A "Sound" Approach to Fourier Transforms: Using Music to Teach Trigonometry

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Abstract

If a large number of educated people were asked, "What was your most exciting class?", odds are that very few of them would answer "Trigonometry." The subject is generally presented in a less-than-exciting fashion, with the repeated caveat that "you'll need this when you take calculus," or "this has lots of applications" without ever really seeing many of them. This manuscript addresses how the author is trying to change this tradition by exposing casual students from kindergarten to college to Joseph Fourier's secret, that nearly any function can be built out of sine and cosine curves. And music serves as a both the bait that entices the student to learn, and the hook.

1 Introduction

I am shameless when it comes to capturing students' interest in mathematics. In talks that I have delivered to math students of all ages, I have taken a digital portrait of myself and warped it to demonstrate matrix multiplication, and have rolled around on the floor aiming my homemade projectile launcher to demonstrate how algebra and trigonometry can be used for targeting the device. And, in an act of gratitude to the teachers who have let me into their classroom, I have made fun of them as well, warping their pictures or shooting cups off of their head with my projectile launcher. My goal has always been to get students interested in a topic first, usually before they are ready for all of the gory details of the mathematics I am using, and then lure them into trying to understand those details.

It is a different approach – no one did that with me as I was learning mathematics, nor did they need to. I was hooked on math at an early age – a dyed-in-the-wool math nerd. However, as my wife is kind to occasionally remind me, I am not normal. My latest foray into the absurd has me dragging my guitar along with my standard computer and projector. By subjecting students to my mediocre singing and guitar-playing, I instantly gain the attention of even the least-interested math student. By linking the topics of music and trigonometry, I at least partially keep the interest of students who are more in tune (pardon the pun) with the arts than with the sciences. By motivating the content with interesting applications like filtering out noise or altering someone's singing voice, I hopefully keep topics like trigonometry from being just an academic exercise.

The following sections provide a light introduction to the physics of sound and the mathematics of Fourier series and discrete Fourier transforms. Then examples are provided of how I use a digital recorder and mathematical software to illustrate the connections between sound, music, and trigonometry. Most of the mathematical calculations are suppressed in favor of graphical representations of the results.

Sound Waves. ***Sound*** is, in the most basic sense, waves of pressure through a medium, like air or water. When most people envision sound waves, they picture up-and-down waves, like the ones shown in Figure 1, called ***transversal*** waves. This is understandable, since many things that produce sound do so by moving as an observable transversal wave, like the strings on a guitar or other stringed instrument. However, sound is actually a ***longitudinal*** wave, moving back and forth parallel to the direction of its movement. Picture a *Slinky*™ being undulated from one end to the other, and you have a fair understanding of the nature of sound waves. Our ears pick up these longitudinal vibrations and convert them into a signal that our brain recognizes as speech, singing, instrumental music, noise, etc. Still, if we think about the vertical difference in the graph of the transversal wave between the wave and the horizontal axis measuring the horizontal difference between the wave position and the equilibrium point with the longitudinal wave, we can still use transversal waves to describe sound.

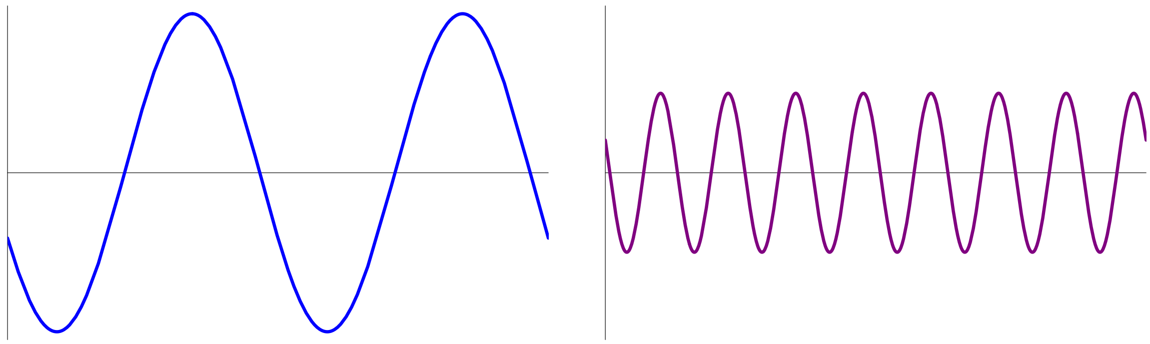

Figure 1: A graphical representation of two different sound waves.

There are two different aspects of sound waves that determine what you hear. The ***amplitude*** of the sound wave determines how loud a sound is, and is shown graphically as the height of the wave above and below the horizontal axis. The curve on the left in Figure 1 has a higher amplitude than the curve on the right, so the sound it represents is louder. The loudness of a sound is measured in ***decibels*** (dB), a logarithmic scale where 0 db is the threshold of hearing, 80 dB is roughly equivalent to the loudness of a vacuum cleaner, and a 160 dB sound will burst your eardrums (see [2]). The other important aspect is the ***frequency*** of the sound wave. The frequency of a sound is measured in ***hertz*** (Hz), or number of cycles per second. The low string on a tuned guitar vibrates at 82.4 Hz, while the high string vibrates at 329.6 Hz. The right-hand curve in Figure 1, while not as loud as the sound represented on the left, represents sound at a higher pitch. Most of our examples will focus on the frequencies of the sounds that we analyze.

Fourier Series. Almost exactly two-hundred years ago, Baron Jean Baptiste Joseph Fourier (1768–1830, France) in his treatise *The Analytical Theory of Heat* proposed an idea that was quite controversial at the time: that any function defined over a closed interval, even those with a discontinuity, could be expressed as an infinite series of different frequency sine and cosine curves (see [4]). He was not completely correct – there are some extreme cases where the statement is not true – but it is safe to say that it is true in most cases. His discovery started a new branch of mathematics, called ***harmonic analysis***, where functions are broken down into their fundamental harmonics. (See [5] for an excellent beginner's guide to Fourier series.)

The infinite trigonometric polynomial mentioned above is called a ***Fourier series***. The Fourier series of a function $f(x)$ over the interval $[c, c+p]$ is defined more precisely by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2\pi}{p} nx\right) + b_n \sin\left(\frac{2\pi}{p} nx\right) \right),$$

where p is the length of the interval and

$$a_n = \frac{2}{p} \int_c^{c+p} f(x) \cos\left(\frac{2\pi}{p} nx\right) dx \quad \text{and} \quad b_n = \frac{2}{p} \int_c^{c+p} f(x) \sin\left(\frac{2\pi}{p} nx\right) dx$$

for integer $n \geq 0$, provided the integrals exist.

There is also a discrete version of the above that does always exist, where if we have a finite-length sequence $\{y_k\}_{k=0}^{N-1}$ of uniformly spaced data values, then

$$y_k = \frac{a_0}{2} + \sum_{n=1}^{N-1} \left(a_n \cos\left(\frac{2\pi}{N}nk\right) + b_n \sin\left(\frac{2\pi}{N}nk\right)\right), k = 0, 1, \ldots, N-1, \tag{1}$$

where

$$a_n = \frac{1}{N}\sum_{j=0}^{N-1} y_j \cos\left(\frac{2\pi}{N}nj\right), \text{ for } n = 0, 1, \ldots, N-1, \text{ and} \tag{2}$$

$$b_n = \frac{1}{N}\sum_{j=0}^{N-1} y_j \sin\left(\frac{2\pi}{N}nj\right), \text{ for } n = 1, 2, \ldots, N-1. \tag{3}$$

The sine and cosine functions in (1) are uniformly sampled, and increase in frequency from 0 to $N-1$. See Figure 2 for an illustration of $\cos\left(\frac{2\pi}{N}nk\right)$ with $N = 16$ and $n = 0, 1, 2, 3$. For a fixed value n, the a_n and b_n measure the amplitude of the waves at frequency n needed to build the data set. It is customary to bind the two coefficients together and merely consider the magnitude of the n^{th} frequency to be $c_n = \sqrt{{a_n}^2 + {b_n}^2}$. The process of finding the Fourier series coefficients in (2) and (3) is called the (discrete) ***Fourier transform***, and the graph of the c_n's is called the ***spectrum*** of the signal.

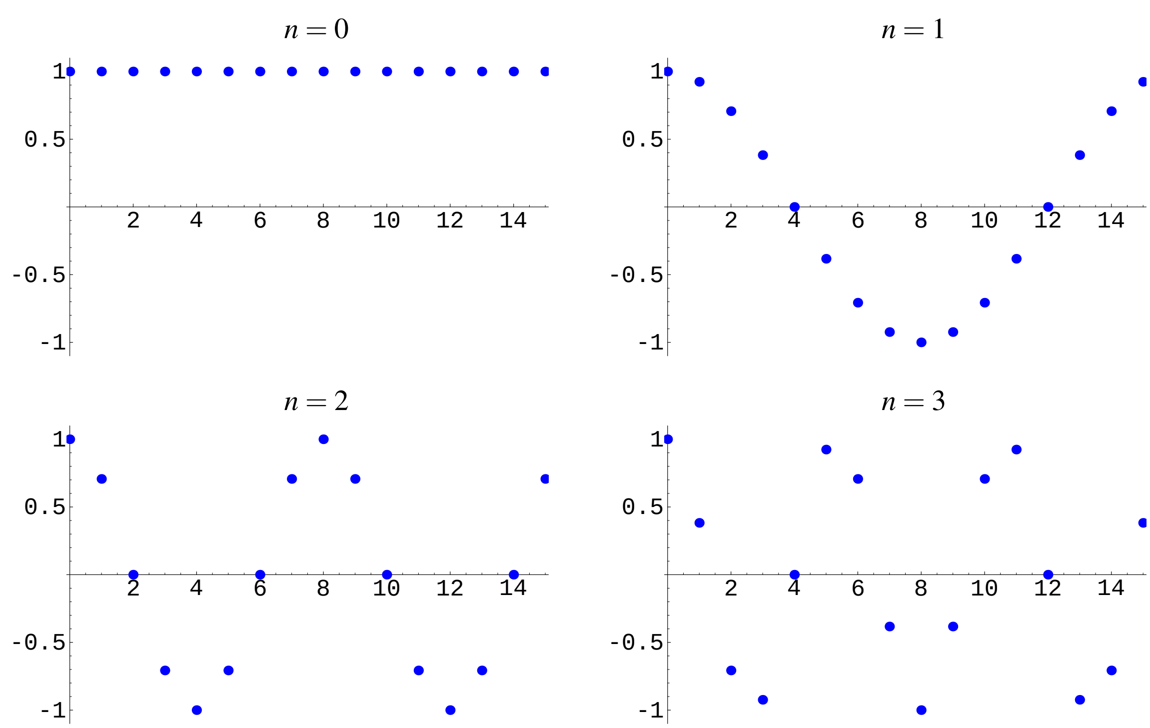

Figure 2: Sets of points from increasingly higher frequency cosine curves.

It is interesting to note that the difference between the continuous Fourier series and the discrete Fourier series is analogous to the difference between analog and digital signals. An analog storage device, like a vinyl album (if you are old enough to remember those things) or a cassette tape, records the actual sound waves. A digital recorder will sample the actual sound waves, usually every $\frac{1}{44100}$ of a second, and store the amplitude of the sound wave at that moment as a 2-byte binary number, sixteen 0's and 1's. While digital recordings, like on a CD or DVD, are generally considered "better," it may surprise you to know that analog sound recordings may initially provide better sound reproduction. However, the physical reproduction of the analog sound (the needle running over the album, or the tape running over the tape head) causes deterioration of the stored signal. Digital storage is superior in the efficiency of storage, allowing compression of the signal (more data in less space) and no loss of accuracy of the signal over time.

2 Examples

The following are some of the activities that I do with students to introduce them to trigonometric concepts. Their initial exposure is mathematically light and intuitive, meant to motivate a deeper study of the topic. For the live experiments, I typically use my Apple PowerBook G4 laptop, connected to a projector and speakers. I use the free software *Audacity*™ to record sounds and save them as a .wav file, and use *Mathematica*™ to analyze, alter, and reproduce sounds.

Listen! Do you see something? Using *Mathematica*™, I am able to play selected frequencies on my computer. I can tune my guitar and play chords, and then quickly calculate the Fourier transform of the signal and graph the spectrum. For example, by sampling the curve $f(t) = \cos(300 * 2\pi t)$ at $t = \frac{k}{10000}$ for $k = 0, \dots, 9999$, I can play a 300 Hz note for one second. The graph of the points for $0 \leq t \leq 0.01$ seconds is shown on the left in Figure 3. Notice that the curve cycles 3 times in one-hundredth of a second, hence 300 times in one second. The spectrum of the tone is shown on the right in Figure 3. Notice how all of the Fourier coefficients are 0 except at 300 and $10000 - 300 = 9700$. (Due to the properties of the sampled curves, the Fourier transform is symmetric across half of the sample-size.)

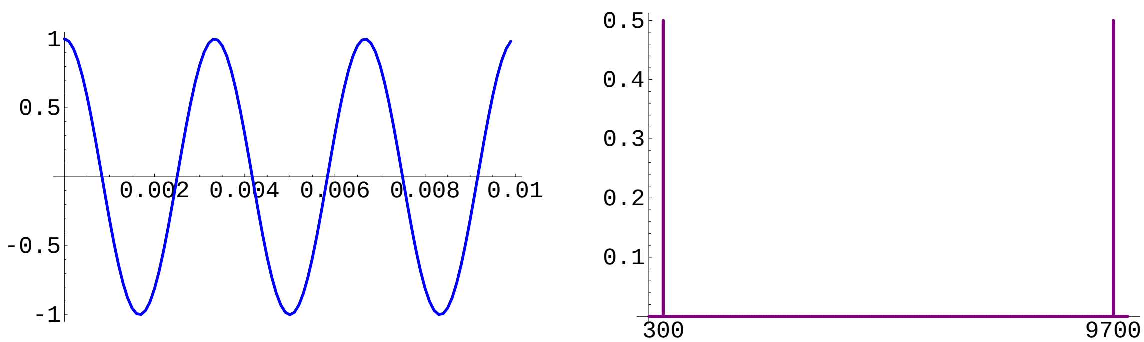

Figure 3: Sampled points for a 300 Hz tone over 0.01 seconds, and the Fourier transform of the entire second of sampled points.

By adding cosine (or sine) curves, I am able to play chords. By sampling the curve

$$f(t) = \cos(146.8 * 2\pi t) + \cos(220 * 2\pi t) + \cos(293.7 * 2\pi t) + \cos(370 * 2\pi t),$$

I can play a D-chord (see [1]). The sample points for the first three-hundredths of a second are shown on the left in Figure 4, along with a portion of its spectrum on the right.

What does music look like? The above tones are pure, synthesized sounds. Compare the sampling of a D-chord (the same four notes) when played on the guitar, shown on the left in Figure 5, and the Fourier transform of the entire one-second sample, shown on the right in Figure 5, to the graphs in Figure 4. The difference is analogous to the difference between a cartoon and a painting: a cartoon has clearly defined areas filled in with one color, while a painting has texture and subtle changes in shading. The irregularities of the curves in Figure 5 show the texture and subtle changes in pitch in the sample from the guitar.

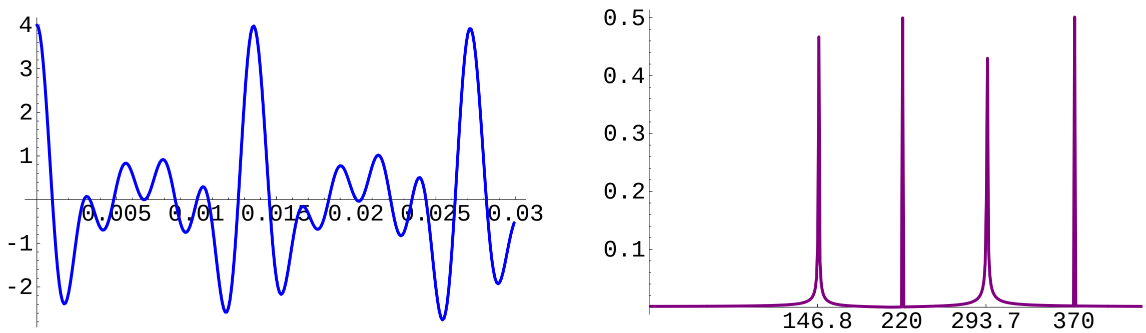

Figure 4: Sampled points for a D-chord over 0.03 seconds, and a portion of the Fourier transform.

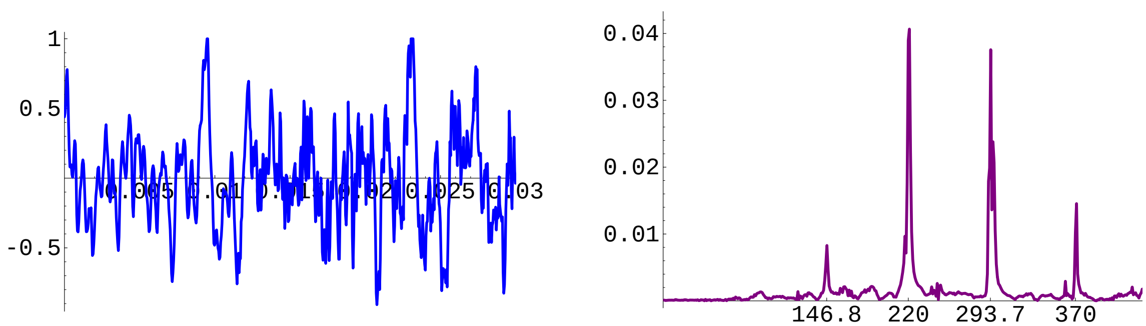

Figure 5: Sampled points for a D-chord played on the guitar over 0.03 seconds, and a portion of the Fourier transform.

As chaotic as the guitar chord may appear in comparison, it is still quite orderly. In Figure 6, you see the spectrum of the sound of the lobby of my building while classes are changing. Comparing this to the right graph of Figure 5, it is easy to "see" the difference between music and noise by looking at the Fourier coefficients.

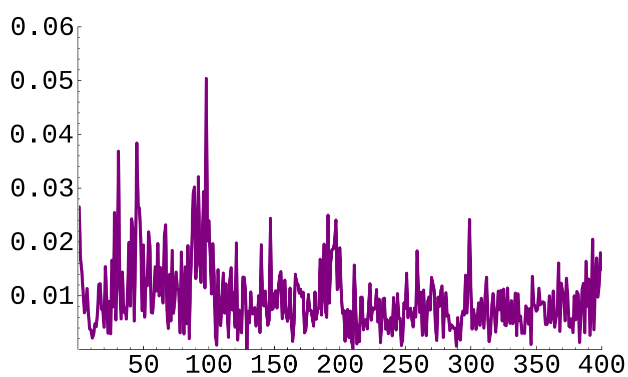

Figure 6: The Fourier transform of the sound from a noisy lobby.

How bad does it hertz? A curious use of sound is the so-called "Mosquito teenager repellant" (see [3]). A British company has a device that emits a loud 17000 Hz sound, and has been selling the device for use in public areas where teenagers congregate to skateboard, etc. A child with normal hearing can detect sounds up to about 20000 Hz, but we lose the ability to hear the higher-pitched sounds as we age. Very few people over the age of 25 can hear the 17000 Hz whine, although it is very irritating to teenagers. I usually play a sample of the "Mosquito" for my students, by sampling the curve $f(t) = \cos(17000 * 2\pi t)$ at $t = \frac{k}{N}$ for $k = 0, 1, \ldots$, where $N > 34000$. (It is curious to note that our sample rate N must be more than twice the highest frequency we want to reproduce, called the ***Nyquist rate***. The curve $g(t) = \sin(2\pi t)$ should generate a 1 Hz tone, but sampling twice in one second, the Nyquist rate for this example, gives $g\left(\frac{1}{2}\right) = g(1) = 0$ and the impression that there is no sound at all. To get the real story, we must sample more than the Nyquist rate.)

In an ironic turn-of-events, several other companies have been marketing an intermittent version of the "Mosquito" tone for download onto cell-phones for use as a teenager-only ringtone that students can use that is not detectable by their parents or teachers.

Can you pick the guitar string? The Fourier transform is an invertible process, meaning that we can reconstruct the exact sound sample from the Fourier coefficients. If we alter the Fourier transform of a sample, then we alter the sound. As a demonstration of this idea, I will take the sample of the D-chord played on the guitar of length N, and set $a_n = b_n = a_{N-n} = b_{N-n} = 0$ for n outside a certain range, and then reconstruct the sound. By keeping the Fourier coefficients between $n = 340$ and $n = 400$, for example (see Figure 7), and zeroing out the rest, the reconstructed signal is just the high string of the D-chord.

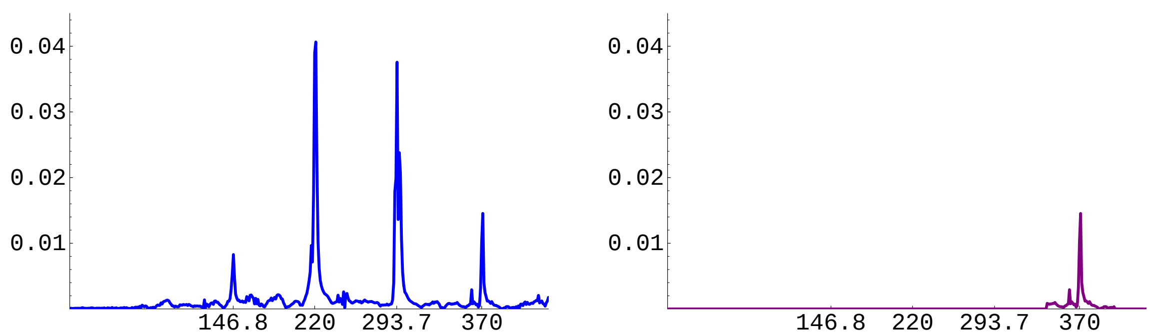

Figure 7: The Fourier transform of the original D-chord and with all but the high-string frequencies removed.

Bass to tenor without surgery. We can raise the pitch of a sound sample by one octave by doubling the frequencies. If we take the Fourier transform of the sample of length N and create a new set Fourier coefficients a_n^* and b_n^* by setting

$$a_{2n}^* = a_{2(N-n)}^* = a_n \text{ for } n = 0, \ldots, N-1 \quad \text{and} \quad b_{2n}^* = b_{2(N-n)}^* = b_n \text{ for } n = 1, \ldots, N-1,$$

and then fill in the odd frequencies with the averages

$$a_n^* = a_{2N-n+1}^* = \frac{a_{n-1}^* + a_{n+1}^*}{2} \quad \text{and} \quad b_n^* = b_{2N-n+1}^* = \frac{b_{n-1}^* + b_{n+1}^*}{2} \text{ for } n = 1, 3, \ldots, 2\left\lceil \frac{N}{2} \right\rceil - 1.$$

The results of this process as applied to the spectrum of the D-chord sample from the guitar are shown in Figure 8. The reconstructed sound sample (which is now twice the length, so it is sampled at twice the rate) is also a D-chord of the same duration, but now one octave higher.

In the same fashion, we can take a sample of someone speaking, and reconstruct it one octave higher. I usually select a male student with a deep voice (or perhaps the teacher if I am visiting a class), and use them as my subject during this demonstration.

Getting rid of a whistle-blower. The following demonstration has the appearance of being a mathematical magic trick, but it actually uses an idea that we have already discussed: removing certain frequencies from

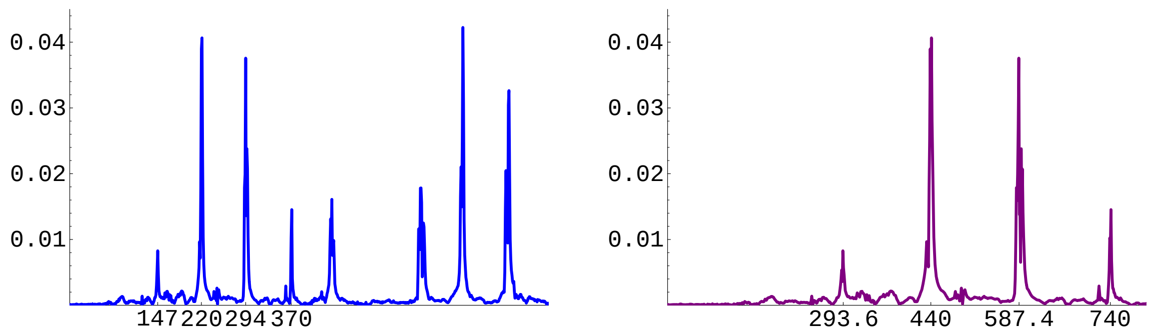

Figure 8: The Fourier transform of the original D-chord and the "stretched" spectrum.

the spectrum causes those tones to disappear from the reconstructed sound sample. For this demonstration, I usually have someone from the audience assist me by singing something for me while I record it. The frequencies present in this sample are relatively low, as shown in the left graph in Figure 8. Then, I have them sing the same song again, while, without warning, I blow a very shrill whistle at the same time. The spectrum of this new sample will be similar to the first, except that it will contain a large spike in the higher frequencies, as shown in the right graph of Figure 9. By zeroing-out the new spike and reconstructing the sound sample, we now have a sample of just the singing, with the sound of the whistle effectively erased.

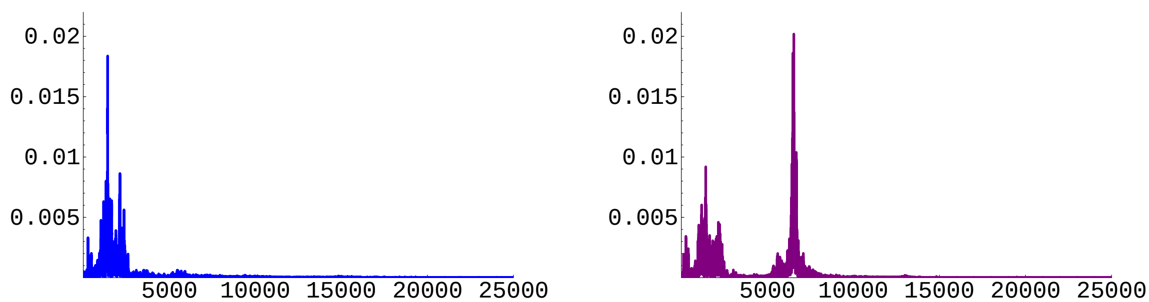

Figure 9: The Fourier transform of a singing voice, and the Fourier transform of the same person singing while a whistle is blown. (Only the left half of each spectrum is shown.)

3 Conclusion

I have spared the reader the actual trigonometry content that the previous demonstrations would lead into in a classroom setting. In practice, each of the activities would be followed by a discussion of the different algebraic and trigonometric principles involved, and perhaps even some traditional lecture and examples, with a chance to revisit the applications for further experimentation once the students have a better understanding of what is happening. While my entire trigonometry course is not built around music, the activities that involve music definitely get the attention of the students.

The important idea is that we do more than just practice skills. When people sign up to play a sport, they will probably have some practices where the coach tries to improve upon their skills with drills and training, but the real (possibly only) reason they signed up was to play the game. Allowing students to apply what they are learning is the academic equivalent of letting them play the game. Incorporating music into the "game" will hopefully encourage even more folks to play.

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