

J-F. Nicéron's *La Perspective Curieuse* Revisited

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Abstract

J-F Nicéron's well known work on the mathematics of anamorphism *La Perspective Curieuse* is a much quoted but perhaps less read classic. In particular the templates he provides for various transformations are commonly used as a starting point by those artists who occasionally practise the anamorphic art. Some of these templates are known to be approximations and some are exact. In the process of casting the mathematical descriptions of these templates into modern notation suitable for computation, a peculiar error has been found in Nicéron's analysis of transformations onto the surface of a cone or pyramid. The correct relationships are presented and possible reasons for the error are discussed.

1. Introduction

Anamorphic Art still engenders a response from the general public on the rare occasion when it is placed on exhibit. A few artists still practise it and produce anamorphoses usually of the most familiar type viz. that resolved in a cylindrical mirror. To make the anamorphic original it is common to use the templates provided by J-F. Nicéron in his book *La Perspective Curieuse* [1] first published in 1638. Every person interested in the field of anamorphism knows and consults this book [2], principally for the working templates. Practitioners probably do not read the text carefully and perhaps do not know where Nicéron's prescriptions are exact and where approximate. This difficulty is compounded by the fact that the book has not been translated in its entirety into English (although there is an Italian translation of the second and third volumes [3]) and Nicéron's 17th century prose style is not exactly easy.

Whether or not the templates are exact or approximations is probably moot to the practising artists who may use them only for a beginning and make subsequent alterations in the finished product to suit their taste and eye. However, in the computer era, when the manipulation of bitmaps is routine, knowing the correct analytical prescription for the various forms of anamorphic transformation is desirable [4]. If the exact analysis is intractable then the effect and magnitude of any approximations must be known.

For example, we know from his own words that Nicéron's template for the cylindrical mirror anamorph is a somewhat crude approximation, whereas his geometrical analysis of the tilted plane viewed from a finite distance is exact [4]. This raises the question then as to which other analyses in Nicéron's classic book are exact and which are approximations since he only rarely comments on this fact. In the course of producing an English translation of the work it has been discovered that most of the matter of the second volume is not only approximate but incorrect in a manner that is very peculiar considering that Nicéron was a skilled mathematician and, moreover, under the tutelage of an expert like Mersenne.

2. Anamorphic Transformation on to the Surface of a Right Circular Cone.

Having produced his version of the standard theory of perspective in the first volume, in the second volume he gives an exact prescription for the plane anamorph viewed from a finite distance. He then begins a lengthy consideration of the perspectively-distorted image on the surface of a right circular cone and right pyramids of various order. Virtually without any preamble and no justification he presents a template and scheme to construct an anamorphic image on the exterior surface of a cone and shows that it is related to the table of tangents. The image is to be resolved by observation along the axis from some previously chosen point above the apex.

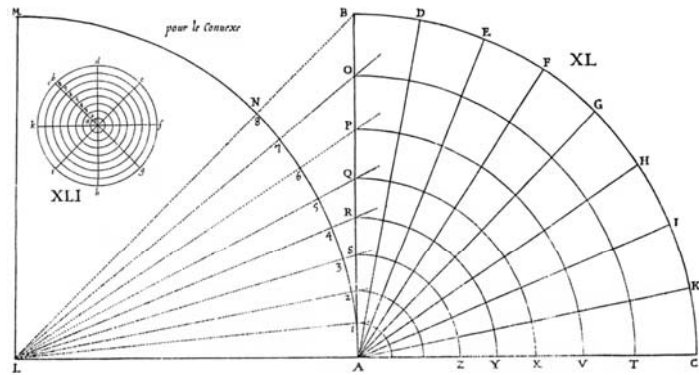


Figure 1: Nicéron's graphical prescription for constructing an anamorphic image on the exterior surface of a cone.

As Nicéron proves nothing in the book but presents everything as a series of assertions it perhaps can go unnoticed that the template is, in fact, incorrect. The proposed scheme is shown in Figure 1 adapted from Plate 14 of the work. The construction is effected in XL by dividing 45° of the left hand sector into N equal parts (here $N = 8$). The equiangular radii are extended onto a line AB which they cut in a manner that produces lengths proportional to the tangent of a series of angles; in this case $45/n$, where $N \geq n \geq 1$. The number N is the number of radial and azimuthal divisions into which the original image is divided in the cone's base (in this case therefore, 8 as shown in XLI). The right hand sector in XL is the actual 90° sector to which the anamorph will be transformed. It is divided into equiangular segments whose width increases radially to account for the perspective. The sector is then rolled into a cone (apex A and lines AC and AB congruent) and observed along the axis from a point which is as much elevated above the apex as the apex is above the base. Nicéron does not aim for generality as the method only applies to a right sector which, when rolled into a cone, has a total apex angle of 28° .

The correct transformation is shown in Figure 2 where the eye at $2h$ observes a point on the conical surface at m' projected at ρ in the base of radius R . Similar triangles show that the relationship between m' and ρ is

$$m' = \frac{\rho}{2R - \rho} \sqrt{R^2 + h^2} = \frac{\rho r}{2R - \rho} \quad (1)$$

or, if ρ is a fraction f of R then,

$$m' = \frac{f}{2 - f} r \quad (2)$$

Nicéron considers only the case of a 90° sector cone and observation at $2h$ but both of these can be easily included in a more generalized formulation (See Appendix). The kernel of this transformation $f/(2-f)$ is to be compared with Nicéron's kernel: $\tan(f \times 45^\circ)$. The anamorphic transformation of a simple square grid onto the surface of a cone obtained by applying Equations 1, 2 is shown in Figure 3.

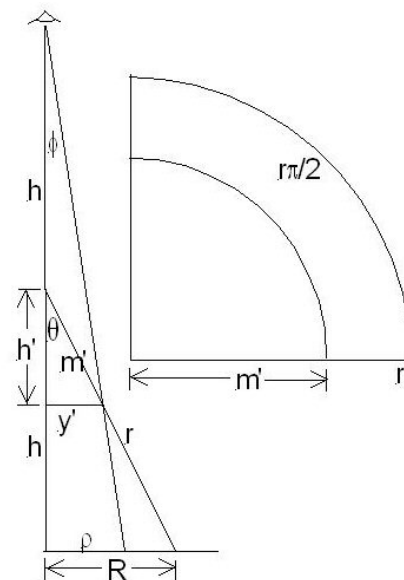


Figure 2: The geometry of the anamorphic transformation to the exterior surface of a cone.

The mystery is to understand how Niceron missed this simple transformation. It is all the more mysterious since the correct transformation was actually at hand in the immediately previous subject where he presented a well known and rigorous template for the transformation of a square tessellation as shown in Figure 4. He apparently failed to recognize that the transformation of the centre line of this figure is precisely the required transformation (Equation. (1) and (2)) if h is set equal to d , i.e. observation takes place at $2h$.

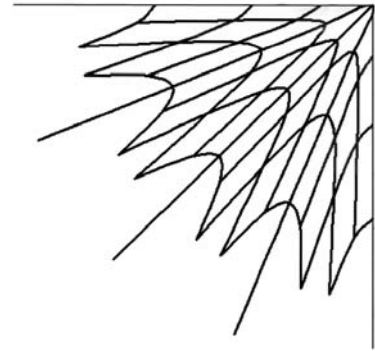


Figure 3: The anamorphic transformation of an 8x8 square grid. The reconstruction is shown in Figure 5.

The differences between the two formulations on the cone are subtle but observable, for a regular array for example, as shown in Figure 5. The grid on the left has been calculated using Niceron's kernel and that on the right by Equations. (1), (2). A paper cone has been prepared and the reconstruction photo-graphed under identical conditions from height $2h$. The irregularities produced by the imperfect closing of the seam are evident, but the figure on the right has straighter lines and tiles which are more accurately square. Similar tests performed with concentric circles yield identical results.

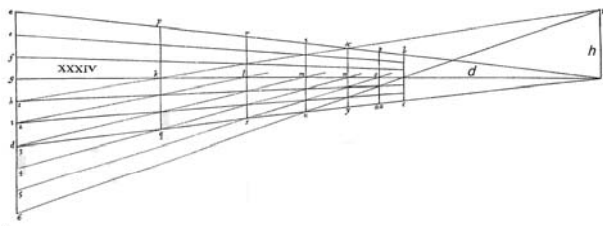


Figure 4: Niceron's exact prescription for the perspective of a square tessellation (The letters d and h have been added.)

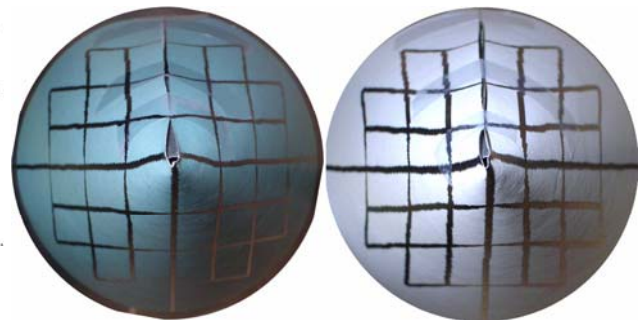


Figure 5: Critical comparison of a square grid transformed according to the kernel of Niceron (left) and Equations 1 and 2 (right).

The next topic in his book is the related one of the anamorphic image on the interior surface of the cone and, using a similar incorrect recipe, he comes to the correct conclusion that the same numerical results apply except in reverse order: widest segments nearest the apex and narrowest at the base.

3. Anamorphic Transformation on to the Surface of a Right Pyramid of any Order.

Niceron next turns his attention to the problem of a pyramid with a square base and also observed along the axis. Again he confines his prescriptions to those pyramids which unfold into a 90° sector and as before the more general equations are easy to derive; the recipe is given in Figure 6, (his XLIX and LI). The recipe is absolutely identical to the cone except for the segmentation of the pyramid base, where the resolved image appears, is even more complex and difficult to use in practise. For the pyramid surface, again it is divided into sectors of equal angle and divided radially using the tangent formulation given previously.

Now the question really arises as to why he did not see the error? Why did he not recognize that each face of this pyramid is exactly the case of the tessellation shown above in Figure 4? He seems to have been blind to everything except the apparent cleverness of his first guess using tangents, the error in which is unlikely to be revealed by transforming artistic images; it requires simple geometric shapes to do that.

For calculation purposes it is simplest to reduce the pyramid case to that of the cone. How this can be done is illustrated in Figure 7 for the case of a 4-sided pyramid. Again the reconstruction is to be viewed from $2h$ along the pyramidal axis and the reconstructed image is taken to be in the pyramid base. The pyramid is characterized by a base element b and an axial height h ; only pyramids that unfold to a right-angle sector are considered. Imagine a vertical, axial slice through the pyramid creating the vertical plane ACE. This is the same as a section through a cone of base radius R and height h . A point P in the base has coordinates x, y or ρ, α ; and as before $f = \rho / R$. Therefore the point P can be transformed from x, y (ρ, α) to x', y' (m', ϕ) using Equations (1), (2). As was done for the cone in Figure 5, the transforms are compared for a 4-sided pyramid in Figure 8. It is even more apparent that the tangent kernel is incorrect as it gives a pattern with converging lines and rectangular rather than square tiles.

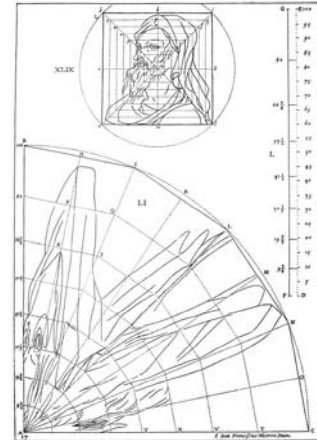


Figure 6: Niceron's template for putting an anamorphic figure on the exterior surface of a 4-sided pyramid. XLIX-the original drawing, XLI-the transformed drawing, XL-the table of tangents.

A topic which Niceron does not address is that of the continuity of lines across pyramid boundaries. In performing the anamorphic transform how does one insure that a line which crosses the boundary between two faces does not show a break or other discontinuity when observed? In the case of the 4-sided pyramid the condition is particularly trivial, which may be the reason he ignores it. From Figure 8 it is obvious that a vertical line in one face becomes a horizontal line when crossing to an adjacent face and vice versa. In Figure 9 is shown the reconstruction of a circle which intrudes on two faces adjacent to the central one.

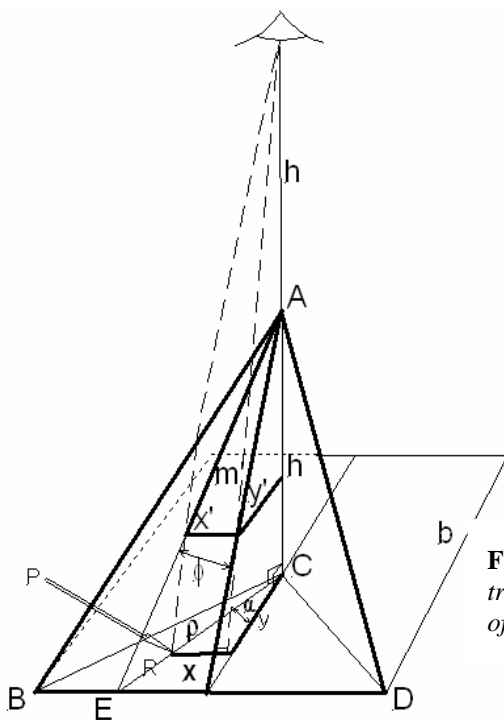


Figure 7: Geometry of the transformation on the exterior surface of a 4-sided pyramid.

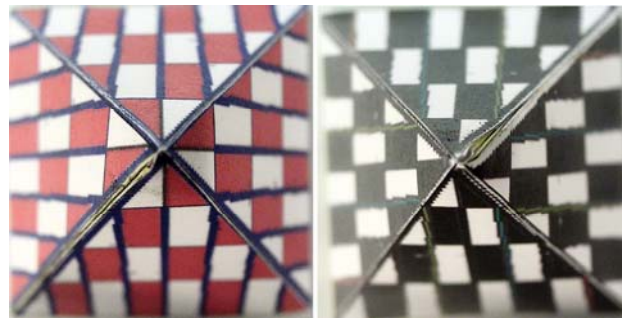


Figure 8: Critical comparison of a square grid transformed according to the kernel of Niceron (left) and Equations 1 and 2 (right)

Niceron mentions pyramids of other orders (3-sided, 5-sided etc.) but shows no examples. In these cases the continuity conditions are not so trivial. They can, however, be avoided (or rather, automatically applied) in every case by using the following recipe:

1. Divide the base into as many equal sectors as the base has sides.
2. Place the image to be transformed in the base and choose one of the sectors as the “calculation” sector.
3. When all the elements in the calculation sector have been transformed, rotate the entire image by one sector and transform the new elements as before.

This procedure guarantees the continuity of the image elements as the conditions are now automatically imposed by the proper rotation of the coordinates. In this way anamorphic pyramids of any order can be generated.

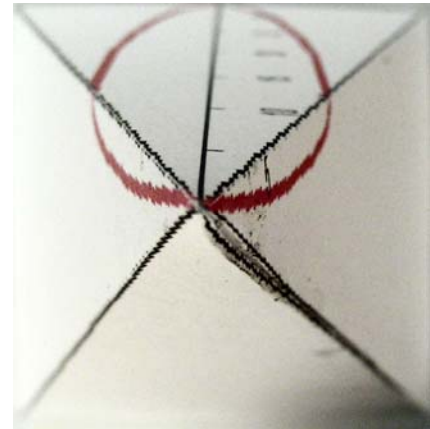


Figure 9: A circle residing on three faces of a 4-sided pyramid with the proper continuity.

4. Discussion of Niceron’s Prescription

In trying to understand Niceron’s construction of Figure 1 and why it is incorrect, it helps to see how it connects to the geometry of the cone or pyramid. This is attempted in Figure 10 where the capital letters indicate points common with Figure 1. The half of the real cone is shown in bold lines as AxC and the base radius R is one quarter the slant side r ; this is necessary to produce a right-angle sector ABC which rolls up to form the cone (apex A , centre x).

The eye point y is placed so that $yA = Ax$ (observation at $2h$). The construction of Figure 1 is obtained by drawing a triangle on AB as ABL with the angle at L and B equal to 45° . Looking at the sight-lines from y we have:

1. yA is directed to the apex ($\theta = 0$, $\tan 0 = 0$) and thus to the centre of the base as required.
2. If $\theta = 45^\circ$ then this determines the point B which tracks by means of the arc BC to the periphery of the cone’s base and thus determines the limiting sight-line yC .

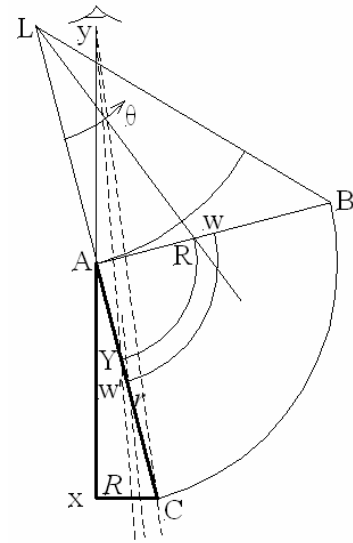


Figure 10: Analysis of Niceron’s tangent prescription for the cone. Dashed lines are sight-lines, capital letters correspond to Figure 1 and italics are dimensions.

An example of Niceron’s prescription is the following: Find where the sight-line to the point half-way between the centre and the periphery of the base, cuts the surface of the cone. Bisect the angle BLA and draw an arc from where the bisector cuts the line AB at R and track that to the cone’s surface at Y . Since $\tan 22.5^\circ = 0.414$, then the point Y is 41.4% distant from the apex along AC . Niceron infers then that the sight-line from y through Y will fall half way along the radius R , which is manifestly untrue.

He would have been aware of the fact that bisecting the line AB to produce the point w , which tracks to w' , produces a sight-line too near the periphery and that a mechanism had to be found to move that point back toward the centre, which the tangent calculation accomplishes but not sufficiently.

Whatever mechanism is used it must give a value of 1 at 45° and 0 at 0° which the tangent does (as does $f/(2 - f)$ of Equation. 2 for $f = 0$ and 1). It is difficult to think that Nicéron could not do the simple calculation of Equation 1 but perhaps he felt constrained to find the best approximation that could be carried out with ruler, compass and pencil alone. Indeed throughout the book he reminds us that he is speaking to the practitioners who, presumably, he did not expect to do calculations and make detailed measurements. However, nowhere in Volume 2 does he indicate that this is an approximation.

Appendix: The Generalized Equations for the Conical Transformation.

Nicéron deals only with the special case where observation is from a point at h above the apex but it can be generalized. If the observation point is at $h + qh$, where q is a dimensionless number then the transform equation (analogous to Equation 2) is,

$$m' = \frac{qf}{1 - f + q} r. \quad (3)$$

Nicéron also considers only a cone rolled from a right sector, i.e. in Figure 10 angle BAC ($= \theta$) = $\pi/2$ radians. To further generalize, the cone can be rolled from a sector of any angle, although, clearly very flat anamorphs produce unspectacular results. If a fraction ε of the circumference $2\pi r$ is used then,

$$\theta = \sin^{-1} \varepsilon, \quad (4)$$

and

$$h = \left(\frac{R}{\varepsilon} \right) \sqrt{1 - \varepsilon^2}. \quad (5)$$

Using Equations (3) to (5) the anamorph of any image can be constructed.

References

1. Jean-Francois Nicéron, *La Perspective Curieuse*, Chez Pierre Billaine, Paris, (1638)
2. Jurgis Baltrusaitis, (Translated by W.J. Strachan), *Anamorphic Art*, Abrams, New York (1977)
3. www.artetoma.it/anamorfofi/niceron.PDF
4. Hunt, J.L., Nickel, B.G., Gigault, Christian. *Anamorphic Images*, Am. J. Phys. **68**, pp 232- 237 (2000).