The Ideal Vacuum: Visual Metaphors for Algebraic Concepts

Jessica K. Sklar *
Mathematics Department
Pacific Lutheran University
Tacoma, WA, 98447, USA
E-mail: sklarjk@plu.edu

Abstract

In this paper, we discuss a series of photographs, created by Shawna Holman, Matthew Fishburn, and myself, that present visual metaphors for abstract algebraic concepts. The series has a two-fold purpose: first, to be available as a learning aid for abstract algebra students, and second, to be a work of art that can perhaps engage the general public in conversations about higher-level mathematics. We provide some examples from the series, and discuss the various ways in which we are sharing our vision.

1 Introduction

When I began studying graduate-level abstract algebra, a typical conversation with my beloved thesis advisor, Frank, would go something like this:

Frank: Look at the bottom of the module.
Me: Huh?
Frank (*gesticulating emphatically*): At the bottom!
Me (*exasperatedly*): Where?!
Frank (*exasperatedly*): At the socle!

Despite my initial confusion, this soon made perfect sense to me. Though I had always thought of myself as a nonvisual mathematician, shying away from curves and surfaces in favor of group algebras, I always approached algebraic structures, such as rings, visually. Working with Frank, I honed these visual perceptions. Now, when I teach abstract algebra, I often draw pictures on the board: for example, I might draw a circle to represent a finite cyclic group. A visual understanding of abstract algebraic concepts is invaluable for me as a teacher.

The question then becomes: what is the best way to share this visual understanding with students? While drawing circles is sufficient for teaching finite cyclic groups, other mathematical concepts require sketching abilities that far outstrip my own; moreover, certain concepts, such as that of an ideal, are not as easily translated into visual terms. Then one day I realized I could use visual metaphors to express complicated mathematical concepts. Just as a poet uses literary metaphors to convey subtle emotions, a mathematician can use visual metaphors to help her students understand subtle (or not so subtle) mathematical objects or ideas.

Visual mathematical metaphors can also engage the general public. "What does that vacuum have to do with algebra?" someone might ask, prompting a conversation about rings and ideals. Of course, to entice a general audience, one's images should be aesthetically appealing. In our art project, Shawna Holman, Matthew Fishburn, and I have chosen to use photographs to convey our visual metaphors. Each photograph

*With thanks to Shawna Holman, Matthew Fishburn, Dr. Katherine Brandl (Centenary College), and Thomas W. Brooks.

demonstrates the algebraic concept in its title, and is associated with a textual “tag” that includes a definition of that concept and an explanation of the relevant metaphor.

In this paper, we provide some examples from our photographic series and discuss the physical manifestations of our project: namely, a framed series of photographs and a collection of interlinked webpages. Our series is primarily aimed at students studying what is usually considered to be college- or graduate-level mathematics. To demonstrate our series’ range, we have included in this paper two metaphors that are appropriate for undergraduates who are beginning to learn abstract algebra, and two that are more appropriate for advanced undergraduates or beginning graduate students. Though our project content is primarily directed at mathematics students, this paper contains some further explanations and examples that may help provide illumination for readers less familiar with this type of mathematics.

2 Selected Metaphors

Consider our first photograph, seen in Figure 1.

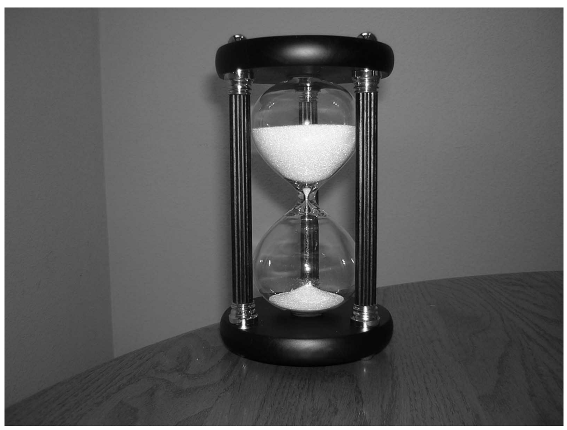

Figure 1: Invertible Function

This photographic metaphor is appropriate for students beginning their first abstract algebra course. Associated with this photograph, we have the following tag:

> **Invertible function:** A function f from a set X to a set Y is said to be *invertible* if there exists a function g from Y to X such that $f(g(y)) = y$ and $g(f(x)) = x$ for every $y \in Y$ and $x \in X$.
>
> In this image, we see a standard hourglass; applying our function f corresponds to flipping the hourglass. When f is applied, the grains of sand (elements) in the top of the hourglass (set X) are sent to the bottom of the hourglass (set Y). This action can then be reversed: we can send the grains of sand (now elements of Y) back to set X by, literally, inverting the hourglass—this action corresponds to applying function g, the inverse of f. While flipping an hourglass is not exactly a function (it does not, in fact, send grains of sand bijectively to other grains of sand), it captures the notion of invertibility: it is an action that can be reversed, ad infinitum, to send elements back from whence they came.

For those who are unfamiliar with the concept of a mathematical function, or would like a refresher on the topic: you can think of a function, f, as a “machine” that takes as input an object, x, and outputs a unique associated object, y. We often express this by saying that f "sends" x to y. For example, we can define a function p that sends an input real number, x, to the number $4x+3$. We write this as $p(x) = 4x+3$. As an example, $p(2) = 4(2)+3 = 11$.

A function is invertible if there is a function that "reverses" it. We demonstrate what we mean by "reverses" by defining a function q that reverses our function p. Let q be the function that sends every real number y to the number $(y-3)/4$. This function reverses our function p in the sense that $p(a) = b$ exactly when $q(b) = a$: for example, $p(2) = 11$ while $q(11) = (11-3)/4 = 2$. Thus, p is invertible (not all functions are).

Now return to the photograph. When you invert an hourglass, the falling grains of sand can be thought of as inputs being sent to outputs by a function. Flipping the hourglass over again in a sense "reverses" that function. As our tag notes, the inversion of an hourglass is not a perfect representation of an invertible function; however, it can help encourage a visual understanding of a fairly abstract idea.

We next present a metaphor that is perhaps appropriate for the end of a first course in abstract algebra, or the beginning of a second.

Figure 2: Ideal

Ideal: Let R be a ring. A subset I of R is said to be an *ideal* of R if it is a subgroup of R under addition, and has the property that for every $r \in R$, we have $rI \subseteq I$ and $Ir \subseteq I$.

One way to recall the definition of an ideal is to remember that ideals "suck." If I is an ideal of ring R, then when any element of R is multiplied by any element of I, the resulting element must be in I: in a sense, the ideal uses multiplication to draw elements into itself. Granted, like our *Invertible Function* metaphor, this isn't a perfect explanation of the situation: if an element r of R is multiplied by an element of I, r itself does not become a member of I. However, this colorful description of I's role in R can be extremely pedagogically useful: many people remember the concept, because they like saying that things suck.

Hence, the photograph.

Our readers are probably familiar with rings and ideals, though some may not realize it. In mathematics, a ring is a collection of objects on which we've defined an addition and a multiplication such that several conditions are satisfied. (We don't discuss those conditions here; see Chapter IV in [2] for an in-depth discussion of rings.) Some examples of rings are the collection of all real numbers under standard addition and multiplication; the collection of all integers (whole numbers) under standard addition and multiplication; and the collection of all 2×2 matrices with real number entries under matrix addition and multiplication.

Next, an ideal I of a ring R is a subcollection of objects in R such that another set of conditions hold. One of these conditions is that whenever you multiply any object in I by any object in R you obtain another object in I. Consider the following example: let R be the collection of integers under standard addition and multiplication, and let I be the collection of all even integers. Since the product of an even integer and any integer is even, we have that I contains the product of any object in I and any object in R. It turns out that I satisfies all the necessary conditions in order to be an ideal. On the other hand, let J be the collection of all odd integers. It is *not* true that an odd integer times an integer must be odd: for instance, even though 3 is odd, $2(3) = 6$, which is even. Thus, J is not an ideal of R.

Many students have trouble remembering the above-described condition; they remember only the weaker condition that the product of two objects in an ideal is also in the ideal. One way to recall the correct condition is to think of ideals as "sucking" objects in R into themselves. Admittedly, this is not exactly what's going on. Objects in R do not *become* objects in an ideal I; it is rather that resulting products are in I. Still, this technique seems to help students remember the correct, stronger condition.

The photograph now probably requires no further explanation.

We end with a pair of photographs that are probably most appropriate for advanced undergraduates or for graduate students. These photographs demonstrate the mathematically dual notions of "essential" and "superfluous" (see pp. 72-74 in [1]). We don't supplement these metaphors' tags with further explanations, but do encourage anyone who might be intrigued but confused to explore abstract algebra using texts such as [1], [2], and [3]. (If you are just beginning your foray into this topic, you may wish to start with [2].)

Figure 3: Essential

Essential: Let R be a ring. A submodule N of an R-module M is said to be *essential* in M if whenever $L \neq 0$ is a submodule of M, then $N \cap L \neq 0$.

The idea behind this photograph is to portray the "expansiveness" of an essential submodule. To use a linguistic metaphor, an essential submodule N of a module M is a submodule that has its fingers in the pots of every one of M's submodules: if L is any nonzero submodule of M, then N and L must have a nontrivial intersection. N need not contain many elements, however; its significance in M is due to its "density" in M, so to speak, rather than to its size. Our photograph attempts to convey this density: the tree (our submodule) fills the picture (our module), leaving no room in which other submodules can play without its leaves getting in the way.

Finally, we consider the photograph in Figure 4:

Figure 4: Superfluous

Superfluous: Let R be a ring. A submodule N of an R-module M is said to be *superfluous* in M if whenever $L \neq M$ is a submodule of M, then $N + L \neq M$.

In contrast to the tree in the photograph *Essential*, this tree is hardly space-filling. As in the case of essentialness, superfluity has more to do with "density" than size: a submodule is superfluous if it is, in some sense, sparse in the module. Specifically, a submodule of a module M is superfluous in M if it is so "small" that it cannot generate all of M even when taken together with any other proper R-submodule. Using trees in the photographs for the concepts of both "essential" and "superfluous" emphasizes these adjectives' natures as mathematical duals.

3 Now What?

Now that we have the photographs, what are we doing with them? At the moment, we are focusing on two specific projects.

- We have created a series of interlinked webpages, which you can access at

  `http://www.plu.edu/~sklarjk/theidealvacuum.html`.

  Our main page consists of captioned thumbnail images of our photographs; clicking on a thumbnail or its caption opens another page, displaying the photograph's tag and a larger version of the image. This format for the project has several advantages. First, it is very accessible, both by students and by the public at large; second, the metaphors' tags can contain hyperlinks associated with other concepts represented in the project; and third, the site can be regularly updated, based upon both ideas of our own and ideas of our site's viewers. Feel free to visit our site and make your own suggestions!

- We are creating high-resolution prints of our photographs. These can play a mnemonic role for students when displayed in a university environment, and serve as aesthetic conversation starters when elsewhere displayed. Our hope is that our metaphors will both encourage intuition among mathematics practitioners and arouse the curiosity of their peers.

References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules* (2nd ed.), Springer-Verlag, New York, 1992.

[2] J. B. Fraleigh, *A First Course in Abstract Algebra* (7th ed.), Addison-Wesley, Boston, 2003.

[3] T. W. Hungerford, *Algebra*, Springer-Verlag, New York, 1974.