Celtic Knotwork and Knot Theory

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Abstract

Celtic knotwork is a form of decoration in use for over a thousand years. The designs fill spaces or borders with a pattern derived from plaiting. The designs have no loose ends and may contain more than one closed loop. As in a plait (or braid) of hair, each strand bounces back and forth like a billiard ball to form a pattern of diagonal lines between the edges of the rectangle while crossing over and under others alternately. The dimensions of a rectangular plaited panel can be expressed as the number of bounces there are along the long and short edges. The number of closed loops, referred to as knots by knot theorists, is the greatest common divisor of these two numbers. This paper shows how one can predict the number of loops there will be as a piece of knotwork is created from the panel by removing some of the crossing places and rejoining the loose ends, without crossing to make a gap either looking like) (to make what will be called a horizontal gap, or like this shape turned through 90° to make a vertical gap. As each of the chosen crossing places is removed and the loose ends rejoined in this way a more intricate interlaced design is formed. It is easy enough to trace round the resulting design with coloured pens to find out how many closed loops there are, but the results proved and demonstrated in the paper enable one to predict how the number of loops will change at each stage of the creation of the interlaced design. Such a prediction is not addressed by current knot theory. The first thing to notice is whether a loop crosses itself at the crossing to be removed or whether two different loops cross there. In the first case one or two questions must be answered before the number of loops can be predicted. In the second case the two different loops get combined into one single loop by the rejoining of the loose ends. The difficult part of the research was to devise questions which could be proved to make reliable predictions possible. One's common understanding of how one might take a shortcut back to the start while following a nature trail provides the last link in the chain leading to the prediction. The method will be applied to successive designs produced as each crossing is removed.

1. Introduction

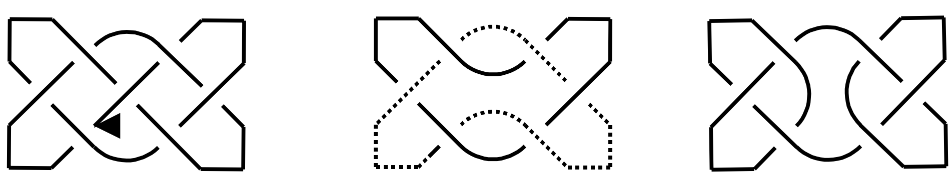

Figure 1: *Two celtic knotwork designs derived from a plaited panel with one loop*

Even a simple plaited panel can be converted to an interlaced design of a type which distinguishes celtic knotwork from the plaits used in Egyptian, Greek or Roman art. Figure 1 demonstrates what happens when the central crossing is removed and the loose ends rejoined to make a gap in the plaiting. The two different ways of rejoining produce different designs, one with two loops and one with one loop. While it is easy enough to determine the number of loops there are even in an intricate interlaced design, with the help of coloured pens, the mathematician seeks an underlying structure which will enable a prediction to be made for the resulting number of loops as each of many crossings is removed. The paper continues with classification of the crossing to be removed and carries on with classifying the ways of rejoining the loose ends to make a gap before proceeding to arguments justifying the predictions.

1. The classification of crossings

A crossing can be:
(a) where a loop crosses itself or
(b) where it crosses another loop.

In case (a) we need to define two types of crossing. To do this we start at the crossing to be replaced and proceed to add arrows along the loop, starting in either direction, until we reach this crossing again. It will then be clear which of the diagrams in Figure 2 below resembles our crossing. The first two resemble part of an upright figure of eight circuit, (not to be confused with what knot theorists call the Figure Eight Knot) while the second two resemble part of a figure of eight lying on its side, ∞; let's call it an infinity circuit. So we can ask whether our crossing to be replaced resembles an 8 crossing or an ∞ crossing, even though the actual loop may have many tangles on either side of the particular crossing to be replaced which disguise its resemblance to one of these two circuits. There is no topological difference between these two circuits; making this distinction will help us to visualise what the different ways of rejoining the loose ends will do to the number of loops.

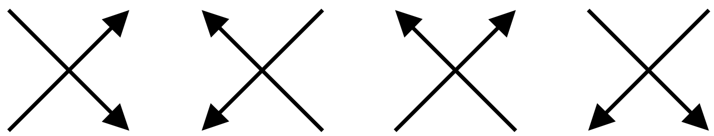

Figure 2: *Different types of crossing.*

2. The classification of gaps

A gap like) (is classified as horizontal because horizontal arrows would indicate its width in a technical drawing, and a gap like this shape turned through 90° as vertical. We can already visualise that a horizontal gap will leave an 8 circuit in tact as one loop and a vertical gap will leave an ∞ circuit in tact as one loop, whereas a vertical gap will cut an 8 circuit into two loops and a horizontal gap will cut an ∞ circuit into two loops. With these classifications of crossing and gaps we can now proceed to arguments supporting a general statement for predicting the number of loops.

3. Arguments justifying the predictions

We start by visualising a simple figure of 8 nature trail shown upright on a map. If one replaced the crossing with a horizontal gap, like) (, one would just follow part of the trail in the reverse direction, whereas a vertical gap would provide a shortcut back to the start. This common understanding can be extended beyond a simple figure of 8.

We will use the plaited panel on the left in Figure 1 to demonstrate such an extension by replacing the central crossing with a gap. The argument, continuing the nature trail analogy, will follow Figure 3 from left to right. Arrows added to the plaited panel show that the central crossing is an 8 crossing. Then, texture added shows the trail as a continuous dotted path followed by a black path. Think of yourself starting at the bottom left hand corner of the trail map and arriving at the central crossing point along the dotted path. If you alter your route so that instead of carrying on straight ahead you turn off to the right, as in the next diagram, then you will find that you have taken a short cut back to the start and left out the black part of the trail. You followed a map with a vertical gap in the middle. If, on the other hand you turn off to the left you will follow the black path, but against the direction of the arrows, as in the right hand diagram, until you reach the central crossing again and can turn left to follow the rest of the dotted path, with the arrows, back to the start. This time you followed a map with a horizontal gap in the middle. We need only join up the second pair of loose ends at the centre of the third diagram and remove all arrows to see that the third and fourth diagrams confirm the effect on the number of loops of replacing the central 8 crossing with vertical and with horizontal gaps, as shown in Figure 1.

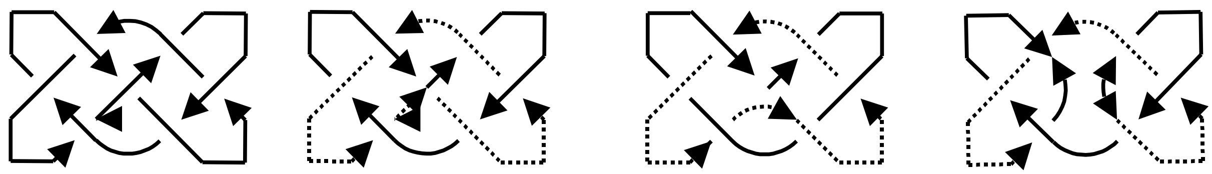

Figure 3: *The nature trail analogy with right and with left turnings instead of crossing ahead.*

We are now ready to summarise our findings in the middle line of Table 1 below. The bottom line arises because an ∞ circuit is merely a rotation of an 8 circuit.

Crossing type	Vertical gap	Horizontal gap
8 crossing	1 extra loop	No extra loops
∞ crossing	No extra loops	1 extra loop

Table 1: *The effects of different types of gap on a loop crossing itself.*

Mathematicians will want a general argument to support this table as a reliable prediction. Consider a nature trail with p consecutively numbered direction posts. Let them be numbered 1 to m as far as the 8 crossing. The original route, on the left in Figure 4, would carry on with numbers $m + 1$ up to n, just before one meets the crossing again, and then from $n + 1$ to p. The effect of a right turn is to miss out posts numbered $m + 1$ up to n; one has kept going in the direction of the arrows, just missing out a piece of the circuit. A trail through these makes one of the two loops, while a trail through those missed out makes the other loop. The effect of the left turn, on the right in Figure 4, is to pass posts 1 up to m in ascending order, then, going against the direction arrows, to post numbered n straight after post numbered m and then posts with decreasing numbers, from $n – 1$ down to $m + 1$, before turning left again to pass post numbered $n + 1$ and then posts with increasing numbers, $n + 2$ up to p. This time all the posts have been passed in a single loop. This generalised argument supports the predictions for the numbers of loops.

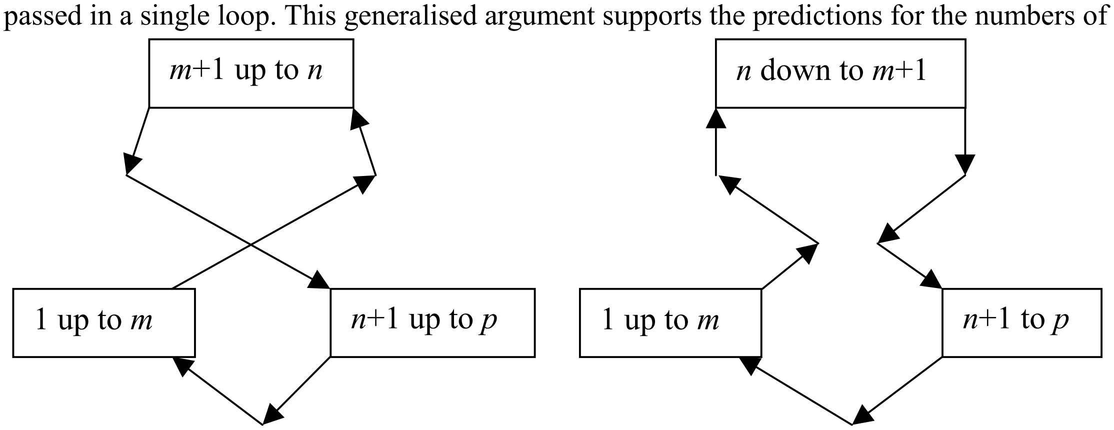

Figure 4: *The nature trail analogy with left turning instead of crossing ahead.*

The generalised argument has focussed on following or going against the direction arrows. Figure 5, with arrows on the gaps enables us to determine the effect on the number of loops with just one question.

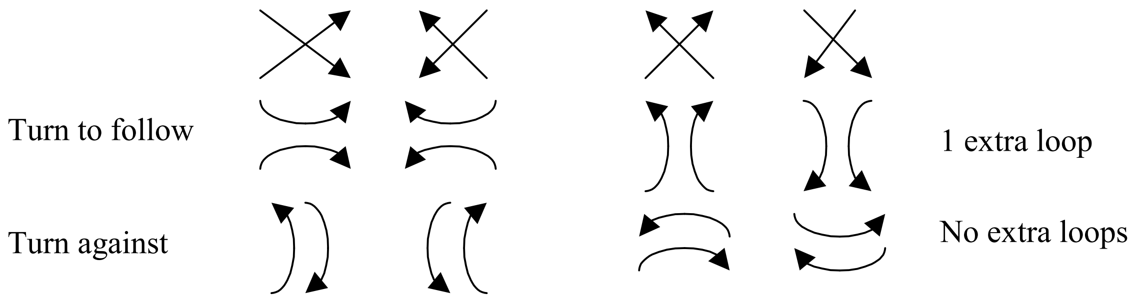

Figure 5: *From crossings to gaps, to number of extra loops*.

So the question is "Does your turn follow the arrow direction or go against it?" Turning to follow makes one extra loop while turning against makes no extra loops.

We now consider the much easier case (b) where a loop crosses another loop as in the middle diagram in Figure 1. The replacement of the left hand crossing with each type of gap is shown in Figure 5.

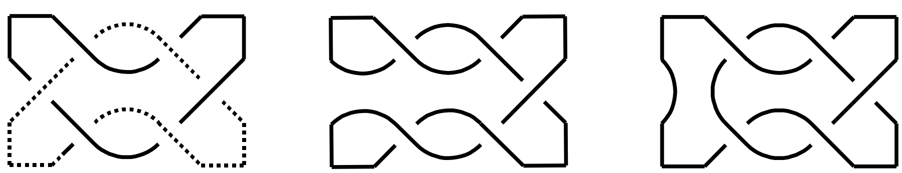

Figure 6: *A place where one loop crosses another replaced with vertical and horizontal gaps*.

In case (b) either type of gap will amalgamate the crossing loops to make one loop because the loose ends consist of two from each loop and if they are to be rejoined without crossing they have to be paired up using one loose end from each loop. So, if the answer to the question of whether a loop crosses itself or another loop at the crossing to be replaced with a gap is that it crosses another loop then one needs no further questions; the replacement will reduce the number of loops in the design by one.

Conclusion

While it is easy to create a knotwork design from a plaited panel by making all the replacements of crossings with gaps at once and then discover how many loops it has by tracing round the thread or threads, the challenge of the research supporting the analysis described in this paper was to find a method of predicting the number of loops there will be at each stage when one replaces one crossing at a time. The mathematician likes to seek out a structure beneath problems which enables many different instances to be analysed with ease. First of all one asks whether the crossing is where a loop crosses itself or another loop. The first case requires matching the crossing to a part of one or other of two simple circuits, and then the orientation of the replacement gap to vertical or horizontal; Table 1 showed the predictions. Figure 5 portrays the same information visually. In the second case the two loops are amalgamated by the replacement gap. This analysis, to predict the number of loops there will be in a Celtic knotwork design brings an area of abstract art born in the Celtic culture into the orbit of knot theory. It provides another bridge linking the joy of the abstract artist to the joy of finding a mathematical structure to predict the answer to a practical question. The analysis can be applied to any knot or link.