Geometric Constructions and their Arts in Historical Perspective

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Abstract

This paper presents the mathematics and history behind the three artwork plates which have been created for display at the Bridges Mathematical Art Exhibit in San Sebastian, July 2007. Their construction serves to complement activities designed to promote the subject of geometry in the mathematics curriculum of colleges and universities.

1. Introduction

In a traditional synthetic geometry course we are introduced to rigorous treatment of axiomatic systems. During this process we become acquainted with historical and philosophical implications of various discoveries in Euclidean and non-Euclidean geometries. In addition, as a part of reasoning or as a mathematical challenge, we learn how to make geometric constructions using a compass and straightedge.

Geometric constructions and the logic of the steps bring excitement while challenging our intelligence to justify the steps to reach a conclusion.

Geometric constructions have formed a substantial part of mathematics trainings of mathematicians throughout history. Nevertheless, today we are witnessing a lack of attention in colleges and universities to the importance of geometric constructions and geometry as whole, including the role of the axiomatic system in shaping our understanding of mathematics. A quick survey reveals that many schools offer a mathematics undergraduate curriculum without geometry, or offer geometry as an option along with other courses in traditional mathematics. Nowadays students may obtain a bachelor in mathematics in some tracks without taking geometry!

The goal of this article is to explore the mathematical ideas in three presented artwork plates at the 2007 Bridges Mathematical Art Exhibit, and to provide historical background. The hope is by visual and artistic presentation of such constructions we may promote the importance of geometry in shaping our education. We hope such activities encourage schools and academia to bring back the subject of geometry to their center of mathematics education.

2. Compass, the Perfect Maker!

As a mental activity and challenge, and also to follow a principle in mathematics to purify a mathematical process from unnecessary steps and assumptions, Greeks set limits on which tools should be permitted to construct geometric shapes. They considered only compass (circle creator) and straightedge (line creator) as essential tools to perform and present geometric ideas. (It is interesting to know that in 1979, an Italian professor, *Lorenzo Mascheroni*, proved that all the problems that are soluble by means of compasses and ruler can also be solved exactly by means of compasses alone. In 1890 *A. Adler* proved this statement in an original way, using inversion. However, later in 1928, the Danish mathematician *Hjelmslev* found an old book by *G. Moher* published in 1672 in Amsterdam that included a full solution of the problem [1]).

Much earlier, during the reigns of Abbasid caliphs in Baghdad, and under Buyid rule, the Greek mathematical tradition was explored by mathematicians in Persia, as well as in the rest of Middle East, the Iberian Peninsula, and North Africa. All of the Greek texts were translated and studied by Arab and

Persian mathematicians and scientists in the Abbasid Empire. They also created their own texts, to be translated along with the Greeks documents in Arabic, to European languages during Renaissance and later periods.

The Greeks ideal of a compass and straightedge for constructions was the use of compasses that cannot be fixed to be used as dividers to transfer a line segment around. This turns out to be not an essential restriction:

2.1. Collapsing Compass. The compasses used in ancient Greek geometry had no hinges. Therefore, it was impossible to fix a compass on a certain distance in order to transfer this distance to another location. Geometric drawings were performed on sand trays. As the compasses were raised from the sand trays they collapsed. Today, these compasses are called *collapsing compasses*.

Consider that $\overline{AB}$ and a point outside of $\overline{AB}$, call it C, are given. The problem is to find another point, call it D, using a collapsing compass, so that $\overline{AB} \cong \overline{CD}$.

This problem simply says that it is possible to transfer a distance using a collapsing compass. Mathematically speaking, it says that whatever one can do with a regular compass is possible to do with a collapsing compass; therefore, a modern compass is not superior to a collapsing one!

We begin by drawing a circle with center A and radius $\overline{AC}$. Then, we draw another circle with center C and radius $\overline{AC}$. These two circles meet at points E and F. Draw a circle with center E and radius $\overline{EB}$ and a circle with center F and radius $\overline{FB}$. These two circles meet at a point, call it D. $\overline{AB} \cong \overline{CD}$ (Figure 1)!

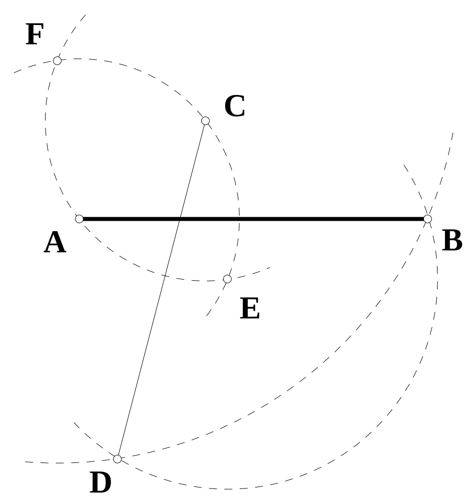


Figure 1

2.2. Rusty Compass. It is interesting to learn that the opposite extreme to the collapsing compass is called the rusty compass, a compass that is rusted into one unmovable radius, has much longer and more exciting story:

The study of the rusty compass goes back to antiquity. However, the name most associated with this compass is Buzjani. Abûl-Wefâ Buzjani (940-998), was born in Buzjan, near Nishabur, a city in Khorasan, Iran. He learned mathematics from his uncles and later on moved to Baghdad when he was in his twenties. He flourished there as a mathematician and astronomer.

The Buyid dynasty ruled in western Iran and Iraq from 945 to 1055 in the period between the Arab and Turkish conquests. The period began in 945 when Ahmad Buyeh occupied the 'Abbasid capital of Baghdad. The high point of the Buyid dynasty was during the reign of 'Adud ad-Dawlah from 949 to 983. He ruled from Baghdad over all southern Iran and most of what is now Iraq. A great patron of science and the arts, 'Adud ad-Dawlah supported a number of mathematicians and Abu'l-Wafa moved to 'Adud ad-Dawlah's court in Baghdad in 959. Abu'l-Wafa was not the only distinguished scientist at the Caliph's court in Baghdad, for outstanding mathematicians such as al-Quhi and al-Sijzi also worked there. Sharaf ad-Dawlah was 'Adud ad-Dawlah's son and he became Caliph in 983. He continued to support mathematics and astronomy and Abu'l-Wafa and al-Quhi remained at the court in Baghdad working for the new Caliph. Sharaf ad-Dawlah required an observatory to be set up, and it was built in the garden of the palace in Baghdad. The observatory was officially opened in June 988 with a number of famous scientists present such as al-Quhi and Abu'l-Wafa [2].

Buzjani's important contributions include geometry and trigonometry. In geometry he solved problems about compass and straightedge constructions in the plane and on the sphere. Among other manuscripts, he wrote a treatise: *On Those Parts of Geometry Needed by Craftsmen*. Not only did he give the most elementary ruler and rusty compass constructions, but Abûl-Wefâ also gave ruler and rusty compass constructions for inscribing in a given circle a regular pentagon, a regular octagon, and a regular decagon [3].

Until recently it was thought that the study of the rusty compass went back only as far as Buzjani. A recent discovery of an Arabic translation of a work by *Pappus* of Alexandria, the last of the giants of Greek mathematics, shows that the study of the rusty compass has its roots in deeper antiquity [3].

Italian polymath *Leonardo da Vinci*, Italian mathematicians of sixteen century *Gerolamo Cardano*, his student *Lodovico Ferrari*, and *Niccolò Fontana Tartaglia* studied construction problems using rusty compasses.

The Russian mathematician A. N. Kostovskii has shown that restricting the compass so that the radii never exceed a prescribed length still leads to all compass constructible points, as does restricting the compass so that the radii always exceed a prescribed length. However, the problem of restricting the radii between a lower bound and an upper bound seems to be still open [1].

Kostovskii showed that by means of a rusty compass one cannot divide segments and arcs into equal parts or find proportional segments. Thus, it is impossible to solve all construction problems, soluble by means of compasses and a ruler, using only compasses with a constant opening [1].

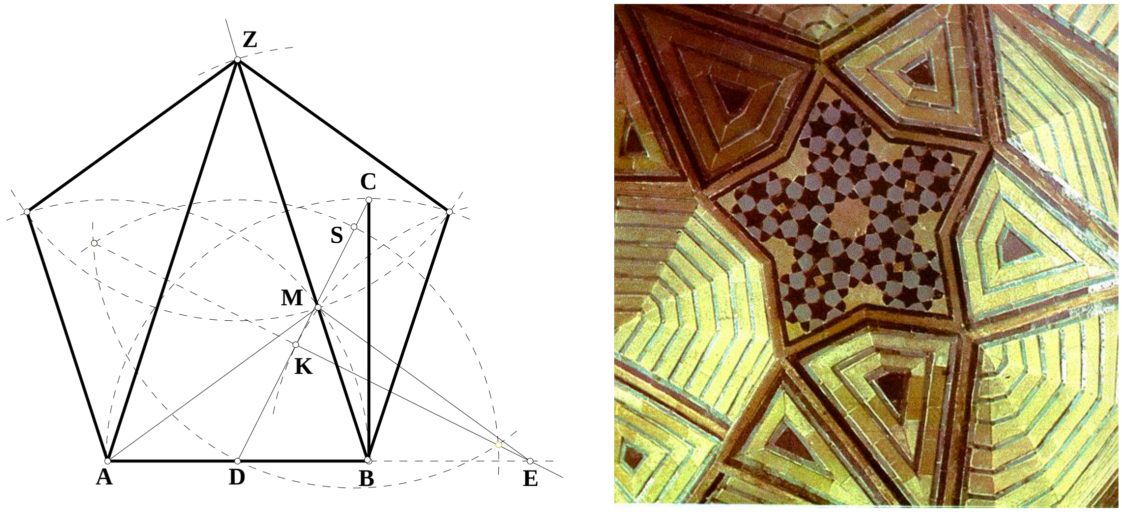
3. Buzjani's Rusty Compass Pentagon Construction

There are four known hand-written copies of the Buzjani's treatise, *On Those Parts of Geometry Needed by Craftsmen*. One is in Arabic and the other three are in Persian. The original work was written in Arabic, the scientific language of the 10^{th} century, but it is no longer exists. Each of the surviving copies has some missing information and chapters. The surviving Arabic, although not original, is more complete than the other three surviving copies. The Arabic edition is kept in the library of *Ayasofya*, Istanbul, Turkey. The most famous of the other three in Persian is the copy which is kept in the National Library in Paris, France. This copy includes an amendment in some constructions, which are especially useful for creating geometric ornament and artistic designs. This is the copy used by *Franz Woepke* (1826-1864), the first Western scholar to study medieval Islamic mathematics.

In Chapter Three of the treatise, Regular Polygonal Constructions, Buzjani, after presentation of simple constructions of the equilateral triangle and square, illustrates the compass and straightedge construction of a regular pentagon. The fourth problem is the construction of a regular pentagon using a rusty compass. To present this problem we use a recent book published in Persian that includes all known Buzjani's documents, *Buzdjani Nameh* [4]:

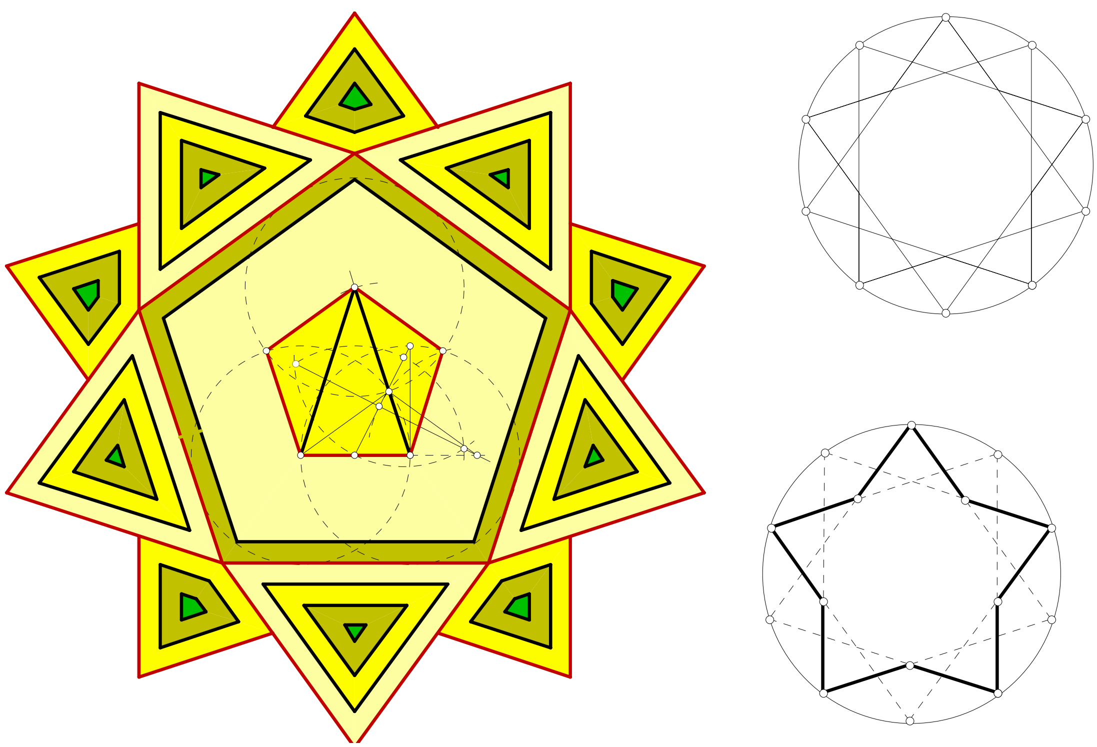
> We would like to construct a regular pentagon with sides congruent to given $\overline{AB}$, which is the same size as the opening of our rusty compass. From B we construct a perpendicular to $\overline{AB}$ (This is simple, therefore, Buzjani didn't perform it) and find C on it such a way that $\overline{AB} \cong \overline{BC}$. We find D the midpoint of $\overline{AB}$ (another simple step dropped from the figure) and then S on $\overline{DC}$ such a way that $\overline{AB} \cong \overline{DS}$. We find K, the midpoint of $\overline{DS}$. We make a perpendicular from K to $\overline{DC}$ to meet $\overleftrightarrow{AB}$ at E. Now we construct the isosceles triangle AME such a way that $\overline{AB} \cong \overline{AM} \cong \overline{EM}$. Now on ray $\overrightarrow{BM}$ we find point Z such a way that $\overline{AB} \cong \overline{MZ}$. $\Delta\ AZB$ is the well-known Pentagonal Triangle (Golden Triangle). On

side $\overline{AZ}$ construct the isosceles triangle *AHZ* the same way as the construction of *AME*. Point *T* will be found with the same procedure.



(a) (b)

Figure 2: (a) *Detailed Construction of a regular pentagon using a rusty compass,* (b) *A Persian mosaic design that inspired the work in Figure 3.*



(a) (b)

Figure 3: (a) The artwork in Plate I which is created by the author using the Geometer's Sketchpad, (b) The geometric structure of the mosaic design, constructed based on a regular (10, 3) star polygon [5].

Woepcke [4] presents the following proof:

It is obvious that $\Delta\ KED \cong \Delta\ BCD$. Therefore, $\overline{ED} \cong \overline{CD}$. This implies $\overline{ED}^2 \cong \overline{BC}^2 + \overline{BD}^2 \cong \overline{AB}^2 + \overline{BD}^2$. So $\overline{AB}^2 \cong \overline{ED}^2 - \overline{BD}^2 \cong \overline{AE}\,.\,\overline{BE}$ (This means B is the Golden cut of $\overline{AE}$). Therefore, $\overline{AE}$ is congruent to the diagonal of the regular pentagon with side $\overline{AB}$ (see Theorem 8, Chapter 13, The Elements, Euclid). (It is also congruent to the legs of the Golden Triangle with the base $\overline{AB}$). Now what is left to prove is to show that $\overline{BZ} \cong \overline{AZ} \cong \overline{AE}$. For this, we consider P on $\overline{AE}$ such a way that $\overline{MP} \perp \overline{AE}$. Then $\overline{MB}^2 - \overline{BP}^2 \cong \overline{ME}^2 - \overline{EP}^2 \cong \overline{EA}\,.\,\overline{EB} - \overline{EP}^2 \cong 2\,(\overline{EB} + \overline{BP})\,\overline{EB} - (\overline{EB} + \overline{BP})^2 \cong \overline{EB}^2 - \overline{BP}^2$. This implies $\overline{MB} \cong \overline{EB}$. So $\overline{AE} \cong \overline{AB} + \overline{EB} \cong \overline{AB} + \overline{MB} \cong \overline{MZ} + \overline{MB} \cong \overline{BZ}$.

Since $\Delta\ BME$ and $\Delta\ MAE$ are isosceles, we have $\angle ZBA \cong 2\angle\ MEB \cong 2\angle MAB$. Also, since $\Delta\ MAB$ and $\Delta\ MAZ$ are isosceles, we have $\angle ZBA \cong \angle AMB \cong 2\angle MAZ$ and therefore, $\angle MAB \cong \angle MAZ$, and $\angle ZBA \cong 2\angle\ MAB \cong \angle MAB + \angle MAZ \cong \angle ZAB$. Hence $\overline{AZ} \cong \overline{BZ} \cong \overline{AE}$ and the proof is complete.

4. A Plate for the Memory of Gauss

During October 2003 a call for entries to an art and design competition was posted by the *Mathematical Sciences Research Institute* (MSRI). This research center is hosted by the University of California, Berkeley, and is founded in 1982. MSRI's primary functions include mathematical programs and workshops, postdoctoral training, development of human resources, communication of mathematics, and education and public outreach.

MSRI is located in the hills above the campus of the University of California, Berkeley, off of Centennial Drive at 17 Gauss Way (named for Johann Carl Friedrich Gauss (1777-1855), who discovered the construction of a 17-gon, the proof for which he published in his *Disquisitiones Arithmetucae*). During 2003 a new building addition was completed. The new building opens onto a pedestrian access, which is along the extension of Gauss Way. The goal of the competition was to provide a work of art, graphics, relief, or sculpture that would serve to enhance the entry forecourt of the building.

The author of this article participated in this competition with the artwork presented in Figure 4 without success. However, the work provided the artwork for the second plate presented at the art exhibit. Along with the design, the author submitted the following note to members of the jury:

The Almighty Gauss!

The ancient mathematicians discovered how to construct regular polygons of 3, 4, 5, 6, 8 and 10 sides using a compass and straightedge alone. The list of other constructible regular polygons known to them included the 15-gon and any polygon with twice the number of sides as a given constructible polygon. No matter how much effort, mathematicians until 1796 were not successful in constructing a regular heptagon by compass and straightedge, nor were they successful in proving the construction is impossible.

After a period of more than 2000 years, Gauss, as a young student of nineteen years, proved the impossibility of its construction. In fact, he proved that in general, construction of a regular polygon having an odd number of sides is possible when, and only when, that number is either a prime *Fermat* number, a prime of the form $2^k + 1$, where $k = 2^n$, or is made up by multiplying together different *Fermat* primes. Such a construction is not possible for 7 or 9.

2^2^2+1 = 17 is a Fermat Prime!

Gauss first showed that a regular 17-gon is constructible, and after a short period he completely solved the problem.

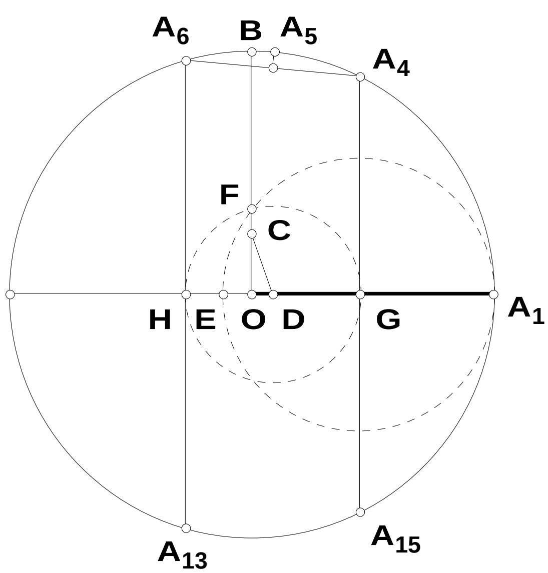
Rosebud and Citizen Kane?

It was this discovery, announced on June 1, 1796, but made on March 30th, which induced the young man to choose mathematics instead of philology as his life work. He requested that a regular 17-sided polygon to be engraved on his tombstone. This shows that for all his contributions, which place him in the circle of three of the world's all-time great mathematicians, Gauss chose his first discovery, a simple 17-gon construction, to identify himself. A wish that was never fulfilled.

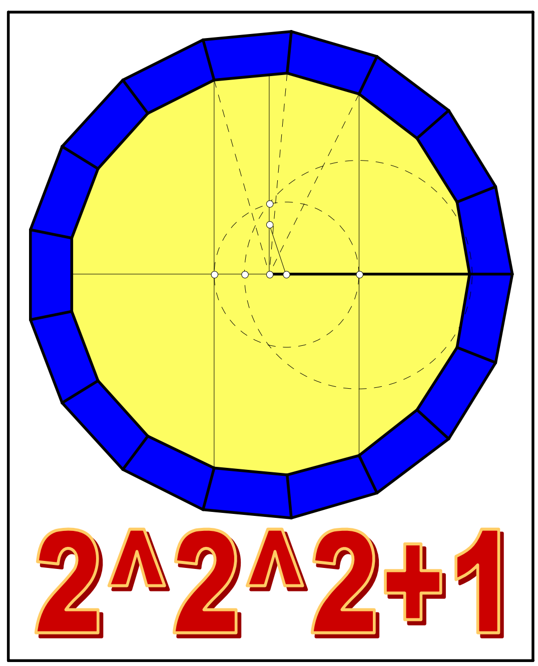
How to Construct Such a Regular 17-gon?

To create a regular 17-gon, we select two random points O and A_1, and construct a circle with center O and radius $\overline{OA_1}$. We find B on this circle in such a way that $\overline{OB}$ becomes perpendicular to $\overline{OA_1}$. We find C on $\overline{OB}$ such that $\overline{OC}$ is one-quarter of $\overline{OB}$. Point D on $\overline{OA_1}$ can be found in such a way that $\angle OCD$ is one-quarter of $\angle OCA_1$. We find E on line OA_1 such that $\angle ECD = \pi/4$. Construct the circle with diameter $\overline{EA_1}$. This circle intersects $\overline{OB}$ at F. The circle centered at D and through F intersects the diameter constructed based on $\overline{OA_1}$ at two points G and H. The perpendiculars to $\overline{OA_1}$ through G and H intersect the original circle at A_4 and A_6 (and also A_{13} and A_{15}). We can find A_5, the point that bisects the arc A_4A_6. The arc A_4A_5 divides the circle into 17 equal parts.

For more information about this geometric construction, the interested reader may visit the internet resources provided at the end of the References section.



(a)



(b)

Figure 4: (a) *The construction of the regular 17-gon,* (b) *The artwork of the second plate created by the author using the Geometer's Sketchpad*

5. A Medieval Approximation to the Regular Heptagon Construction

Let us return to Buzjani's treatise, *On Those Parts of Geometry Needed by Craftsmen*, to find an approximation to the construction of the regular heptagon. For this, we present the image and the constructions' steps to this problem according to [6].

Figure 5(a) is from this book. Figure 5(b) – (f) illustrate steps that are taken in Figure 5(a): Side $\overline{AB}$ of the heptagon is given. We find point C such a way that $\overline{CA} \cong \overline{AB}$. We construct the equilateral triangle with side $\overline{CB}$ and its circumcircle. We find point H on this circle so that $\overline{HB} \cong \overline{AB}$. After finding the midpoint M we find N on the circle such a way that $\overline{NM} \perp \overline{HB}$. We then find O, the midpoint of $\overline{AB}$ and construct $\overline{PO}$ such a way that $\overline{PO} \cong \overline{NM}$ and $\overline{PO} \perp \overline{AB}$. The circle that passes through the three points A, B, and P, which is congruent to the mentioned circumcircle, is the circle that circumscribes the regular heptagon with side $\overline{AB}$.

If the radius of the inscribed circle is 1, then $\overline{AB} = \sqrt{3}/2 \approx .8660$. The exact measure of one side of a heptagon is 2 Sin $\pi/7 \approx .8678$. That is the reason that even a modern software utility such as the Geometer's Sketchpad can not pick up the error.

The treatise does not indicate whether or not the author knew that his construction was an approximation and not an exact construction. Based on what is known about Buzjani and his thorough study of geometry of his time, which included all the geometry produced by Greeks, we may conclude that he was aware of this fact.

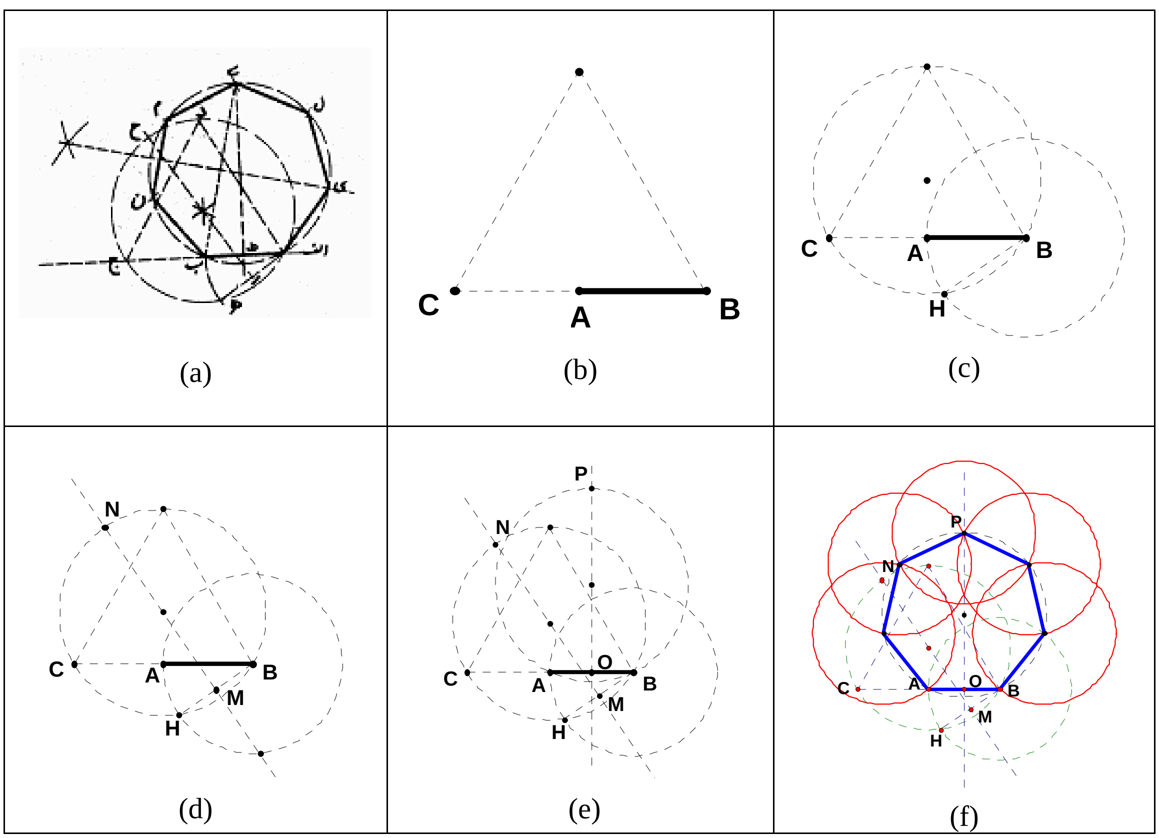
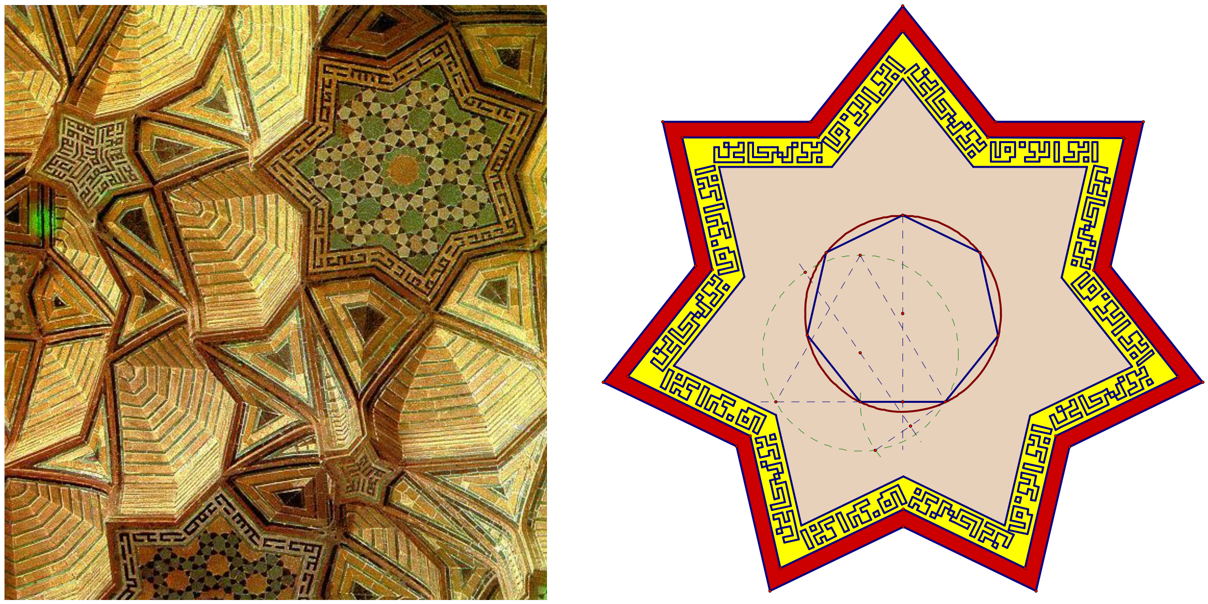


Figure 5: (a) *The construction of the heptagon, (b)-(f) The steps of the construction.*



(a) (b)

Figure 6: (a) *A (7, 2) star polygon Persian mosaic and*, (b) *the generated art based on the Buzjani's approximation of heptagon, the writings on the edges repeats the name of Buzjani in Farsi.*

Conclusion

Poincaré said: The mathematician does not study pure mathematics because it is useful; he studies it because he delights in it and he delights in it because it is beautiful [7].

Geometry and geometric constructions induce aesthetic pleasure in a manner that mathematicians need to experience through the processes of investigation, challenge, and mental exercise. By appropriate uses of art and geometric constructions, the new generation of mathematicians should be introduced to the beauty of mathematics.

Note: The three artwork plates are created as a joint project of the author and the Bridges Mathematical Art Exhibit curator, Robert Fathauer.

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