An Introduction to Medieval Spherical Geometry for Artists and Artisans

Reza Sarhangi
Department of Mathematics
Towson University
Towson, Maryland, 21252
rsarhangi@towson.edu

Abstract

The main goal of this article is to present some geometric constructions that have been performed on the sphere by a medieval Persian mathematician, Abul Wafa al-Buzjani, which is documented in his treatise *On Those Parts of Geometry Needed by Craftsmen*. These constructions, which have been illustrated as flat images, could be considered the bases of the arts and designs that artists and artisans have created on both the exterior and interior surfaces of a dome. Therefore, such a dome art design is a result of cooperation between mathematicians and artists.

This article also shows that the construction of the icosahedron on a sphere presented in that treatise is not mathematically correct. However, the construction of the spherical dodecahedron is exact. The article also presents flat images of constructions of some Archimedean solids according to the treatise.

1. Introduction

The purpose of this article is to analyze a part of medieval mathematics, a study of the sphere, which has been conducted by a Persian mathematician: Abul Wafa al-Buzjani.

Abul Wafa al-Buzjani was born in Buzjan, near Nishabur, a city in Khorasan, Iran, in 940 A.D. He learned mathematics from his uncles and later on moved to Baghdad when he was in his twenties. He flourished there as a great mathematician and astronomer. He was titled *Mohandes* by the mathematicians, scientists, and artisans of his time, which meant “the most skillful and knowledgeable professional geometer”. He died in 997/998 A.D. [1]

Buzjani’s important contributions are in the areas of geometry and trigonometry. In geometry he solved problems about compass and straightedge constructions in the plane and on the sphere. Among some other manuscripts which have been disappeared throughout the history of mankind he wrote a treatise: *On Those Parts of Geometry Needed by Craftsmen*.

The treatise by Buzjani was originally written in Arabic, the academic language of the Islamic world of the time, and was translated to the Persian language of its time in two different periods: 10th and 15th centuries. There are only a few known original translations of this book in the world; two of them are kept in two libraries in Iran, *Tehran University Library* and *Astan Ghods Razavi Library*, and another in a library in Paris. The book *Applied Geometry,* which includes a contemporary Persian language translation of Buzjani’s treatise appeared first in 1990 and then was republished with some corrections in 1997 [2]. The book also includes another treatise of later centuries: *Interlocks of Similar or Corresponding Figures.* Even though the book introduces a 15th century Persian mathematician as the author of this treatise, there are documents that suggest the possibility of a much earlier time writer, around the 13th century, for this work [3].

To construct shapes and patterns on a sphere, it is essential to have a basic understanding of a type of geometry that does not necessarily follow the foot-steps of Euclidean geometry: spherical geometry. Much formal study of spherical geometry occurred in the nineteenth century. However, some properties of this geometry were known to the Babylonians, Indians, and Greeks more than 2000 years ago. Euclid, in his *Phenomena,* discusses propositions of spherical geometry.

2. Study of Regular Tessellation on a Sphere

In Euclidean geometry of the plane, if p indicates the number of sides of a regular polygon and q the number of copies of the regular polygon about each vertex point, then it is elementary mathematics to show that $(p-2)(q-2) = 4$. Therefore, the number of regular tessellations on the Euclidean plane is three; the equilateral triangle tessellation, $\{3, 6\}$, the square tessellation, $\{4, 4\}$, and the regular hexagonal tessellation, $\{6, 3\}$.

It is interesting to study regular tessellations on a sphere. Since the sum of the angle measures of a spherical triangle is more than 180°, then for the tessellation of a regular p-gon with q copies about each vertex we have $(p-2)(q-2) < 4$.

An important fact about the above formula is the assumption of $p > 2$ for the Euclidean case. However, on the sphere, we may construct regular polygons of only two sides, which are called biangles or lunes. If the angle measure of a biangle is $360°/q$, then we are able to tessellate a sphere with q copies of the biangle.

A surprising fact about the spherical geometry is we don't have the similarity concept of polygons the way that we understand in Euclidean geometry. The following theorem gives a formula for the area of a spherical triangle:

Girard's Theorem: *If ABC is a spherical triangle with interior angles* α, β, *and* γ, *then the area of the triangle will be* $\pi r^2 \left(\frac{\alpha+\beta+\gamma-180}{180}\right)$, *where r is the radius of the great circle.*

In the Euclidean plane if two triangles have identical angles then they do not necessarily have the same area. On the sphere, however, if two triangles have congruent corresponding angles, then according to the above theorem they must be congruent. This means that we do not have any non-congruent similar objects on the sphere!

An observation related to this theorem is that there exists infinite number of regular p-gons with different angle sizes. For example, on a sphere, we may construct equilateral triangles with interior angles 70°, 85°, 90°, or any other angle with a value between 60° and 300°.

Going back to our discussion of tessellation, we are interested in determining all of the possible regular tessellations on the sphere. The above few paragraphs have made clear for us that unlike the case for Euclidean geometry, it is not sufficient to say, for example, equilateral triangles tessellate the sphere: It is possible that an appropriate equilateral triangle tessellates the sphere; however, another equilateral triangle with a different angle size does not.

It seems that the problem is much more complicated than its Euclidean version. However, with some information coming from another part of Euclidean geometry, we may be able to find the solution relatively easily.

Suppose that a certain spherical regular p-gon, $p > 2$, with angle measure $360°/q$, can tessellate a sphere. This means a finite number of this regular p-gon can cover the sphere without any gaps or overlaps. Now if all the vertices stay at their places but the sphere becomes flattened on its p-gons, then the result will be a Platonic solid: a polyhedron with identical regular faces, and identical vertices. Therefore, for the case $p > 2$, the problem of the regular spherical tessellation and the Platonic solids are identical.

Platonic solids were known to humans much earlier than the time of Plato. There are carved stones (dated approximately 2000 BC) that have been discovered in Scotland. Some of them are carved with lines corresponding to the edges of regular polyhedra. Specifically among them is a dodecahedral form that shows that the dodecahedron was known to humans much earlier than it appears in any written document. In addition, Icosahedral dice were used by the ancient Egyptians.

Evidence shows that Pythagoreans knew about the regular solids of cube, tetrahedron, and dodecahedron. A later Greek mathematician, Theatetus (415 - 369 BC) has been credited for developing a general theory of regular polyhedra and adding the octahedron and icosahedron to solids that were known earlier.

The name "Platonic solids" for regular polyhedra comes from the Greek philosopher Plato (427 - 347 BC) who associated them with the "elements" and the cosmos in his book *Timaeus*. "Elements," in ancient beliefs, were the four objects that constructed the physical world; these elements are fire, air, earth, and water. Plato suggested that the geometric forms of the smallest particles of these elements are regular polyhedra. Fire is represented by the tetrahedron, earth, cube, air, the octahedron, water the icosahedron, and the almost-spherical dodecahedron, the universe.

The Greeks only considered polyhedra with planar faces. As far as we know from the surviving texts, Buzjani was the first mathematician to consider the projections of polyhedra onto a sphere

3. How to Tessellate a Dome? A Medieval Persian Approach

In the absence of metal beams, domes have been an essential part of the architecture of both official and religious buildings around the world for several centuries. Domes were used to bring the brick structure of the building to conclusion. Based on their spherical constructions, they provided strength to the buildings' foundations and also made the structure more resistant against snow and wind. Besides bringing a sense of strength and protection, the interior designs and decorations resemble sky, heaven, and what a person may expect to see beyond "seven skies" [4].

When we encounter a structure such as a Persian dome and are amazed by the striking beauty and harmony of the varieties of patterns that have been constructed inside and outside of the surface of the dome, a natural question that may come to our minds is if rigorous geometry was involved, either by knowledgeable artisans, or mathematicians of old time, to design the dome. The treatise *On Those Parts of Geometry Needed by Craftsmen* reveals the direct involvement of mathematicians in the study and performance of tiling on the sphere.

Buzjani mentioned in his treatise about the interactions of artists and artisans with mathematicians on topics such as geometric constructions of ornamental patterns and the application of geometry to architectural construction. In Chapter Twelve of his treatise, Buzjani presents the ways that a sphere can be tessellated using properties of Platonic and some Archimedean solids (solids with two or more regular faces with identical vertices). One interesting remark about the original illustrations of this chapter is that all of the spherical constructions have been presented flat, with the hope that the reader uses his imagination to "see" them three-dimensionally.

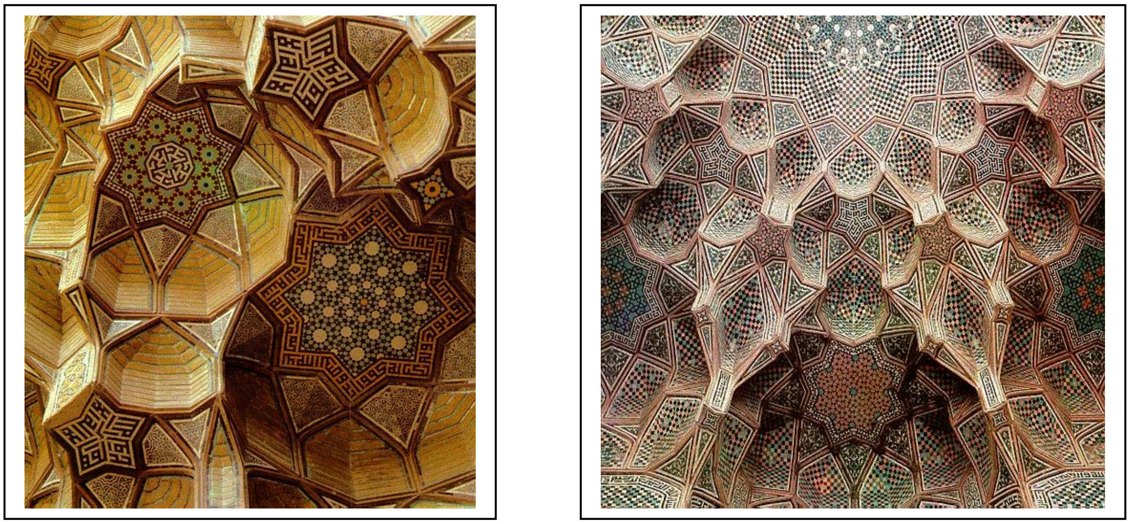

Figure 1: *Two dome interiors.*

3.1. Tiling with Eight Equilateral Triangles. After explanations of some elementary spherical constructions, in the problem numbered 174 of the treatise, Buzjani illustrates the construction of a spherical octahedron as follows: We want to construct three pair-wise perpendicular great circles. For this, we first construct two perpendicular great circles that meet at *A* (الف) and *C* (ج), and then divide one of the circles into four equal arcs of *AB*, *BC*, *CD*, and *DA*. The great circle that passes through *B* (ب) and *D* (د) — and meets the other circle on *E* (ه) and *F* (ر)—is the desired one. Now we notice that the sphere has been divided into eight spherical equilateral right triangles. This is the spherical octahedron.

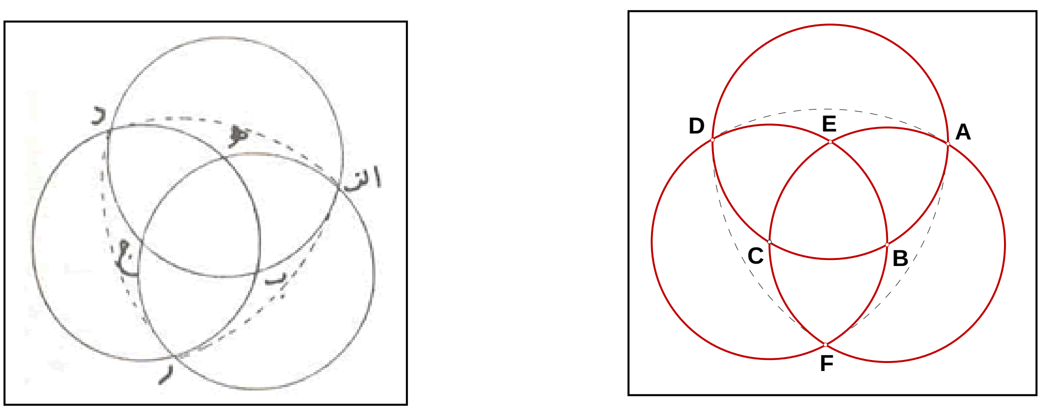

Figure 2: *Tiling of the sphere with 90° equilateral triangles*

3.2. Tiling with four equilateral triangles. Problem numbered 176 is about the construction of a spherical tetrahedron. For this, we need to divide the sphere into four congruent equilateral triangles. The first step is to construct the spherical octahedron as was illustrated in the previous problem.

Figure 3 begins with the construction of Figure 2. Three perpendicular great circles are illustrated that meet pair-wise at *A* and *C*, *B* and *D*, and finally *E* and *F*, and have created eight congruent equilateral triangles. Now let H (ح) be the centroid of the triangle ABE. We then construct three great circles that pass through AH, BH, and EH, respectively. Each circle passes through the centroid of a neighboring triangle, which has only a vertex in common with the original triangle. Name them K (ک), L (ط), and M (ع). Then Δ KLM is one face of our spherical tetrahedron. This process will divide the sphere into four congruent equilateral triangles: a spherical tetrahedron.

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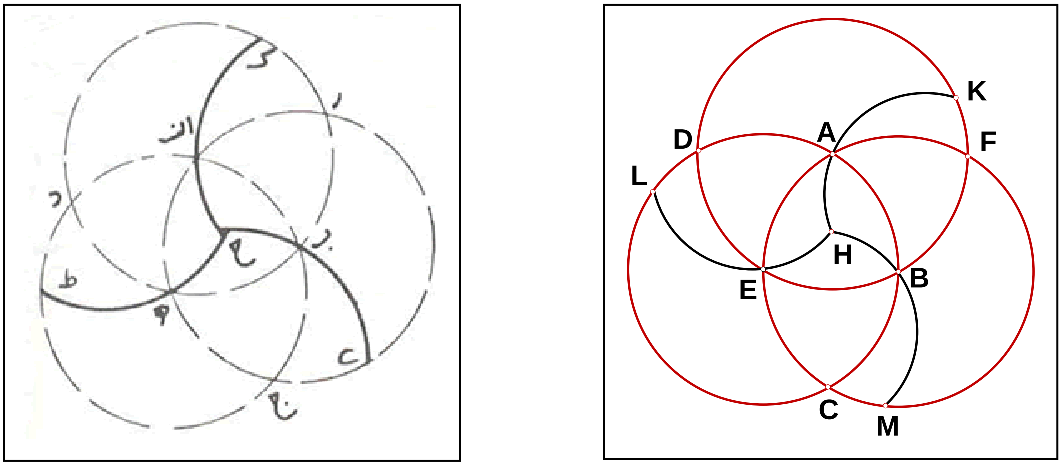

Figure 3 *Tiling of the sphere with 120° equilateral triangles*

3.3. Tiling with six squares. To construct a spherical hexahedron (cube), we first construct the spherical octahedron. Then we connect the centroid of each pair of neighboring triangles that share a common side by an arc from a great circle. This will divide the sphere into six congruent spherical squares.

For a person familiar with the idea of duality of platonic solids, the above construction is an immediate property of the octahedron (or hexahedron), but projected on to a sphere.

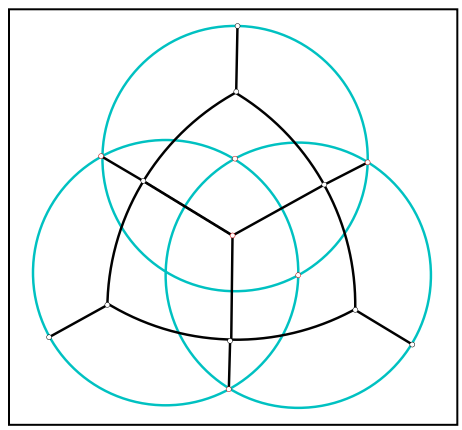

Figure 4: *Tiling with six spherical squares*

3.4. Tiling with twenty equilateral triangles. The next spherical construction is for the spherical icosahedron. But as we will see below, it contains an error. In problem numbered 180 the construction has been performed as follows:

Let E and F be the two poles of a sphere (only E (ه) can be seen in the following figure) and the great circle ζ is perpendicular to EF. Divide circle ζ into ten equal arcs C_iC_{i+1}, where $i = 1\ldots9$, and $C_{10}C_1$).

With the centers of C_1 and C_2 and with the radius of C_1C_2, we construct two arcs to meet at A_1 on the E side of the great circle C. With the centers of C_2 and C_3 and with the radius of C_2C_3 we construct two arcs to meet at B_1 on the F side of the great circle ζ. We repeat this procedure to find five points A_i on the E side and five points B_i on the F side of the great circle. Then using great circles passing through A_1 and B_1, B_1 and A_2, and A_2 and A_1 we will construct the equilateral triangle $A_1B_1A_2$. With this procedure we will construct ten equilateral triangles of $A_iB_iA_{i+1}$, $i = 1\ldots4$, $A_5B_5A_1$, $B_iA_{i+1}B_{i+1}$, $i = 1\ldots4$, $B_5A_1B_1$. Now on the E side we have a spherical regular pentagon of $A_1A_2A_3A_4A_5$ and on the F side another spherical regular pentagon of $B_1B_2B_3B_4B_5$. Using great circles that pass through E and A_i, $i = 1\ldots5$, we are able to divide pentagon $A_1A_2A_3A_4A_5$ into five congruent triangles. We do the same on the other side of the great circle ζ to find five more equilateral triangles. This will conclude our construction of a spherical icosahedron.

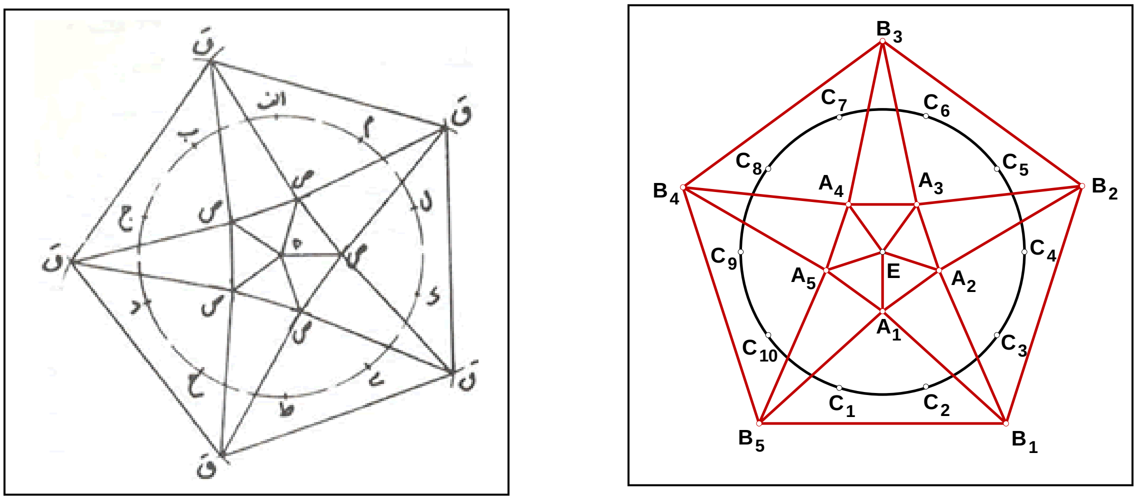

Figure 5: *Tiling with twenty 72° equilateral triangles*

3.5. Tiling with twelve regular pentagon. To construct the spherical dodecahedron, the book gives two methods. The first method uses the duality property of the Platonic solids. In Problem numbered 182 we read that we need first to construct the spherical icosahedron and then find the centroid of each triangle and connect the centroids of the neighboring triangles that have common sides in order to construct a spherical dodecahedron.

The other approach for constructing a spherical dodecahedron that the book illustrates is as follows: We have a sphere S with a given diameter. We first construct segment AB congruent to the diameter and divide it into three congruent segments of AC, CD, and DB. With center D and radius AD we draw a half circle that meets the perpendicular line to AB passing through B at point E. We find H on AB in such a way that $BH = 1/2\ BE$. With the center H and radius HE we find point L on ray AB.

BL is a side of a spherical pentagon that covers the sphere. Now we choose M (here it is ی) on the sphere and draw a circle with radius BL. We divide the circumference into three congruent arcs M_1M_2, M_2M_3, and M_3M_1 (here M_i's are ک, ل, and م).

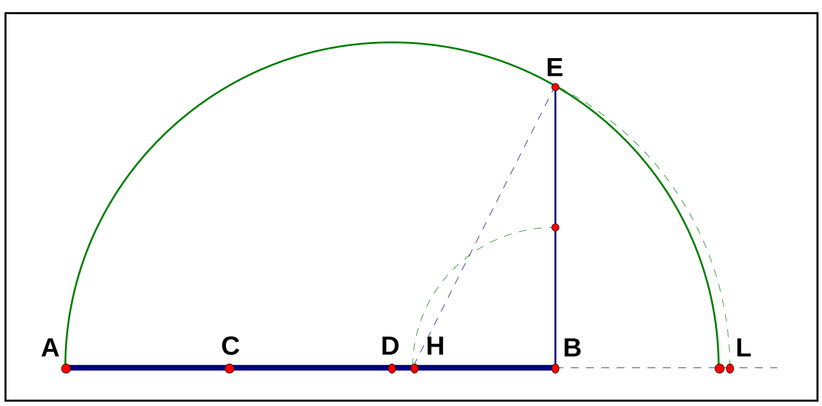

Figure 6: *Geometric construction of a side of spherical pentagon*

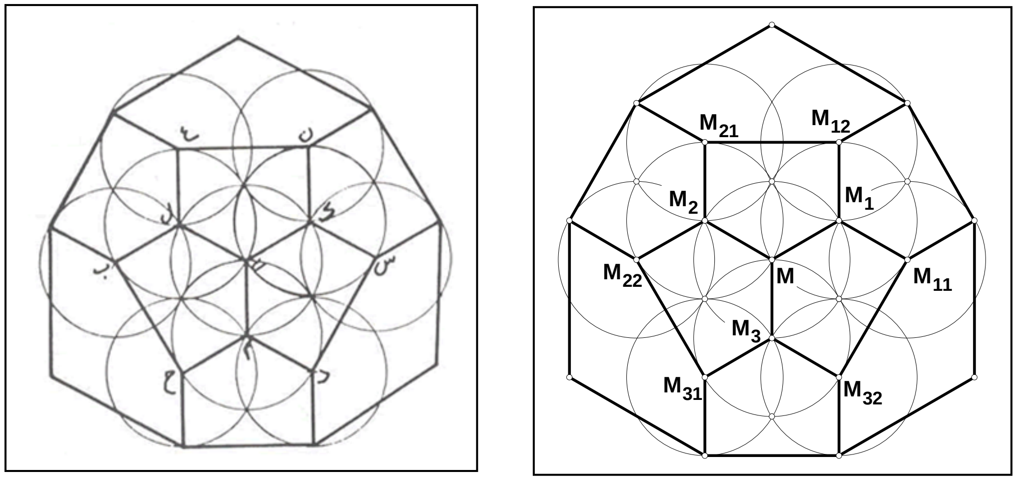

Figure 7: *Tiling of the sphere with twelve 120° regular pentagons*

From each M_i ,i=1,2,3, we make a circle with radius M_iM and divide it into three congruent arcs starting with point M as MM_{i1}, $M_{i1}M_{i2}$, and $M_{i2}M$ (these points are س, ن, ع, ب, ح, and د in the original figure). Now we have three spherical pentagons of $MM_1M_{12}M_{21}M_2$, $MM_2M_{22}M_{31}M_3$, and $MM_3M_{32}M_{11}M_1$. We continue this process to complete the tiling. It is understood by the reader that the sides of each pentagon is a part of a great circle that passes through two vertices.

4. Some Notes about the Buzjani's Constructions

An interesting project, rather than using a software utility, would be to construct these tessellations on a real sphere. This will give us a feeling of how a mathematician of ancient or medieval time could realize an abstract geometric construction on such a surface. Perhaps the Buzjani's tool for construction was a wooden sphere. Nevertheless, we may use a modern sphere, the *Lenart Sphere*, which is manufactured and sold as a pedagogical "spherical blackboard"[5]. The sphere comes with a smooth spherical surface on which we can draw with water-soluble pens, a torus on which to rest the sphere, hemispherical transparencies that fit over the sphere, a spherical ruler/protractor, and a spherical compass and center locator.

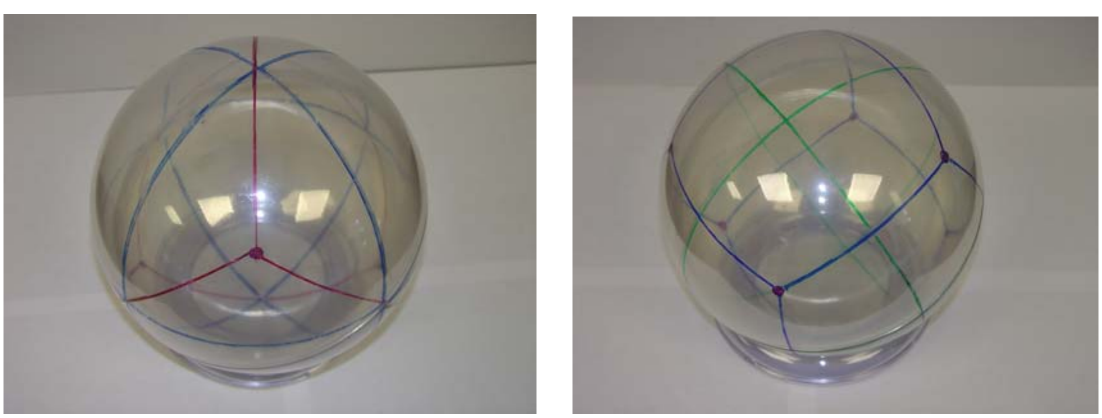

Figure 8: (a) *Tiling with four equilateral triangles*, (b) *Tiling with six squares.*

Figure 8.a presents the spherical tetrahedron and Figure 8.b presents the spherical hexahedron based on the Buzjani's approaches. To understand the logic behind these two constructions, we may take a look at Figure 9. In this figure, the octahedron and its dual, cube, are illustrated. If we start from a vertex of a cube and construct the diagonals of the faces that share that vertex, and connect the other vertices with diagonals of other faces, we will complete a tetrahedron inside the cube. Figure 9 presents this relationship as well and justifies the construction in Figure 3.

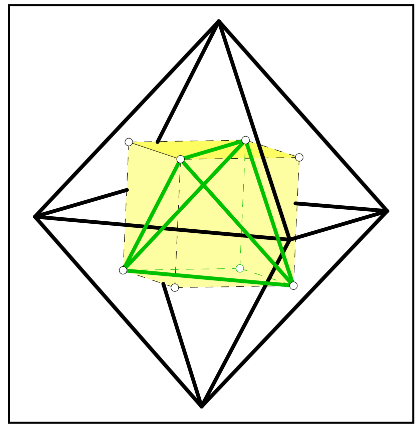

Figure 9: *The octahedron, its dual, the cube, and the tetrahedron.*

To understand the procedure of the mentioned construction of the spherical icosahedron, we need to study Figure 10.a. Points $A_1, A_2, \ldots, B_1, B_2, \ldots$, and $C_1C_2 \ldots C_{10}$ are on an icosahedron that seem to be corresponding to the vertices of the Buzjani's constructions of spherical icosahedron in Figure 10.b. In Figure 10.a the triangles $C_1C_2A_1$, and $C_2B_1C_3$ and so on are among the upside-down equilateral triangles that are parts of faces of the icosahedron of $B_5B_1A_1$, $B_1A_2A_1$, and so on.

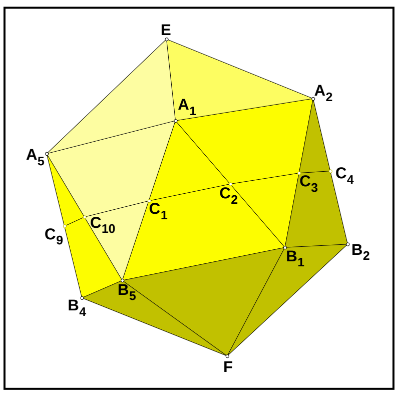

Figure 10: *(a) An icosahedron (b) Buzjani's Tiling with twenty equilateral triangles.*

The logic behind the construction in section 3.4 is since in Figure 10.a, the triangles $C_1C_2A_1$, $C_2B_1C_3$, and so on are equilateral triangles –which are similar to equilateral triangles $B_5B_1A_1$, $B_1A_2A_1$, and so on– by constructing them we are able to find the locations of the vertices A_1, A_2, A_3, A_4, A_5 and B_1, B_2, B_3, B_4, B_5 on the sphere. However, as it was mentioned before, in spherical geometry, the similarity property of the non-congruent triangles fails. This means what we have in Figure 10.a is not an accurate tiling of the sphere with twenty congruent equilateral triangles! In fact the triangles that constitute the antiprism of $A_1A_2A_3A_4A_5B_1B_2B_3B_4B_5$ in Figure 10.b are only isosceles triangles, which are larger than other isosceles triangles that constitute the two pyramids of $EA_1A_2A_3A_4A_5$ and $FB_1B_2B_3B_4B_5$. The problem is when we project the icosahedron in Figure 10.a on the ball that circumscribes it, then the image of $C_1C_2A_1$ on the ball is no longer equilateral. The central angles are the measure of length along the surface. Points C_1 and C_2 are edge midpoints, and are closer to the center of the sphere than the vertex A_1. So different central angles are subtended by equal length segments.

Now we would like to analyze the tiling with twelve regular pentagons presented in section 3.5. Let us assume that the diameter of the sphere is d. Then we find that $EB = \sqrt{3}\, d/3$, $LH = \sqrt{15}\, d/6$, and finally $BL = (\sqrt{5} - 1)\sqrt{3}\, d/3$. This can be expressed as $d = (\sqrt{15} + \sqrt{3})/4\ BL$. But this is the exact value of the circumradius of the dodecahedron with the side congruent to the segment *BL* [6]. Therefore, the Buzjani's construction of the dodecahedron is an exact construction and not an appropriate approximation (He has presented approximations in his treatise for the cases of the non-constructible regular polygons, such as the heptagon and nonagon).

But then there is a question that whether the same mathematician that came up with the exact construction of the spherical dodecahedron—in such an interesting approach that needs a deep investigation to find out how he successfully related the details in Figure 6 to the properties of the dodecahedron—has made such a faulty mistake in the construction of the spherical icosahedron! One possibility would be the addition of new parts to the original manuscript by mathematicians who are supposed to make new copies from the treatise. In fact what we have today are all copies, and perhaps the original document does not exist today. Writing the entire document was the only way of publishing a new book, and, therefore, the correctness of a new copy of a book was totally in the hands of the copy writer and his honesty.

5. Construction of Some Spherical Archimedean Solids

The treatise includes constructions of some spherical Archimedean solids such as cuboctahedron (tiling with eight equilateral triangles and six squares), truncated icosahedron (tiling with twelve pentagons and twenty hexagons), and icosidodecahedron (tiling with twelve pentagons and twenty triangles).

The other Archimedean tilings illustrated by Buzjani include the construction of truncated octahedron—which is done by joining the one-third points of each side to appropriate neighboring one-third points using arcs from great circles and then erasing old vertices of the octahedron—and the truncated tetrahedron—with the same procedure as the former construction. The first construction results in a division of six squares and eight hexagons. The second construction divides the sphere into four triangles and four hexagons.

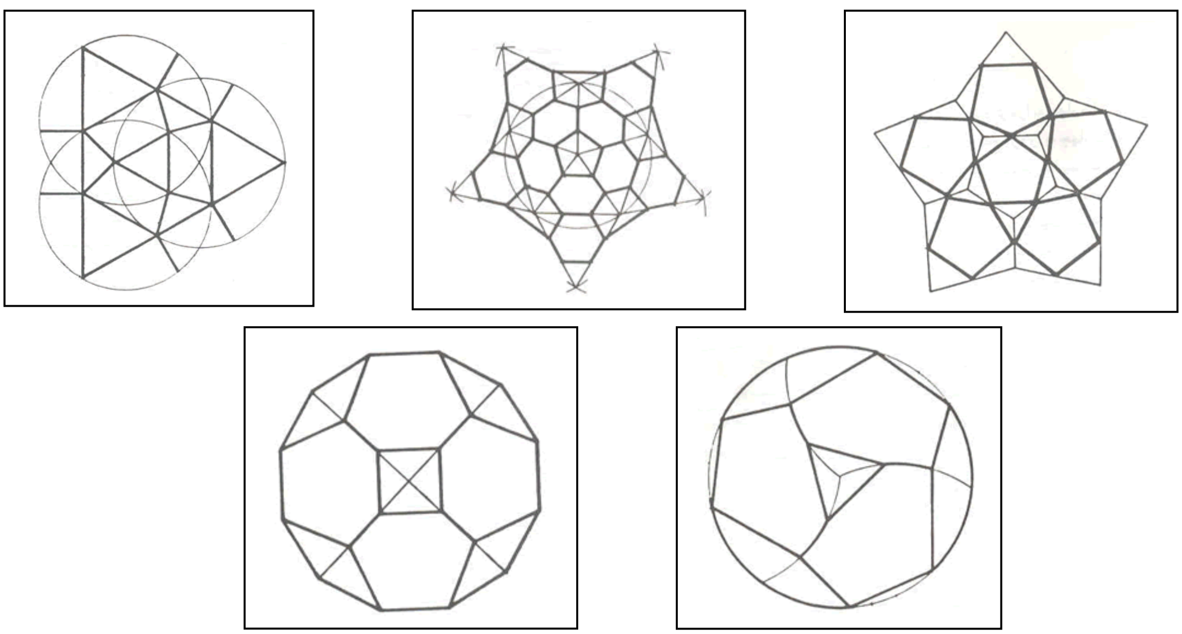

Figure 11: *The flat presentations of the constructions of cuboctahedron, truncated icosahedron, icosidodecahedron, truncated octahedron, and truncated tetrahedron.*

5. Conclusion

In the medieval Persian culture, the structure of objects, such as domes, necessitates the artists and artisans to rely on mathematicians. Such a relationship has occurred during this period as documented in the treatise *On Those Parts of Geometry Needed by Craftsmen.* Even though the observer of such an artwork may fantasize about it by relating the harmony and symmetry of the piece to some sort of revelations, the above explanations present the direct involvement of mathematicians, solely based on the scientific approaches available in those times, for creation of such a work.

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