Visualizing Escape Paths in the Mandelbrot Set

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Abstract

This paper describes a method for producing a striking animation of the explosions that take place as the parameter *c* that defines the Mandelbrot Set is allowed to traverse a path from inside the large cardioid component of the Mandelbrot Set into one of the attached "bulbs" or other regions just outside the set. The presentation will include the animation itself, as well as some of the colorful images obtained by stopping the animation at various points.

1. Introduction

Mathematicians and artists are familiar with the famous orbit diagram for the real-valued function $f_c(x) = x^2+c$ where we see successive period-doubling bifurcations and eventual chaos as the real parameter *c* decreases along the real number line from 1/4 to -2. In this case the parameter *c* is actually traveling one of the many paths from $c = 1/4$ to the outer reaches of the Mandelbrot Set at $c = -2$. Because we are dealing with real numbers we can plot the fixed or periodic points of f_c on one axis as a function of the parameter *c* on the other axis. Suppose we want to examine the dynamics by letting *c* travel along a different path from the origin to some point outside the Mandelbrot Set; in this case both the parameter *c* and the fixed point *w* will be complex numbers. How are we to visualize the changing dynamics in this case? By expressing both *c* and *w* as functions of a single parameter, *r*, we can make an animation by plotting several thousand points in the orbit of 0 under $f_c(z)= z^2+c$ for each *r*, as *c* travels various escape routes from the Mandelbrot Set, *M*. By assigning the color of the points in the orbit as a function of *r* we can create some quite spectacular animations. In [2] I explore the escape routes from *M*; each route comes with its own unique dynamics and its own amazing graphics illustrating the transition from order to chaos.

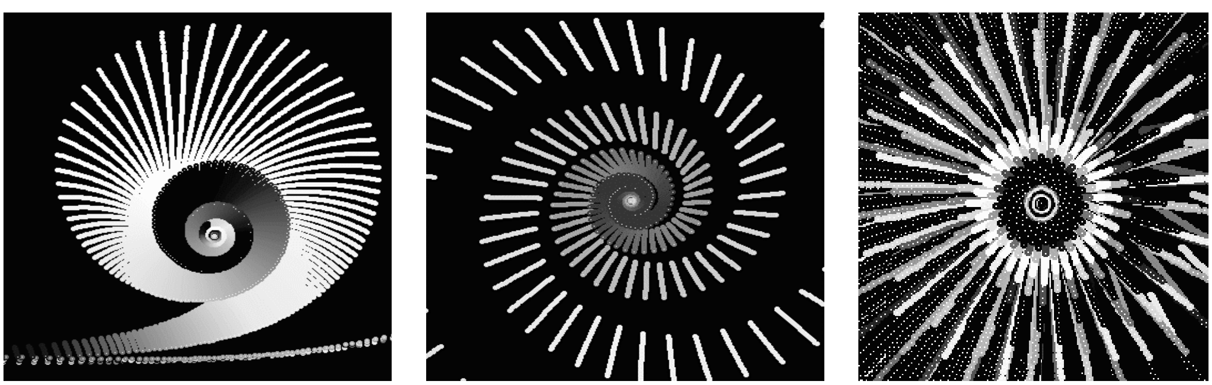

Figure 1: *Examples of images obtained by stopping the animation*

2. How to find the escape routes

The first step in finding the escape routes is to express both c and the fixed point w as a function of a single parameter r. To find a fixed point, w, of $f_c(z) = z^2+c$, we solve the equation

(1) $w^2+c = w$, and we note that w is attracting if
(2) $|f'(w)| = 2|w| < 1$, or $|w| < 1/2$.

We are going to be interested in the set $M_1=\{c \mid f_c(z)$ has an attracting fixed point$\}$. Solving (1) for c we get $c = w - w^2$ and from (2) letting $w = (r/2)e^{i2\pi\theta}$ gives

(3) $c = \dfrac{re^{i2\pi\theta}}{2} - \dfrac{r^2e^{i4\pi\theta}}{4}$, $0 \le r < 1$, $0 \le \theta < 1$.

The boundary of this region ($r = 1$) is the large cardioid (the boundary of M_1) in the Mandelbrot Set. It is well known (see any of the references) that for c on this boundary the dynamics of f_c are determined by the value of θ.

If $\theta = \dfrac{p}{q}$, p and q natural numbers with $p < q$ and $\gcd(p,q) = 1$, then at the point c on the boundary of M_1 a hyperbolic component (or what Devaney calls a "bulb") is attached to M_1. We will call this bulb $M_{p/q}$. The corresponding fixed point is neutral and is a parabolic fixed point. At this parabolic fixed point the attracting fixed point bifurcates into an attracting periodic cycle of period q. For $r < 1$ the corresponding w is attracting and there is a repelling cycle of period q surrounding w.

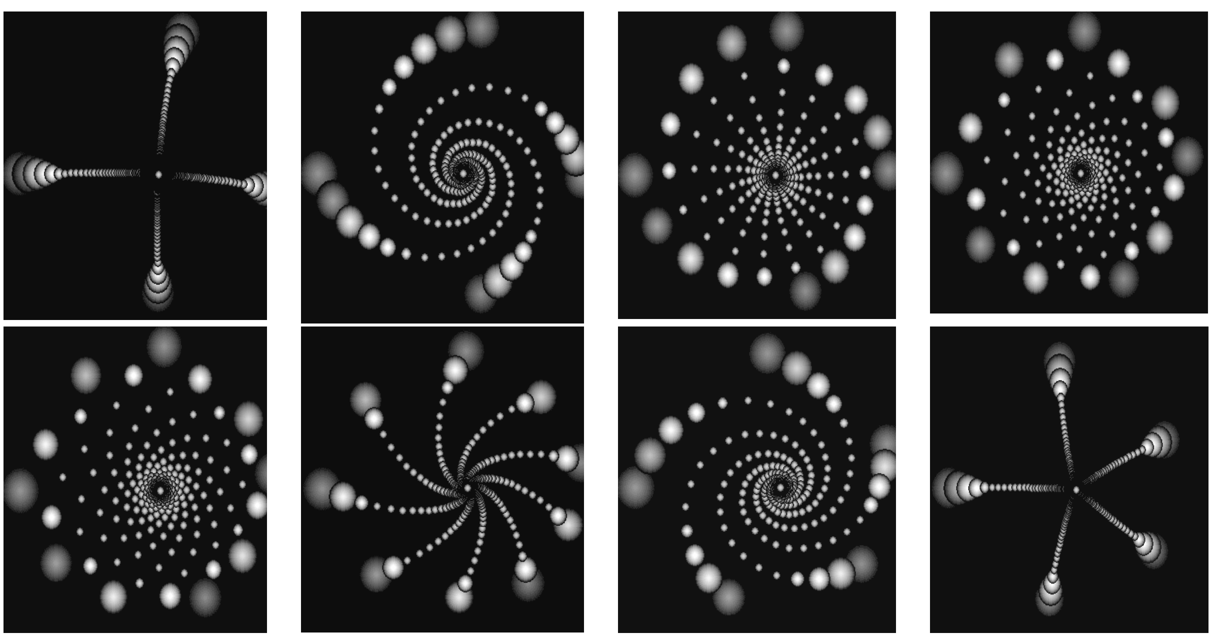

Figure 2: *Transition from 1/4 to 1/5*: *Orbits keeping r fixed at .975; θ varies from 1/4 to 1/5*

(Note that for the real orbit diagram $\theta = 1/2$ and the bifurcation is period-doubling i.e. $q = 2$.) As r approaches 1 from below, the fixed point w and the repelling cycle coalesce into the neutral fixed point. As r increases beyond 1, c moves into $M_{p/q}$, the fixed point w becomes repelling and the period q cycle becomes attracting. Smaller bulbs are attached to each $M_{p/q}$ and for c in one of these bulbs f_c has an attracting cycle of some finite period (the period will be a multiple of q).

For θ "sufficiently" irrational there is a neighborhood of $w = (1/2)e^{i2\pi\theta}$ called a Siegel disk. In this neighborhood orbits of nearby points look like deformed circles surrounding the fixed point w [4]. There are many books containing beautiful pictures of fractals illustrating the complicated dynamics for values of c near the boundary of the Mandelbrot Set. What we want to do here is to describe how to create an animation of the orbits as c follows the path (3) from inside M_1 across the boundary and into one of the bulbs, or near one of the bulbs.

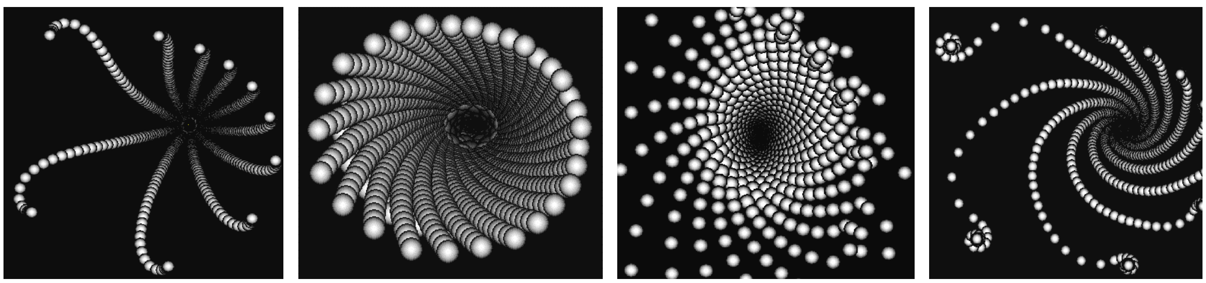

Figure 3. *Transition from 1/10 to 1/9*: *Orbits keeping r fixed at 1.00438; θ varies from 2π/10 to 2π/9*

3. Orbits for different values of *r* and *θ*

In Figures 2 – 4 we show orbits of a single point for various values of r and θ. In Figure 2 we have kept r fixed at .975 and we let θ vary in increments from 1/4 to 1/5. Since $r < 1$, in each case there is an attracting fixed point. In each frame 2000 points in the orbit of 0 were plotted. In the first frame ($p/q = 1/4$) points in the beginning of the orbit surround the fixed point in a "4-pattern". This is very clear when we color every 4^{th} point the same color.

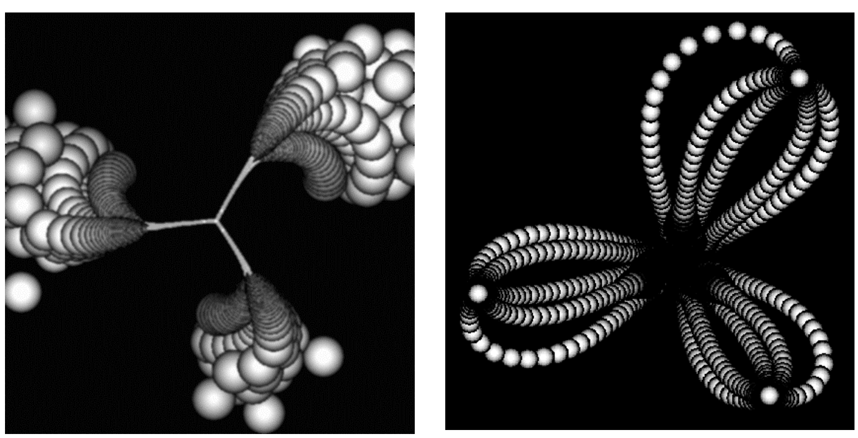

Figure 4. *Transition from attracting fixed point to attracting 3-cycle; θ is fixed at 2π/3; first picture r = .99 (r < 1), second picture r = 1.025 (r > 1)*

Similarly in the last frame (p/q = 1/5) we can see the "five"ness of the pattern. In between we see other types of orbits; for example in the third frame p/q = 4/17 and the orbits fall into a "17-pattern". Incredibly, as p/q varies from 1/4 to 1/5, there is an uncountably infinite variety of orbit patterns. In Figure 3 we have let r be slightly greater than 1 and p/q vary from 1/10 to 1/9. In the first frame we see an attracting cycle of period 10 and in the fourth frame we see an attracting cycle of period 9. In between we see an attracting cycle of period 29 in the second frame and another kind of behavior in the third frame

In Figure 4 we see the orbits of 0 where θ is fixed at $2\pi/3$; in the first picture $r < 1$ and there is a fixed point in the center of the screen; in the second frame $r > 1$ and there is an attracting cycle of period 3. Using three colors and coloring every third point in the orbit the same color gives us a better picture of how the orbits behave.

4. The relationship between the fixed and periodic points

To find the cycles of period n for Q_c we have to solve $Q_c^{\mathrm{n}}(w) = w$. There will be 2^n of them. When c = 0, this means $w^n = w$. The solutions will be $w = 0$ (the attracting fixed point) and the 2^{n-1} th roots of 1. So they will be distributed around the unit circle. For example, if n = 3 there are two fixed points, w = 0 (attracting), w= 1 (repelling) and two 3-cycles: $\{e^{i2\Pi/7}, e^{i4\Pi/7}, e^{i8\Pi/7}\}$ and $\{e^{i6\Pi/7}, e^{i12\Pi/7}, e^{i10\Pi/7}\}$.

If we let $w = (r/2)e^{i2\pi p/q}$, p and q relatively prime natural numbers with $p < q$, be the fixed point, and $c = w - w^2$, when r = 0, w = 0 and there will be at least one repelling q-cycle distributed around the unit circle. In the last paragraph p/q = 1/3 and there were two repelling 3-cycles. As r increases to 1, the fixed point moves along the ray $(r/2)e^{i2\pi p/q}$ and one of the repelling q-cycles surrounds the point w, moving ever closer to w. At r = 1 the fixed point w and the q-cycle coalesce. At r = 1 $|Q_c'(w)| = 1$ and the multiplier of the cycle, $\prod_{i=1}^{q} Q'_c(w_i) = 1$. This point is called a parabolic fixed point of Q_c. As r increases beyond 1, the q-cycle becomes attracting and the fixed point becomes repelling. (See Figure 5)

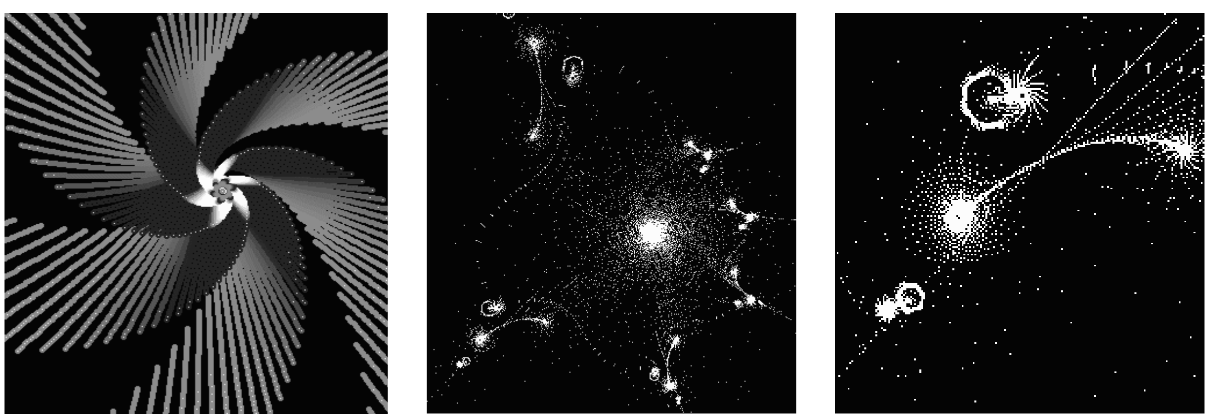

Figure 5: *Three stages in the animation where $\theta = 2\pi(1/6)$. The first frame shows a close-up of the center of the screen after r has increased to about .987; in the second frame r has increased to 1.04. The third frame is a close up of the action near one of the points in the attracting 6-cycle seen in the second frame.*

5. Building the Animation

For each value of θ, where $w = re^{i2\pi\theta}/2$ and $c = w - w^2$, we animate the scene by choosing the center of the computer screen to be the point where the fixed point and cycle coalesce. We allow r to increase in small increments and for each r we plot some number (to be chosen by the user) of points in the orbit of 0. Each orbit is plotted in a different color determined by r, using a continuous ramp of colors. The results are quite spectacular.

After writing the program animating the orbits, we incorporate that program into a larger program where we continuously change the value of θ and observe the amazing changes in dynamics as θ travels around the unit circle. Some of the more spectacular pictures occur when θ is not rational. In this case the parameter c exits M for a brief moment before re-entering in one of the bulbs. Figure 5 shows three stages in the animation where $\theta = 2\pi(1/6)$. In Figure 6 we have illustrated what the screen looks like for $\theta = 2\pi p/q$, where p and q are successive Fibonacci numbers. It is known that the values $\theta = \dfrac{-1+\sqrt{5}}{2}$, $r = 1$ admit a Siegel Disk where orbits behave like deformed rotations about the fixed point w.

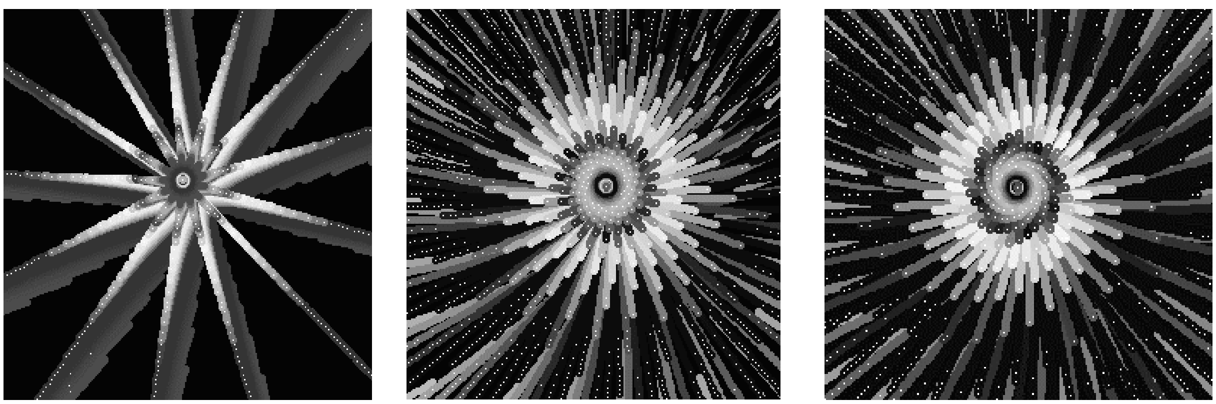

Figure 6: *$\theta = 2\pi p/q$, where p and q are successive Fibonacci numbers. In the first frame p = 8, q = 13; in the second frame p = 55, q = 86; in the third frame p = 141, q = 227. In all frames we let r increase to about .99026 before stopping the animation*

Figure 7 shows some stills from the animation for different values of r and θ.

6. Conclusion

Expressing the fixed point and the parameter c as a function of r and θ allows us to produce virtually any kind of orbit that we choose for the function $f_c(z) = z^2+c$. Then by varying the parameters in small increments, we can animate the continuous change that takes place in the nature of the orbits as the parameters change. I have found that this parameterization helps students in a fractals class to find values that yield interesting orbits as in Figure 8. It might suggest other ways of animating different dynamical systems. We might explore using color as a function of the parameters in other ways. For many people visualizing mathematics is vital to understanding it; conversely, understanding the math allows us to create visually compelling images.

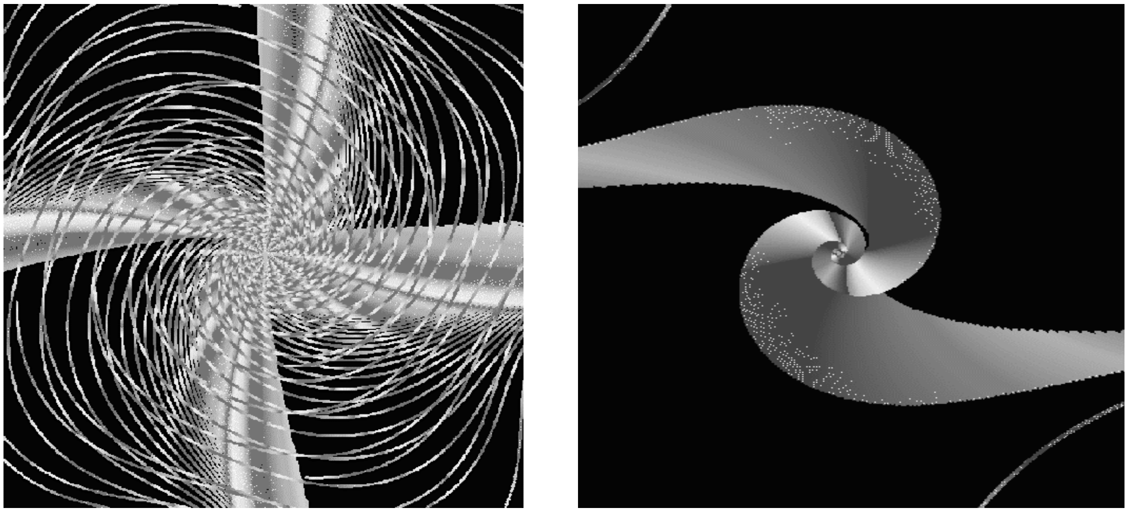

Figure 7. *In the first frame* $\theta = 2\pi p/q$ *where* $p = 1251$ *and* $q = 5000$*; stopped at* $r = 1.1987$*. In the second frame* $\theta = 2\pi p/q$ *where* $p = 251$ *and* $q = 500$*; stopped at* $r = .99459$

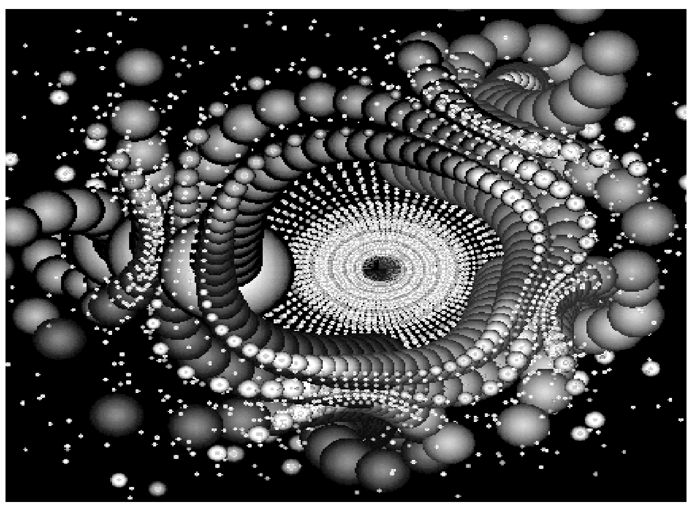

Figure 8. *Orbits of* $z^2 + c$

References

[1] B.Branner, "The Mandelbrot Set"*, in R. Devaney and Linda Keen, editors, Chaos and Fractals (*Proceedings of symposia in Applied Mathematics, vol. 39) American Mathematical Society, Providence, RI, 1988

[2] A. Burns, "Plotting the Escape", *Mathematics Magazine,* Vol. 75, No. 2, April, 2002

[3] R. Devaney, *An Introduction to Chaotic Dynamical Systems,* Addison-Wesley, 1987

[4] H.-O. Peitgen and P.H. Richter, *The Beauty of Fractals,* Springer-Verlag, 1986