Patterned Polyhedra: Tiling the Platonic Solids

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Abstract

This paper examines a range of geometric concepts of importance to the further understanding of two- and three-dimensional design. A brief explanation is given of symmetry in patterns and tilings, and attention is focused on a particular set of polyhedra, known as the Platonic solids. The difficulties encountered in attempting to apply two-dimensional repeating designs to regular polyhedra, avoiding gap and overlap and ensuring precise registration, are recognized. The results of ongoing research at the University of Leeds are presented. The symmetry characteristics of importance to the process are identified, and the patterning of each of the five solids is explained and illustrated. Avenues for further research are suggested.

Introduction

It is well established that regularly repeating patterns (or tilings) exhibit symmetry characteristics. That is, certain geometric actions, known as symmetries, of which there are four (translation, reflection, glide-reflection and rotation), combine to produce seventeen possibilities across the plane. In the context of this paper, patterns are best considered as those repeating designs that comprise a motif (or motifs), which repeats at regular intervals across the plane. An individual repeating unit may comprise both motif and background. A tiling is best considered as a restrictive category of patterns that covers (or tessellates) the plane without gap or overlap. Both categories of repeating designs conform largely to the same symmetry rules and structural geometry. A polyhedron has been defined as "…a finite, connected set of plane polygons, such that every side of each polygon belongs also to just one other polygon, with the provision that the polygons surrounding each vertex form a circuit"[1]. The regular polyhedra also exhibit symmetry characteristics. The polygons that join to form polyhedra are called faces. These faces meet at edges, and the edges meet at vertices. There are five regular convex polyhedra (known as "Platonic solids"), each comprised of combinations of one specific type of regular polygon. Within each Platonic solid the faces are thus identical in size and shape, and the same number of faces meets at each vertex. The five Platonic solids are as follows: the tetrahedron (four faces), the octahedron (eight faces), the cube or hexahedron (six faces), the dodecahedron (twelve faces) and the icosahedron (twenty faces). The patterning (or tiling) of regular solids in ways that ensure precise registration and the absence of gaps or overlaps is neither a trivial nor a straightforward matter. When regular repeating patterns, which perform to satisfaction on the Euclidean plane, are folded several times, into different planes, their component parts will not readily correspond. Only certain pattern types, with particular symmetry characteristics, are suited to the precise patterning of each Platonic solid. This paper identifies a systematic means by which appropriate pattern types can be identified. An understanding of the symmetry characteristics of patterns and polyhedra can help provide a means by which the latter can be covered by the former in a systematic and complete way, avoiding gaps or overlaps and ensuring precise registration.

Symmetry in Patterns and Polyhedra

As noted in the Introduction, it is well recognized that regularly repeating patterns (or tilings) exhibit symmetry characteristics, involving certain geometric actions, which combine to produce seventeen possibilities across the plane. Translation moves a figure over a given distance and direction whilst maintaining the same orientation. A translated figure (motif or tile) may undergo repetition horizontally, vertically or diagonally. Reflection produces a mirror image across a reflection axis. Glide-reflection combines reflection and translation across a glide-reflection axis. Rotation in all-over patterns or tilings may be two-fold (180 degrees), three-fold (120 degrees), four-fold (90 degrees) or six-fold (120 degrees). Proof of the existence of only seventeen all-over pattern classes is provided by Weyl [2], Coxeter [3] and Martin [4].

A further geometrical element of importance to pattern structure is the underlying framework or lattice. Each lattice (of which there are five distinct types) is comprised of unit cells of identical size shape and content. Each cell contains the essential repeating unit or element of the pattern or tiling, as well as the symmetry instructions for the pattern's construction. When translated in two independent non-parallel directions, the full pattern or tiling is produced.

Designs possessing the same symmetry combinations are said to belong to the same symmetry class, and may be classified accordingly. The seventeen all-over pattern combinations, of the four symmetry operations, have been illustrated on numerous occasions elsewhere. Associated with each class is a notation comprised of up to four symbols, indicating the constituent symmetry operations of each class. Further accounts of the classification and construction of regularly repeating patterns and tilings are been given by Woods [5], Schattschneider [6], Stevens [7], Washburn and Crowe [8] and Hann and Thomson [9].

A polyhedron consists of polygonal faces, with these faces meeting at edges, and edges joining at vertices. Composed of faces identical in size and shape, equally surrounded vertices and equal solid angles, the regular convex solids, known as the *Platonic solids*, are highly symmetrical polyhedra. The symmetry characteristics of reflection and rotation that govern the properties of regularly repeating patterns and tilings, are also of importance to three-dimensional solids. This study aims to consider the links between these symmetry properties in two- and three-dimensions and to investigate how these characteristics govern the application of repeating patterns to the Platonic solids. Further accounts of the properties of polyhedra are given by Coxeter [3], Critchlow [10], Pearce [11], Cundy and Rollett [12], Holden [13] and Cromwell [14].

Patterning the Platonic Solids

In regularly repeating plane patterns each unit cell is surrounded by identical cells of exactly the same size, shape and content. The purpose of this investigation is to find which of the seventeen pattern classes can regularly repeat around the Platonic solids, applying only the restriction that the unit cell must repeat across each face in exactly the same way that it does in the plane pattern. In order to assess the geometric characteristics of importance in the regular repetition of pattern around the Platonic solids, the initial investigation has focused on the application of areas of the unit cell (capable of building-up the repeating pattern) to act as a tile when applied to the faces of the polyhedra. This placed emphasis on the pattern's underlying lattice structure and the symmetry operations contained within it.

The first step involved matching the polyhedral faces to a suitable lattice type. Considering the polygons that comprise the faces of the Platonic solids, the equilateral triangle, square and pentagon were seen to be of greatest significance. All of the seventeen all-over pattern classes may be constructed from either a square or hexagonal lattice and some may be constructed on both lattice types [15]. Suitable patterns applicable to the tetrahedron, octahedron and icosahedron, whose faces are equilateral triangles, must therefore be constructed on grids of equilateral triangles. This indicates that patterns created on a hexagonal lattice, where the unit cell comprises two equilateral triangles, are considered appropriate candidates to tile these three solids. The cube is the only Platonic solid that is composed of square faces. All pattern classes constructed on square grids are therefore considered suitable to repeat around the surface of the cube. Patterning the dodecahedron, composed of regular pentagonal faces, requires a different approach. The regular pentagon, with five-fold rotational symmetry, cannot tile the plane without gap or overlap. There are however various equilateral pentagons that can tessellate the plane. Probably the best known is the Cairo tessellation, formed by convex equilateral pentagons (equal-length sides, but different associated angles). Using knowledge of the Cairo tessellation, the method used by Schattschneider and Walker [16] presents one solution to this problem, but this is an area requiring substantial further investigation.

Initial constructions have utilized the whole square unit cell to tile the cube and half the hexagonal unit cell to tile the tetrahedron, octahedron and icosahedron. An example of a p6 design used to tile the icosahedron is shown in Figure 1, where the area used to tile each face is equal to half the area of the unit cell. Figure 2 illustrates the same p6 pattern used to tile the tetrahedron. Further developments have involved extracting smaller areas of the unit cell for use as tiles. The area constitutes either a square or equilateral triangle and must be capable of creating the full repeating pattern when symmetries are applied. The smaller the area of the pattern used to tile the polyhedron, the larger the scale of the pattern on the solid. Figure 3 illustrates pattern class p6 tiling the octahedron. The area used to cover each face of the solid is equal to one-sixth of the area of the unit cell.

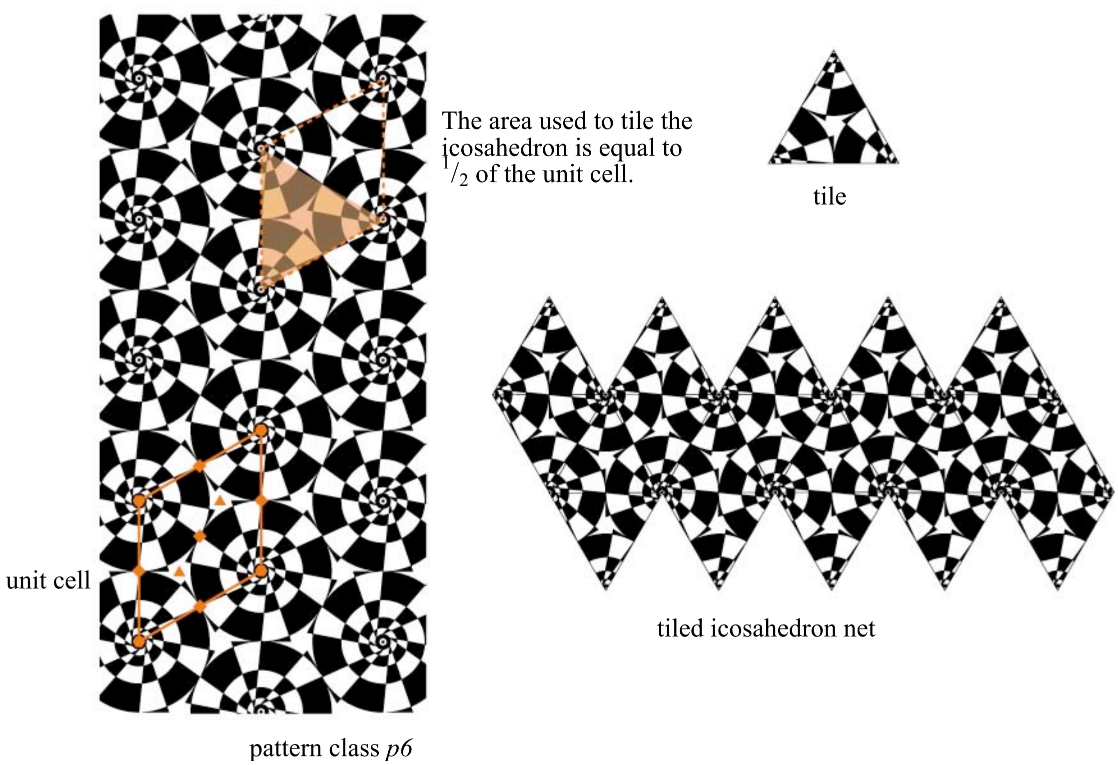

Figure 1 *Design for the icosahedron regularly tiled with pattern class p6.*

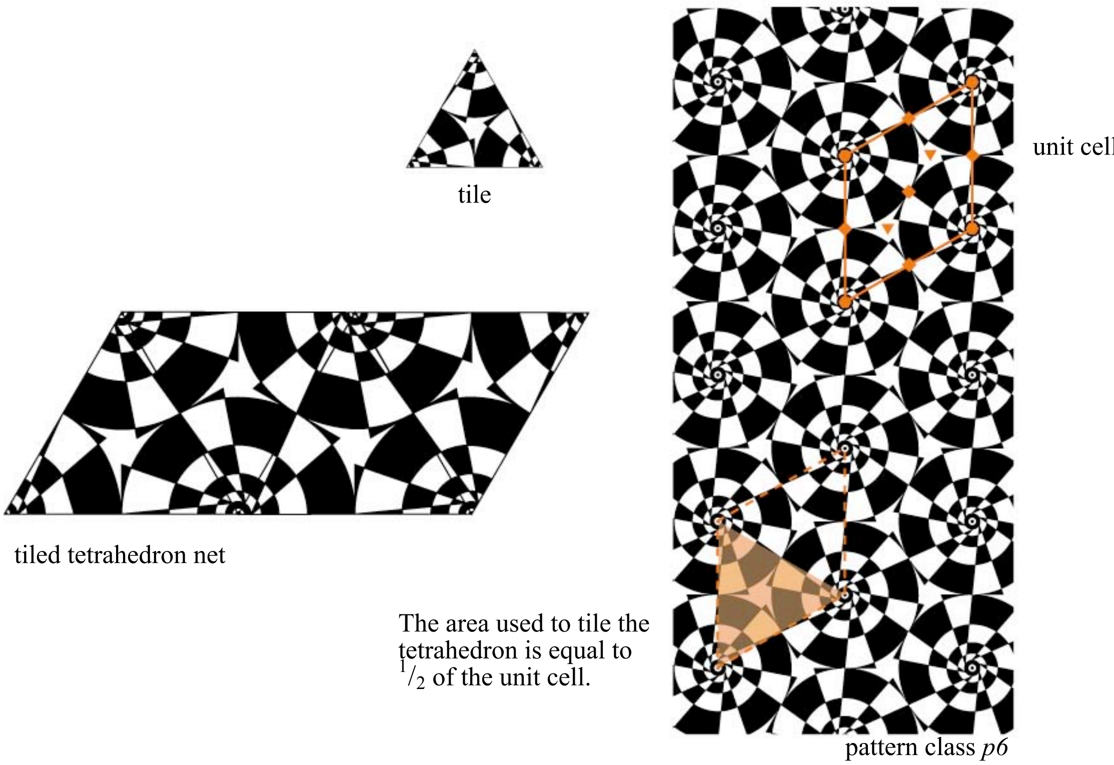

Figure 2 *Design for the tetrahedron regularly tiled with pattern class p6.*

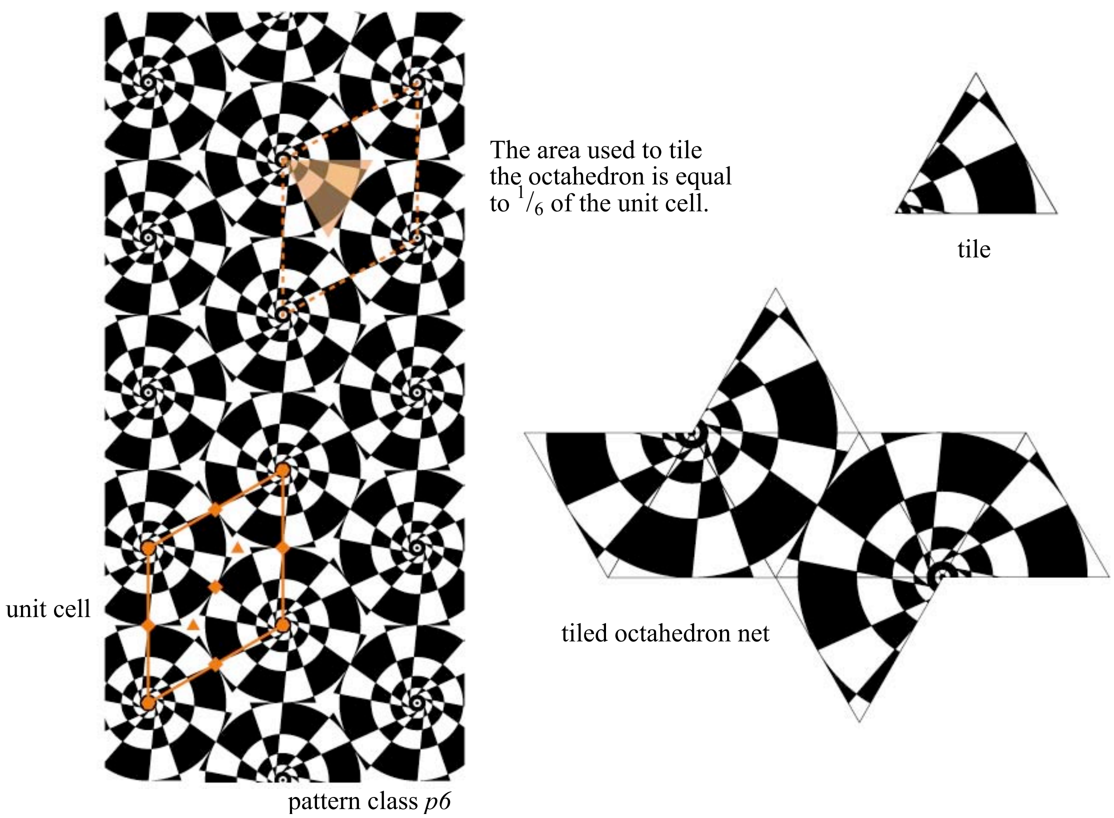

Figure 3 *Design for the octahedron regularly tiled with pattern class p6.*

Results

The interim results of the investigation into the tiling of the Platonic solids are summarized in Table 1. All-over pattern classes are identified along with their constituent lattice structures and the area of unit cell applied to each face. The symmetry characteristics of these regularly patterned solids are described further below, considering the symmetry characteristics of each polyhedron and the changes that occur when regularly tiled.

Table 1 Summary of the symmetry characteristics exhibited by the regularly tiled tetrahedron, octahedron, icosahedron and cube

Platonic solid	Pattern class	Lattice structure	Area of unit cell on face	Vertices equivalent	Rotation present on vertices	Rotation present on edges	Rotation present on faces	Reflection present on edges	Reflection present on faces
Tetrahedron	p2	hexagonal	1/2	✔		2-			
	c2mm	hexagonal	1/2	✔		2-		✔	✔
	p6	hexagonal	1/2	✔	3-	2-	3-		
	p6mm	hexagonal	1/2	✔	3-	2-	3-	✔	✔
Octahedron	p3	hexagonal	1/2	✔	2-		3-		
	p31m	hexagonal	1/2	✔	2-		3-	✔	
	p3m1	hexagonal	1/2	✔	2-		3-		
	p3m1	hexagonal	1/6		a) 2- b) 2- c) 2-			✔	
	p6	hexagonal	1/2	✔	4-	2-	3-		
	p6	hexagonal	1/6		a) 4- b) 2-	2-			
	p6mm	hexagonal	1/2	✔	4-	2-	3-	✔	✔
	p6mm	hexagonal	1/6		a) 4- b) 2-	2-		✔	✔
Icosahedron	p6	hexagonal	1/2	✔	5-	2-	3-		
	p6mm	hexagonal	1/2	✔	5-	2-	3-	✔	✔
Cube	p4	square	1	✔	3-	2-	4-		
	p4mm	square	1	✔	3-	2-	4-	✔	✔
	p4gm	square	1	✔	3-	2-	4-		

It is possible to tile the tetrahedron with certain patterns possessing two-fold and six-fold rotational symmetry. An area equivalent to half the unit cell forms the repeating tile in each case. Tetrahedra tiled with patterns possessing six-fold rotation maintain the original rotational properties of the solid. Centers of six-fold rotation, characteristic of the two-dimensional plane pattern, become axes of three-fold rotation at each vertex on the regularly patterned solid. Centers of two-fold and three-fold rotation in the pattern remain unchanged when applied to the tetrahedron; they locate at the midpoint of each edge and the center of each face respectively. A tetrahedron tiled with class p6mm also presents the six reflection planes of the tetrahedron. Figure 4a shows an illustration of pattern class p6 tiling the tetrahedron in which an area equal to half the unit cell tiles each face. When a p2 or c2mm pattern is applied to the tetrahedron, the pattern exhibits no rotation at the vertices but tetrahedral axes of two-fold rotation are maintained through the mid-point of opposite edges. The edges of the solid tiled with c2mm, however, are not equivalent as two edges exhibit a plane of reflection, which slices through the relevant edge and divides the solid into two equal parts. A reflection plane therefore runs through the center of each face. In the case of the tetrahedron, it seems that a pattern must possess reflection and rotational properties equal to or higher than those of the tetrahedron, in order to maintain the rotational symmetries of the solid. Pattern classes p3, p3m1 and p31m are not applicable to the tetrahedron due to the two types of tile created from the equilateral triangular halves of the unit cells and the reflection axes created in class p31m. For these patterns to repeat around the surface of a polyhedron there must be an even number of faces meeting at each vertex.

Pattern classes p3, p3m1, p31m, p6 and p6mm are each suited to patterning the octahedron. These possibilities incorporate repeating tiles of areas equivalent to half and one-sixth that of the unit cell. Considering the tiles which comprise half the area of the unit cell, it appears that patterns with three-fold rotation must consist of two independent halves of the unit cell or possess reflection planes which correspond with the edges of the solid, in order to permit for the four octahedral faces meeting at each vertex. The two tiles created by each half of class p3 and p3m1 unit cells result in two-fold rotation at each vertex but no rotation at the edges. Pattern class p31m also presents axes of two-fold rotation at the vertices as a result of the reflection planes present at each edge. Axes of three-fold rotation, preserving the symmetries of the solid, are also evident at the center of each face. Pattern class p6 and p6mm maintain the rotational symmetries of the octahedron with four-fold rotation present at each vertex; two-fold rotation is present at the mid-point of each edge and three-fold rotation at the center of each face. Class p6mm also retains the reflection planes of the octahedron. Comparable with the tetrahedron, it seems that a pattern must possess symmetry characteristics equal to or higher than those of the solid in order to preserve the underlying symmetries of the octahedron.

As stated previously, it is also possible to tile the octahedron with an area equal to one-sixth of the unit cells of pattern classes p3m1, p6 and p6mm. The use of a small component of the unit cell imparts different characteristics to the octahedron than previously seen. When p3m1 is applied to the octahedron three different vertex types are evident, with identical types found in opposite pairs each exhibiting two-fold rotation. Pattern class p3m1 requires an even number of faces to meet at each vertex in order for reflection at each edge to form the whole unit cell. This unit is then repeated through rotation or successive reflection to tile the octahedron. The application of classes p6 and p6mm results in the presence of two different types of vertices, in each case producing one axis of two-fold rotation and two axes of four-fold rotation. Axes of two-fold rotation are also found at the mid-point of the four edges that connect the vertices exhibiting two-fold rotation. The reflection present in pattern class p6mm also imparts reflection planes that slice through the center of each face and through each edge. An octahedron tiled with classes p6 and p6mm resembles two identically patterned pyramids placed base to base. Figure 4b illustrates pattern class p6 tiling the octahedron, where an area equal to one-sixth of the unit cell tiles each face. When a component equal to one-sixth of the unit cell is applied to the octahedron the pattern must be capable of repetition through rotation or reflection around an even number of faces. A pattern possessing higher rotational properties than the octahedron will rotate around vertices. Patterns possessing lower rotational characteristics must contain reflection axes that correspond with the edges of the octahedron to allow repetition around the four faces meeting at each vertex.

Only pattern classes containing six-fold rotation are applicable to regularly tiling the icosahedron. As the icosahedron possesses the same rotational properties found in pattern class p6, the icosahedral rotational symmetries are preserved in the tiled solid. In addition, the application of pattern class p6mm also maintains icosahedral reflection properties. Figure 4c illustrates pattern class p6 tiling the icosahedron, where an area equal to half the unit cell tiles each face.

Connecting points of four-fold rotation in pattern classes p4, p4gm and p4mm will produce a square grid, which will lend itself readily to the tiling of the cube. Application of these pattern classes maintains cubic rotational symmetry. Three-fold rotation is preserved at the vertices, two-fold rotation at the mid-point of each edge and four-fold rotation at the center of each face. A cube tiled with class p4mm also retains the reflection symmetry of the cube with four reflection planes slicing through each face and each of the twelve edges.

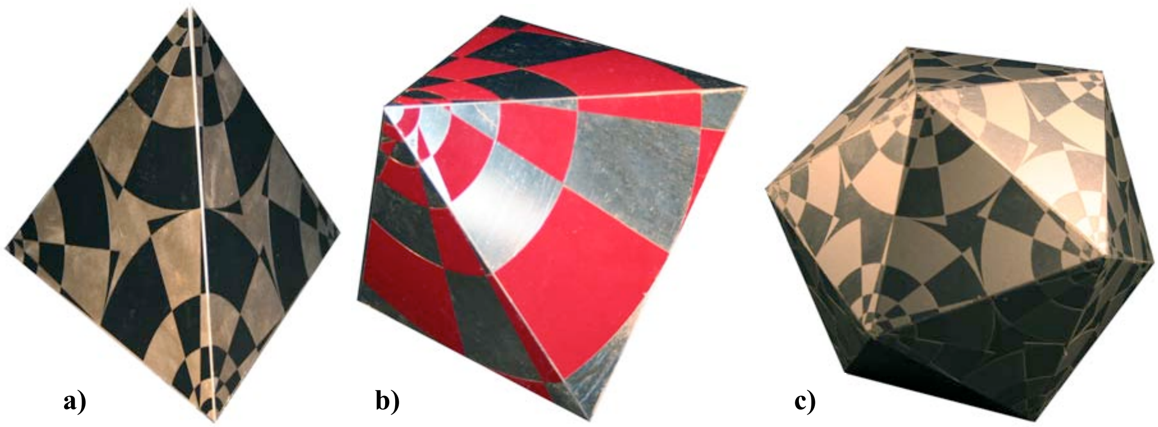

Figure 4 *a) the tetrahedron, b) the octahedron and c) the icosahedron, tiled with pattern class p6.*

Discussion and Avenues of Further Research

The first stages of this investigation into the regular tiling of the Platonic solids have identified several areas for further exploration. The importance of the symmetry properties of the polyhedra, particularly the order of rotation axes present at the vertices, has become readily apparent. The reflection symmetry of the solid is often disguised by the application of pattern, but rotation axes operational at each vertex remain evident, except when a tile is composed of an area smaller than half that of the unit cell. The evidence of rotation axes on the faces and edges of the solid depends entirely on the symmetry characteristics of the two-dimensional pattern. A regularly tiled polyhedron, therefore, can exhibit different symmetry characteristics to the underlying polyhedral structure.

Where solids are tiled with pattern components equal to half or the whole unit cell, vertices are found to be equivalent in terms of their symmetries. Where a smaller component of the unit cell has been manipulated on the surface of a polyhedron, several types of vertex are evident. For example, multiple vertex types are seen when areas equal to one-sixth of the unit cells of pattern classes p3m1, p6 and p6mm are used to tile the octahedron. It should also be noted that the octahedron is the only Platonic solid capable of being regularly tiled with areas of a repeating pattern smaller than half the unit cell. When there is no rotation or reflection present at the edges of a regularly tiled solid, two types of face are evident. On a hexagonal lattice, this is equivalent to no reflection or rotation present at the center of the unit cell, as illustrated by the tiling of the octahedron with p3 and p3m1 (half unit cell).

It is apparent from the results in Table 1 that for a plane pattern to tile the regular solids it must possess rotational symmetry axes of two or higher. This contradicts an earlier study by Pawley [17], which states that plane patterns must have a symmetry axis higher than the second order to fit onto polyhedra, as both classes p2 and c2mm are suited to tiling the tetrahedron. These are the only two pattern types (discovered to date) that exhibit the highest rotational symmetry of order two and are capable of tiling a Platonic solid. When applied to the tetrahedron these patterns are the only types that do not exhibit rotation at vertices.

It should be stressed that this paper reports on the outcome of research completed to date, and that it is recognized by the authors that the results presented are not all encompassing. For example, further analysis of the lattice structures underlying regularly repeating patterns and identification of potential tiling areas within the unit cell need to be undertaken. Attention should be focused on irregular pentagonal tessellations and other such tilings, particularly with respect to their suitability in tiling the dodecahedron. From the results presented above, it is the intention to develop a system of classification for regularly tiled Platonic solids, thus enabling the systematic application of pattern to regular polyhedra following certain geometric rules. It is the intention to extend this enquiry to take account of the Archimedean solids, as well as applying concepts such as scale symmetry and color symmetry to both regular and semi-regular patterned polyhedra.

In Conclusion

It has been noted that the patterning of regular solids in ways that ensure regular repetition and precise registration is not a straightforward matter. When regularly repeating patterns are folded into different planes, their component parts may not readily correspond. As is the case in two-dimensions, the patterns rely on the symmetries present to repeat by rotation (at edges and vertices) and/or through successive reflection present at the edges of the solid. This paper presents and discusses a method by which appropriate pattern types can be identified. An understanding of the symmetry characteristics of patterns and polyhedra can help provide a means by which patterns may be applied to polyhedra in a systematic and complete way, avoiding gaps or overlaps and ensuring precise registration. Preliminary results of the enquiry are very encouraging and have given rise to the creation of a series of remarkable mathematical solids. Several prototypes of these polyhedra are shown in Figure 4.

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