A meditation on Kepler's Aa

Craig S. Kaplan
David R. Cheriton School of Computer Science
University of Waterloo
csk@cgl.uwaterloo.ca

Abstract

Kepler's *Harmonice Mundi* includes a mysterious arrangement of polygons labeled Aa, in which many of the polygons have fivefold symmetry. In the twentieth century, solutions were proposed for how Aa might be continued in a natural way to tile the whole plane. I present a collection of variations on Aa, and show how it forms one step in a sequence of derivations starting from a simpler tiling. I present alternate arrangements of the tilings based on spirals and substitution systems. Finally, I show some Islamic star patterns that can be derived from Kepler-like tilings.

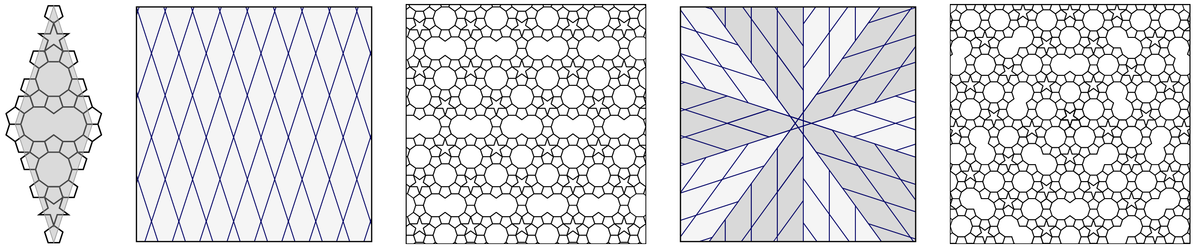
1. Introduction

Can a tiling of the plane be produced in which every tile has fivefold rotational symmetry? The question seems first to have been explored by Johannes Kepler in *Harmonice Mundi*. Although he did not resolve the question one way or the other, he did produce some remarkable tilings in the process of seeking an answer. They are reproduced in the frontispiece of Grünbaum and Shephard's *Tilings and Patterns* [3]. The general problem remains unsolved to this day [1, 2, 7], although solutions exist in which no bound is placed on the sizes of tiles.

Every finite drawing of a tiling requires the viewer to make a tacit assumption that the drawing could in theory be continued in an obvious way to cover the whole plane. In each case, that assumption may conceal a mathematical problem of lesser or greater difficulty. In many of Kepler's simpler drawings, the manner of continuation may indeed be considered obvious. But the same is not true for his arrangements that include shapes with fivefold symmetry.

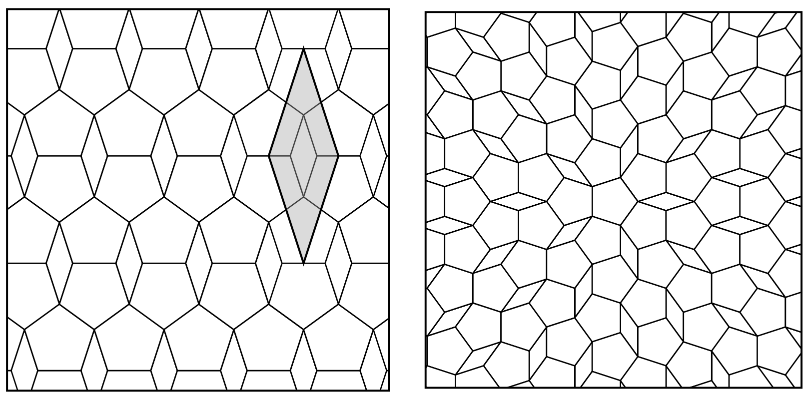
Of these arrangements, the drawing labeled Aa is perhaps the most intriguing. It contains more tiles than its brothers Z and Bb; it seems as if Kepler was more confident of the continuation of this tiling. Indeed, we now know that Aa can be extended to a tiling of the plane. It is far less obvious what Kepler had in mind for Z and Bb (though Grünbaum and Shephard mention a possible solution for Bb).

One possible method of tiling the plane as an extension of Kepler's Aa is attributed to Dessecker and described by Grünbaum and Shephard [3, Section 2.5]. Consider the 36° rhomb in Figure 1(a). The interior of the rhomb can be marked with whole and partial polygons. It turns out that when laid out edge-to-edge in a periodic pattern, copies of this rhomb generate a periodic tiling by pentagons, pentacles, decagons, and "fused decagon pairs" that Kepler called "monsters". Remarkably, this rhomb can also fit snugly with a rotated copy of itself staggered by one half of a rhomb edge (and not by the golden ratio, as Grünbaum and Shephard claim). Moreover, the rhombs can then be put into an arrangement with finite symmetry group d_5 in such a way that the markings on them form a non-periodic tiling of the plane. This new non-periodic tiling contains Kepler's Aa patch as a subset. The rhomb tiling and Kepler tiling are given in Figure 1(d) and (e).



(a) (b) (c) (d) (e)

Figure 1 A demonstration of the construction of Kepler's Aa tiling. The rhomb in (a) forms a periodic tiling in (b), and its markings form a corresponding periodic tiling in (c). Because this rhomb can also meet a copy of itself staggered by half an edge length, the arragement of rhombs in (d) also produces a valid tiling, as shown in (e).



(a) (b)

Figure 2 A simple tiling by pentagons and rhombs, with one possible translational unit shaded. In (b), the translational units are rearranged into a d_5-symmetric tiling in the spirit of Kepler's Aa.

In this paper, I use Kepler's Aa as a starting point for a process of exploration. First, I show how the Aa tiling is but one step in a larger progression of related tilings based on $36°$ rhombs. I then show how to create spiral arrangements of rhombs, giving attractive new tilings. I show how these rhombs can be used as a basis for developing substitution systems, producing additional related tilings without being tied to the underlying rhombs. Finally, I explore some of the decorative possibilities that arise when these tilings are used as templates for the construction of Islamic star patterns.

2. Other Kepler rhombs

Kepler's rhomb in Figure 1(a) can be derived in a fairly natural way from a simpler tiling. In fact, this derivation produces an infinite sequence of marked rhombs, each of which can then generate tilings of the plane in the same ways as Kepler's. In this section, I explain the derivation and demonstrate the first few members of the sequence.

We begin with a well-known periodic tiling by regular pentagons and $36°$ rhombs, as shown in Figure 2(a). One natural translational unit for this tiling is a larger rhomb centred on a rhombic tile. This translational unit also contains fragments of pentagonal tiles adding up to two whole pentagons. We refer to this marked rhomb as $k1$. Observe that copies of $k1$ can meet in the same ways as the rhomb used in Kepler's Aa, and indeed can produce a similar d_5-symmetric tiling, as shown in Figure 2(b).

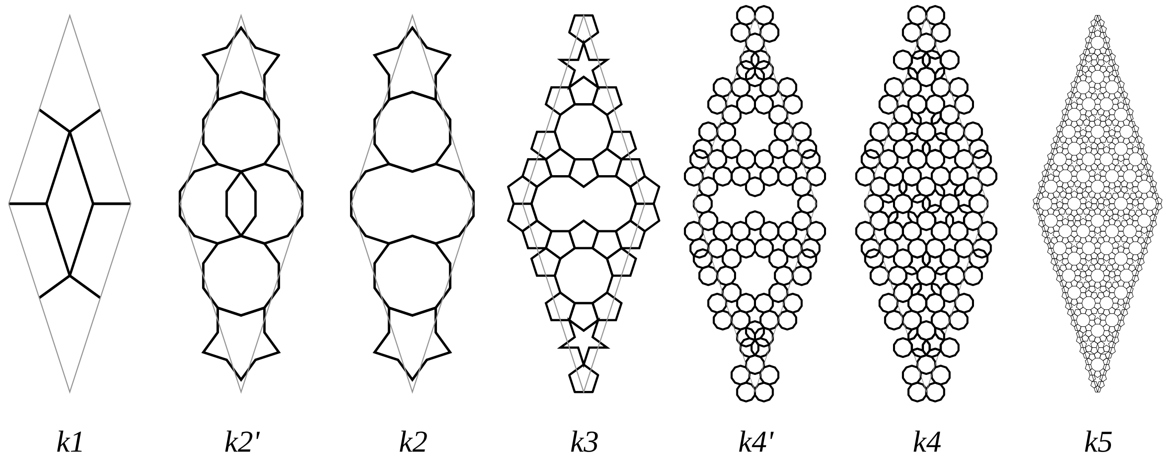


Figure 3 A sequence of marked rhombs, all of which lead to Kepler-like periodic and non-periodic tilings.

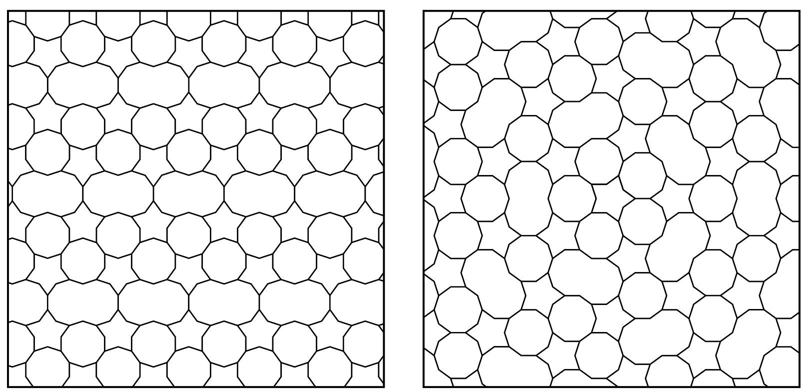


Figure 4 Periodic and non-periodic tilings by marked rhomb $k2$.

In every tiling generated by the markings on $k1$, the angles between adjacent edges at a vertex are multiples of $36°$. It is therefore possible to surround every vertex by a regular decagon in such a way that the vertex's edges are perpendicular bisectors of the decagon edges. Thus we place a decagon at every vertex, scaled so that they meet edge-to-edge when the underlying vertices are adjacent. This substitution yields the marked rhomb $k2'$ in Figure 3. This rhomb produces a tiling with decagons (some of which overlap) and five-pointed stars. We can produce a simpler design, $k2$, by deleting the regions where decagons overlap, resulting in a single large shape that is the union of two decagons. Note that the decagons overlap differently than in the arrangement of Figure 1(a). Inspired by Kepler's use of the name "monster", in previous work I classified Kepler's shape as a $(10, 2)$-monster and the new shape in $k2$ as a $(10, 3)$-monster [4, Section 3.10]. Periodic and d_5-symmetric tilings by $k2$ are shown in Figure 4.

In $k2$ and $k2'$, the angles at every vertex are multiples of 72 degrees. We can therefore perform a similar process as above, this time placing scaled regular pentagons around every vertex. Performing this substitution on $k2$ yields the rhomb $k3$, which is none other than the generator of Kepler's Aa.

We can continue this process indefinitely, alternately placing decagons or pentagons at vertices to produce finer subdivisions of a $36°$ rhomb. At every step, we get a new source of Kepler-like tilings. Figure 3 shows the first few steps of this substitution sequence, ending in $k5$.

Occasionally we will want to adjust the outcome of the substitution. In some cases I replace overlap-

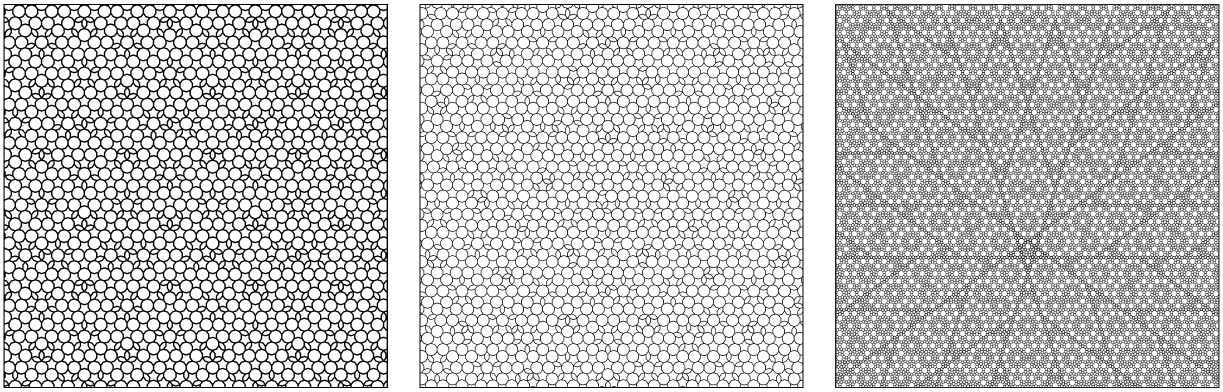


Figure 5 Periodic and non-periodic tilings by marked rhombs $k4$ and $k5$.

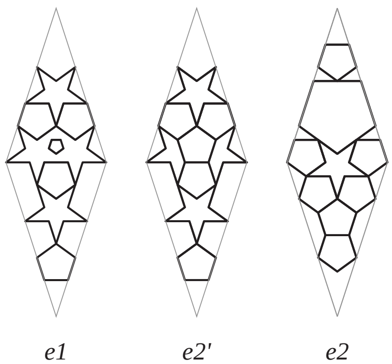


Figure 6 Other marked rhombs that yield interesting tilings. The rhomb labeled $e1$ is due to Eberhart. Rhomb $e2'$ is a simple modification suggested by Grünbaum and Shephard. On the right, $e2$ is the same as $e2'$ but with the enclosing rhomb shifted slightly relative to the underlying periodic tiling.

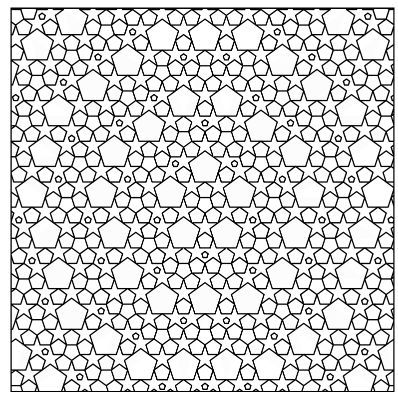


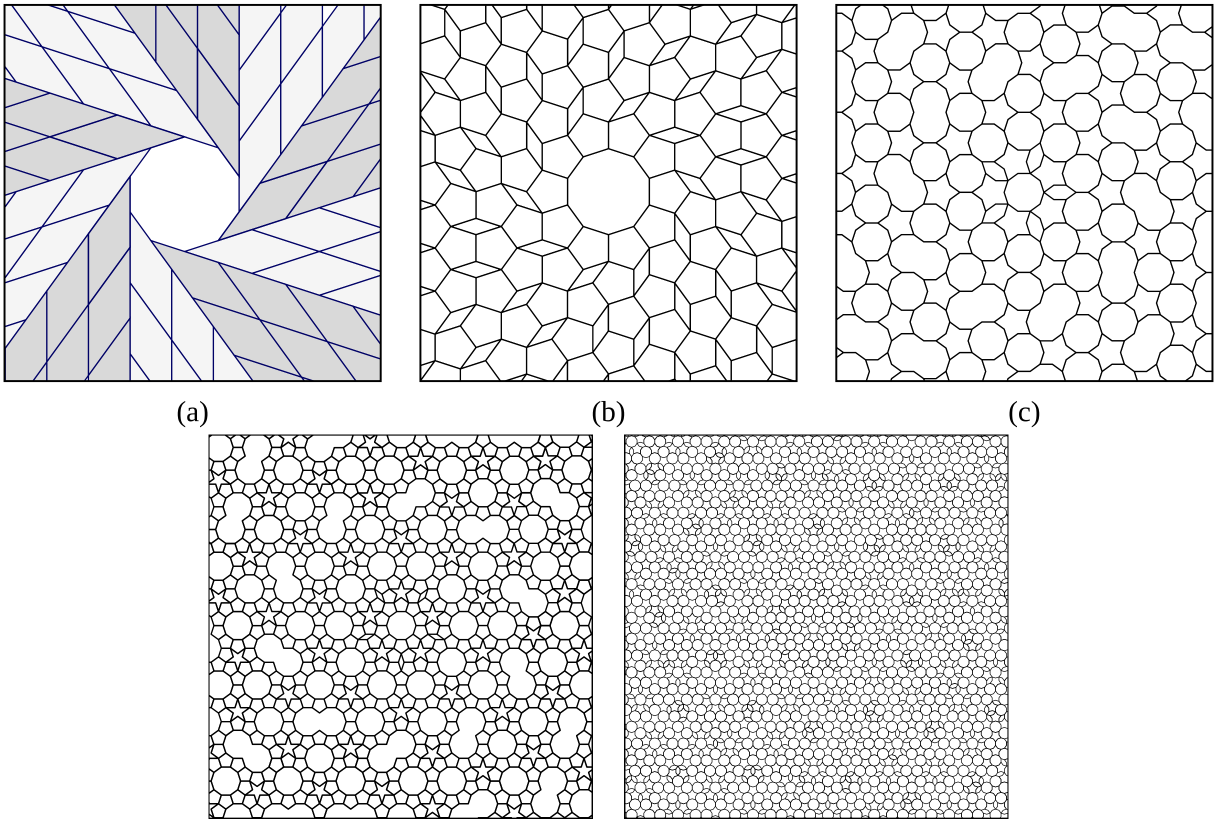
Figure 7 A d_5-symmetric tiling constructed using copies of $e1$.

ping decagons with an alternate arrangement of smaller tiles; in others I fill large empty regions with tiles. Both adjustments can be seen in the transition from $k4'$ to $k4$ in Figure 3. Tilings by $k4$ and $k5$ appear in Figure 5.

Grünbaum and Shephard also present an alternate rhomb, due to Stephen Eberhart. It is labeled $e1$ in Figure 6. This rhomb generates two sizes of regular pentagon, and uses a fused pair of pentacles. Because $e1$ is not centrally symmetric, it cannot be derived in an obvious way from $k1$. Unlike $k3$, Eberhart's rhomb really does fit with copies of itself staggered by the golden ratio. Note that a slight variation of $e1$ is also possible in which the fused pentacles are replaced by two half-pentacles and a pentagon. This attractive alternative is labeled $e2'$ in Figure 6. Figure 7 shows a d_5-symmetric tiling generated by copies of $e1$.

3. Spiral arrangements

In all of the cases above, rhombs can meet their neighbours either edge-to-edge or staggered by some amount. These possibilities lead naturally to periodic tilings or tilings with finite symmetry d_5. The staggered layout suggests that a third arragement of rhombs should also be possible, one where rhombs spiral outward from a centre. Figure 8(a) shows a spiral arrangement of $36°$ rhombs that can meet their neighbours staggered by half an edge length. The only additional question is how to fill the leftover space in the middle



(a) (b) (c)

(d) (e)

Figure 8 The tiling in (a) is a spiral arrangement of $36°$ rhombs that can meet their neighbours staggered by half an edge length. This arrangement leaves an unfilled regular decagon in the middle of the tiling. Tilings (b), (c), (d), (e) are examples of spiral arrangements using rhombs $k1$, $k2$, $k3$, and $k4$ respectively.

of the tiling. I have developed plausible arrangements of tiles through experimentation. In every case, I have tried to avoid introducing new tile shapes. Examples of spiral tilings based on $k1$, $k2$, $k3$, and $k4$ are given in Figure 8.

Figure 9(a) shows a spiral arrangement based on $e1$. Because $e1$ meets itself staggered by the golden ratio, the central decagon is not regular, but the region can still be filled by introducing half pentacles. The spiral tiling by $e2'$ is evidently similar in structure. Interestingly, the arrangement of tiles in $e2'$ can be translated relative to the surrounding rhomb to produce a new rhomb $e2$. This new rhomb generates the same periodic and d_5 tilings of the plane as $e2'$. But the resulting spiral tiling is different, as shown in Figure 9(b).

4. Related substitution tilings

If we superimpose any of the rhombs of Figure 3 on top of any other, a recursive structure suggests itself. For example, every tile from $k3$ can be seen as filled by smaller tiles from $k5$. Any of these superpositions leads to a substitution system that produces new tilings from old ones. These systems are reminiscent of Penrose's early experiments that ultimately led to his discovery of the aperiodic tile set that Grünbaum and Shephard refer to as $P1$ [6]. Two examples of substitution systems inspired by the Kepler rhombs are given in Figure 10. In Figure 11, a related tiling is coloured in.

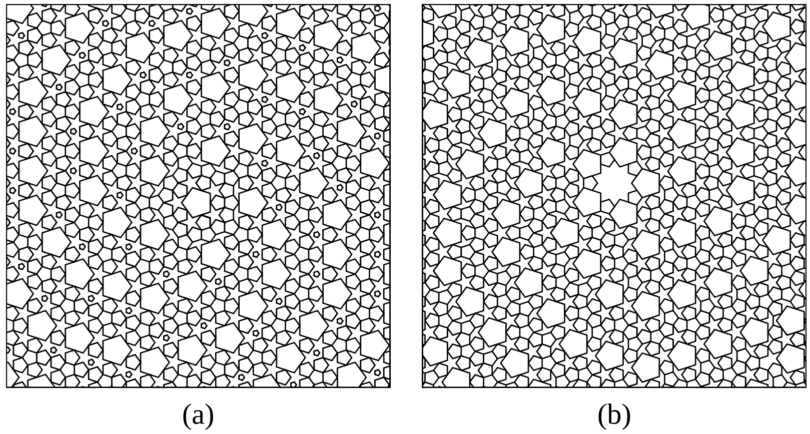


Figure 9 Examples of spiral tilings based on rhombs $e1$ and $e2$.

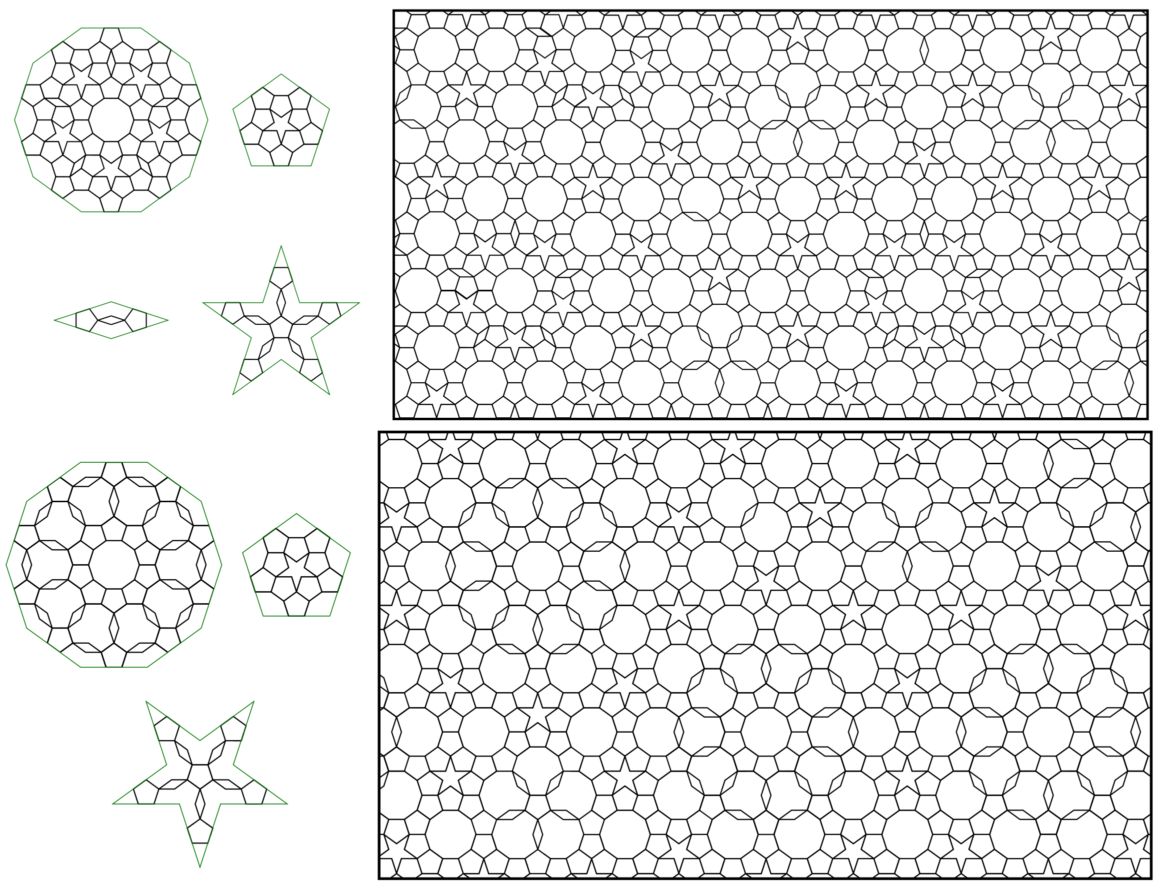


Figure 10 Examples of substitution tilings inspired by examination of the Kepler rhombs $k1$, $k3$, and $k5$. The top diagram shows a simple substitution system that permits decagons to overlap in 36° rhombs. In the bottom diagram, this overlapping is extended to the point where explicit rhombs are unnecessary, and the entire tiling can be regarded as composed of pentagons, pentacles, and overlapping decagons.

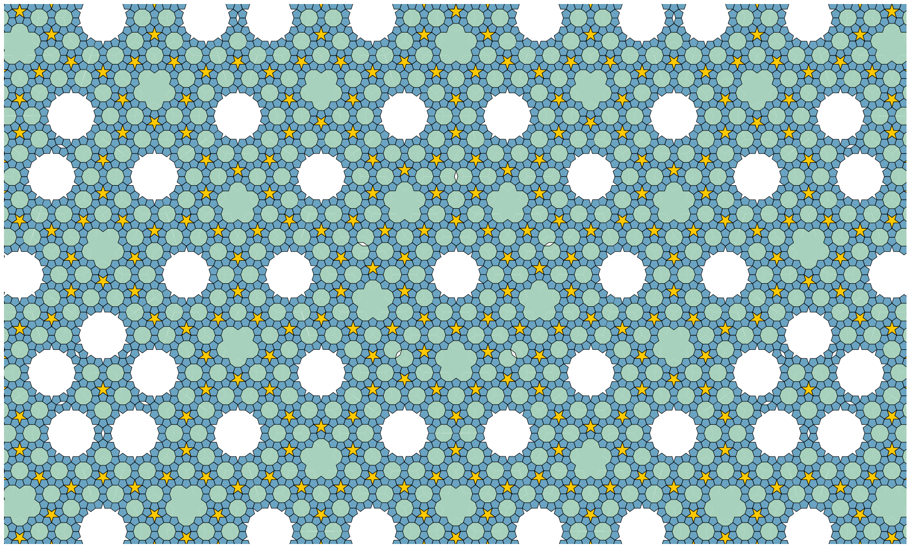


Figure 11 A colourful example of a substitution tiling. This example uses the substitutions defined in the top diagram of Figure 10, but includes a final step that replaces some repeated arrangements of smaller tiles with large symmetric ones.

5. Kepler star patterns

Because these tilings contain many regular polygons, it is natural to consider them as templates for producing Islamic star patterns via the polygons-in-contact method [5]. Figure 12 gives two examples of Kepler-based star patterns. The first is derived from the original Kepler Aa tiling. The second is based on the substitution system at the bottom of Figure 10. In both cases, small adjustments must be made so that attractive motifs can be found for pentacles. Novel motifs must also be supplied for overlapping decagons (or, equivalently, monsters). Details on both adjustments are given in previous work [4, Section 3.10].

The use of Kepler's tilings as a base for constructing Islamic star patterns is a satisfying mix of art and mathematics from across different centuries and cultures.

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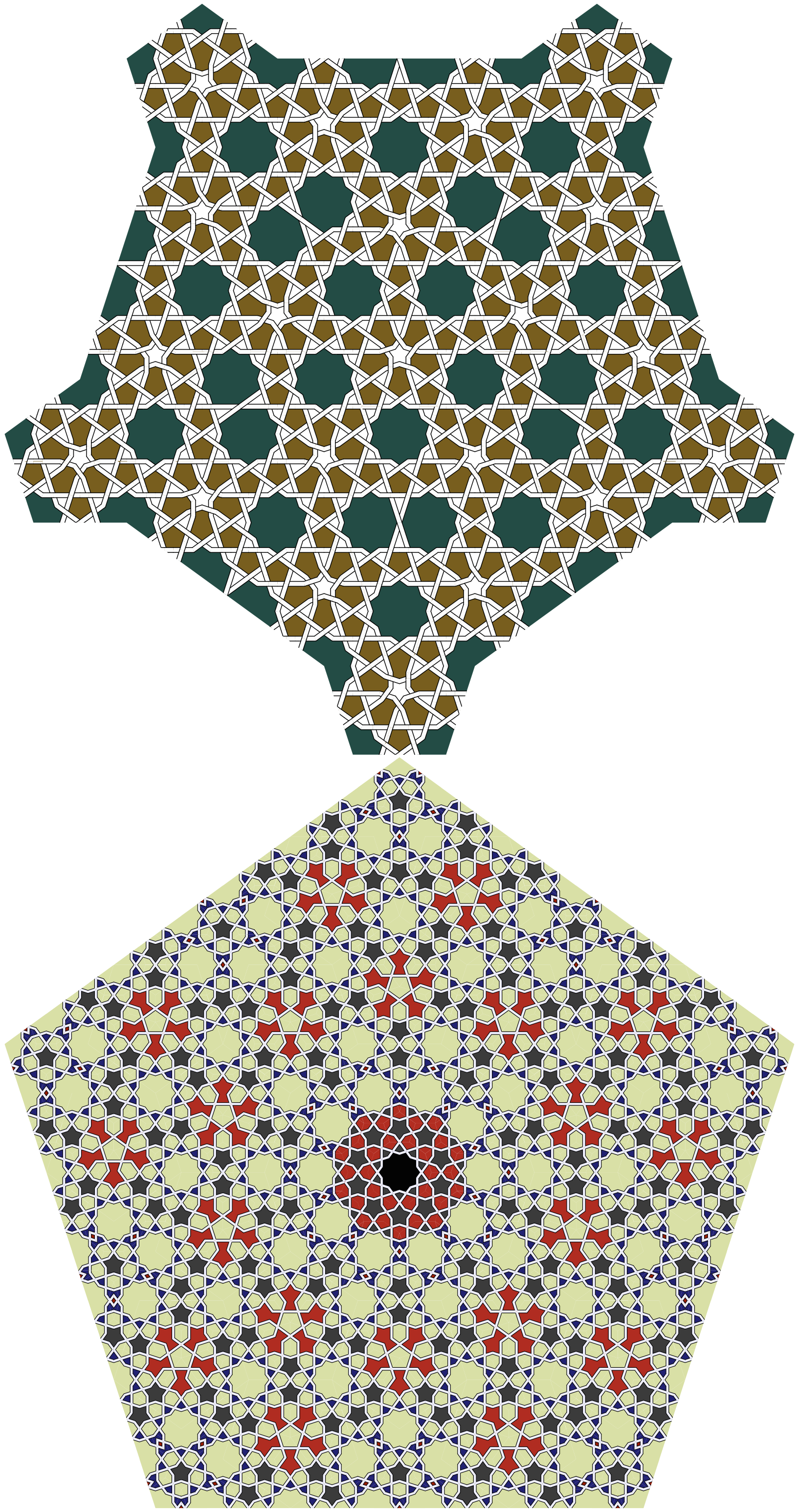

Figure 12 Rendered star patterns that use Kepler tilings as templates.