 1 implies that all metrics agree on how to minimize an inversionally symmetric voice leading. Therefore, all metrics agree about the minimal voice leadings between two perfect fifths, two major or two minor triads, and two dominant or two half-diminished sevenths. These are among the most common voice leadings we find in Western tonal music.

If $[A]$ and $[B]$ are chord types sufficiently far from the boundary of the orbifold, the set of chord types $\mathbf{T}$-closer than $[B]$ to $[A]$ is a polytope centered at $[A]$ in **OPT** space; it is congruent to the projection of the intersection of the submajorization balls for minimal voice leadings[5] of the form $A\to\mathbf{T}_x(B)$. In trichordal **OPT** space, these polytopes are the intersections of two hexagons, one rotated 30° from the other.

5. Evenness

In this section, we develop an application of $\mathbf{T}$-closeness to measuring the "evenness" of a chord type. For any n, we refer to the chord $\{0, 12/n, 24/n, \ldots, 12(n-1)/n\}$ as the *perfectly even chord*; its chord type consists of the chords that divide the octave into n equal parts.

Definition 5 *The chord type $[A]$ is* more even *than the chord type $[B]$ if $[A]$ is $\mathbf{T}$-closer than $[B]$ to the perfectly even chord type.*

Figure 4 shows the evenness ordering on the orbifold $\mathbb{T}^2/(\mathcal{S}_3 \times \mathbb{Z}_2)$ representing trichords modulo inversion. Arrows indicate contours along which $\mathbf{T}$-closeness to the perfectly even chord type (augmented triads) increases. $\mathbf{T}$-closeness imposes a univocal ordering on trichords in twelve-tone equal temperament, with the next-most-even chord type being $[\{0, 3, 7\}]$, which represents both the major and minor triads. However, outside of 12-tet there exist incomparable trichords (for example, $[\{0, 2.2, 4.3\}]$ and $[\{0, 1, 5\}]$). We depict the $\mathbf{T}$-closeness tetrachord ordering on the orbifold $\mathbb{T}^3/(\mathcal{S}_4 \times \mathbb{Z}_2)$ in figure 5, left; a portion of the same partial ordering is abstracted in the lattice at right. The diminished seventh $[\{0, 3, 6, 9\}]$ is perfectly even,

[5] There is a slight subtlety here. Since we are also dealing with the quotient operation $\mathbf{O}$, there may not be a unique class of crossing-free voice leadings between chord types $[A]$ and $[B]$ that contains all the minimal crossing-free voice leadings from instances of $[A]$ to instances of $[B]$. For example, let $A = \{0, 0, 0, 0\}$ and $B = \{0, 4, 5, 9\}$. There are two classes of voice leadings from A to B that contain minimal voice leadings: $(0, 0, 0, 0)\to\mathbf{T}_x(0, 4, 5, 9)$ and $(0, 0, 0, 0)\to\mathbf{T}_x(0, 1, 5, 8)$.

and the next-most-even chord type is $[\{0, 2, 5, 8\}]$, which includes both the dominant seventh and the half-diminished seventh. In contrast to the situation with trichords, the evenness ordering on 12-tet tetrachords is not univocal.

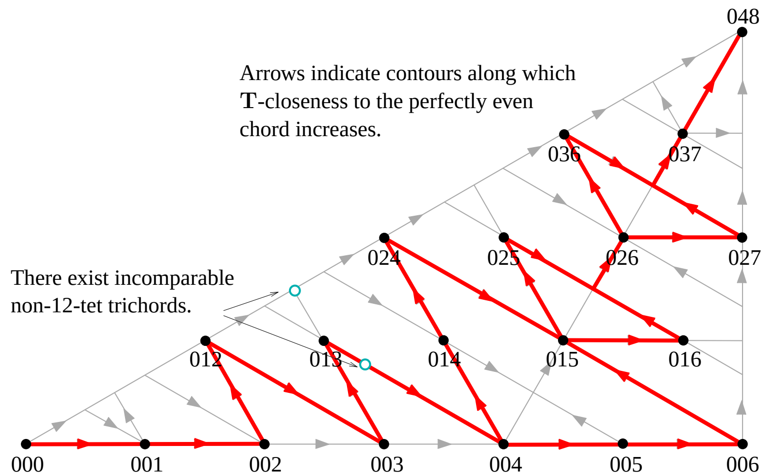

Figure 4: *Evenness ordering on trichords.*

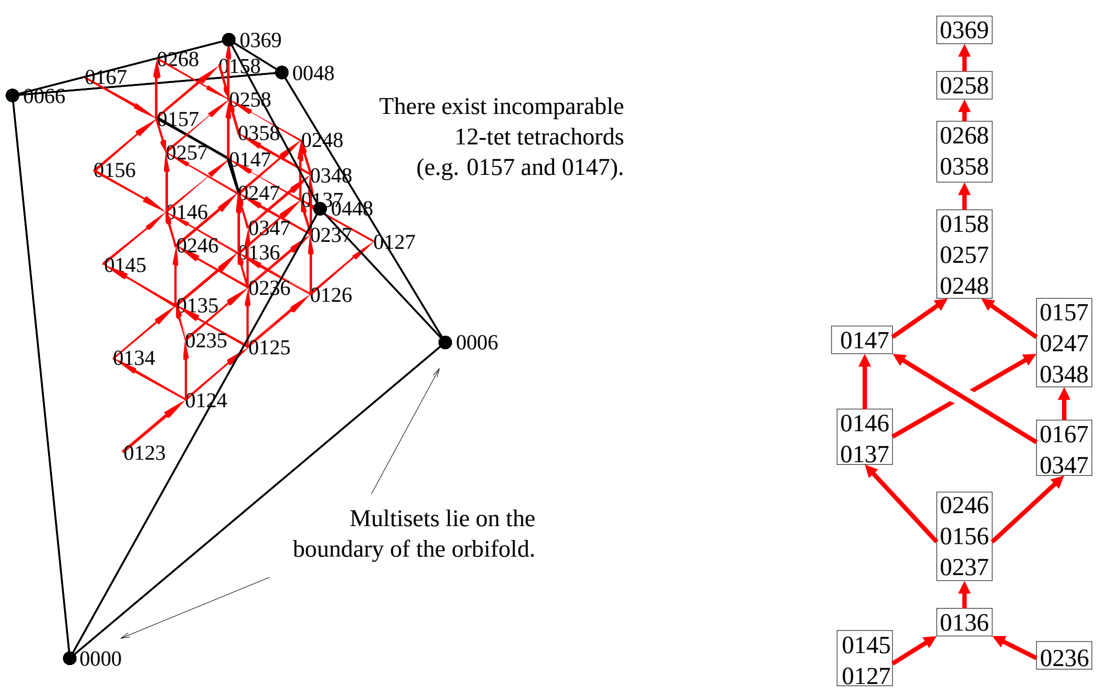

Figure 5: *Evenness ordering on tetrachords: orbifold model (left) and directed graph for the most even tetrachords (right).*

5.1. Maximally even sets. Clough and Douthett [4] define the *maximally even n-note chord* as an approximation of the perfectly even chord in k-tet: $\{0, \lfloor k/n \rfloor, \lfloor 2k/n \rfloor, \ldots, \lfloor k(n-1)/n \rfloor\}$ (the octave has size k). This definition can be understood in terms of voice leading. Since the nearest integer to a real number x is given by $\lfloor x + 1/2 \rfloor$, Clough and Douthett's algorithm maps each note of a perfectly even chord to the nearest integer-valued pitch class. The collection of distances $\{|x - \lfloor x + 1/2 \rfloor|\}$, where x ranges over the notes of the perfectly even chord, determines the size of the voice leading. It follows from the requirement that any acceptable method of comparing voice leadings be nondecreasing in each of its elements that no other voice leading to an integer-valued pitch class set can be smaller. This generalizes to our definition of evenness and only makes use of intrinsic features of pitch-class sets.

In later work, Block and Douthett [1] propose a general definition of evenness. They represent chord types by sets of points on the unit circle; the "evenness" of a chord type equals the sum of the Euclidean lengths of the $n(n-1)/2$ line segments between these points (so, for example, a semitone contributes $2\sin(\pi/12) \simeq 0.52$ and a tritone contributes 2). Using this measure, the perfectly even chord is indeed maximal, unison is minimal, and the maximally even chord is the most even among equally tempered chords. Block and Douthett's measure agrees with our evenness ordering for chords close to the perfectly even chord; however, there is some disagreement for less even chords. To use the lengths of line segments between points on the unit circle seems arbitrary; in the endnotes to their paper [1, p. 40], they provide more flexibility by permitting other choices of weights for distances between notes in a chord, and they propose a set of general constraints on the choice of an evenness measure. Our evenness ordering based on $\mathbf{T}$-closeness satisfies these constraints.

6. Applications to economics

As noted earlier, there is a striking connection between the problem of measuring voice-leading size and problems considered in welfare economics. An individual's wealth or income can be measured as a real number, using units of dollars or utility (or log-dollars or log-utility). Points in $\mathbb{R}^n$ can represent the wealth (or income) of groups of individuals. Thus $(x_1, \ldots, x_n)$ indicates that person number one has x_1 dollars, person number two has x_2 dollars, and so on.

Like music theorists, economists are interested in quotients of $\mathbb{R}^n$. The space $\mathbb{R}^n/\mathcal{S}_n$ results from a basic principle of nondiscrimination or "anonymization": what is important to economics is the distribution of wealth in society, not which individuals have which net worths; hence, the distributions (3, 2) and (2, 3) are identical. Similarly, if we measure wealth in log-dollars, then points in $\mathbb{R}^{n-1}/\mathcal{S}_n$ can represent anonymized net worths modulo inflation. Here, the points (3, 2) and (4, 3) are equivalent, since inflation, represented by addition in log-income space, transforms one into the other. The musical operations of octave equivalence and inversion have no natural analogue in economics.

Points in $\mathbb{R}^n/\mathcal{S}_n$ can be called *states*; they are the economic analogues to multisets of pitches. Points in $\mathbb{R}^{n-1}/\mathcal{S}_n$ can be called *inflation-adjusted states*; they are the analogues to transpositional set-classes of multisets of pitches. The economic analogue to a voice leading might be called a *change:* a mapping from one state to another, showing what happens to individual assets. Thus the change $(3,2,4)\to(2,3,4)$ indicates that the individual who had 3 units of wealth loses one unit, the individual who had 2 units of wealth gains one unit, and nothing happens to the individual with 4 units. Changes can result from transfers between individuals, though they need not: the change $(3,2)\to(2,2)$ might result from an individual losing or destroying a unit of wealth.

Both musicians and economists face significant uncertainty about how to measure distances in their quotient spaces. We have proposed that the distribution constraint represents a reasonable minimal constraint on musical measures of voice-leading size. Might it be the case that it also represents a reasonable constraint on changes in economic states?

We suggest that it may be. The "no crossings principle" of theorem 1 states that there is a minimal change between states that does not disturb the order of individuals by net worth. In other words, there is always a minimal-size change between any two states that satisfies the following condition: if Anne is richer than Bob in the initial state, then Anne is at least as rich as Bob in the final state. From an economic standpoint, this is an extremely attractive assumption, as it ensures that there will always be a minimally disruptive redistribution scheme that does not provide incentives to abandon money. (We assume that in general an ideal redistributive policy should move from an initial distribution to a "target" distribution along a minimal-length path—minimizing the amount of economic "work" required to get between the two states.) Suppose the condition were violated: Alice would have a pre-redistribution incentive to trade economic places with Bob, giving him the difference between their net assets. But Bob would have no inclination to accept Alice's gift—her excess money would be a "hot potato" that neither individual wanted. As a result, there would be severe conflicts between the goal of minimizing economic "work" and the goal of providing individuals with a reason to keep their money.

The music-theoretical problem of measuring the *evenness* of chords is closely related to the economic problem of measuring *inequality*. In section 5, we proposed measuring evenness using the size of the smallest voice leading to any chord that divides the octave perfectly evenly. In the same way, we can measure the inequality of an income distribution using the size of the smallest change to a perfectly equal distribution of wealth. (Note that economic equality is analogous to musical unevenness: a perfectly even distribution of wealth, like {4, 4, 4}, is like a perfectly *uneven* chord such as {E, E, E}, which is maximally distant from the perfectly even chord.) In doing so, we make a significant departure from economic tradition. Economists use submajorization to compare *states:* the state {6, 6, 2, 2} submajorizes {5, 5, 3, 3}, and is therefore considered to be more unequal. We propose using submajorization to measure *changes* between states: we say that {6, 6, 2, 2} is more unequal than {5, 5, 3, 3} because it is *farther from the nearest equal division according to the submajorization partial order*. That is, the minimal change from {6, 6, 2, 2} to the nearest equal division point is $(6, 6, 2, 2) \rightarrow (4, 4, 4, 4)$ which has displacement multiset {2, 2, 2, 2}. By contrast, the minimal change from {5, 5, 3, 3} to a point of equal division is $(5, 5, 3, 3) \rightarrow (4, 4, 4, 4)$, which has displacement multiset {1, 1, 1, 1}. It is because {2, 2, 2, 2} submajorizes {1, 1, 1, 1} that we consider {6, 6, 2, 2} to be more unequal than {5, 5, 3, 3}.

We stress that our thoughts here are preliminary: our goal here is simply to point to an interesting formal analogy between problems in music theory and problems in economics. Whether this analogy is truly useful, or merely an interesting curiosity, remains a matter for further inquiry.

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