Green Quaternions, Tenacious Symmetry, and Octahedral Zome

David A. Richter
Department of Mathematics, MS 5248
Western Michigan University
Kalamazoo, MI 49008, USA
E-mail: david.richter@wmich.edu
Website: http://homepages.wmich.edu/~drichter/

Scott Vorthmann
1317 Lacresta Dr
Freeport, IL 61032, USA
E-mail: scott@vorthmann.org
Website: http://www.vorthmann.org/zome/

Abstract

We describe a new Zome-like system that exhibits octahedral rather than icosahedral symmetry, and illustrate its application to 3-dimensional projections of 4-dimensional regular polychora. Furthermore, we explain the existence of that system, as well as an infinite family of related systems, in terms of Hamilton's quaternions and the binary icosahedral group. Finally, we describe a remarkably tenacious aspect of H_4 symmetry that "survives" projection down to three dimensions, reappearing only in 2-dimensional projections.

1. Introduction

This is a report on a journey of discovery we have shared over the past year, as we endeavored to gain deeper insights into the mathematics surrounding the Zome System of Zometool, Inc. This collaboration has borne some intriguing fruit, specifically the results we describe in this paper concerning generalizations of Zome, and a surprising aspect of symmetry and projection.

Our interest in Zome centers on the way that it hints at deep underlying mathematics: serendipity becomes commonplace, and startling coincindences come to be expected. The "octahedral Zome" and "green quaternions" of our title were quite surprising initially, but turn out to have a very straightforward explanation, detailed below. Nonetheless, the explanation has opened up new vistas, by giving us the ability to characterize some generalizations of Zome. The mechanism of "tenacious symmetry" is not so easy to explain, so we must content ourselves with merely describing its characteristics and some examples.

Our collaboration was also fruitful as a synergy between pure mathematics and the applied science of visualization and modelling software. Scott's Zome modelling software, "vZome", gained a number of rich capabilities as he learned more mathematics from David, and David gained a deeper insight, and even learned some geometry, by working with and talking about vZome. A highly productive feedback loop thus developed. Since this is as much a story of a collaboration as it is about beautiful mathematics, we have decided to tell it in roughly chronological order.

Along the way, we have seen many novel views of some classical polytopes, and here we present a few of these as well. It is important to stress that, with all of our static art being 3-dimensional, it is notoriously difficult to visualize the complicated and beautiful objects which exist in higher dimensions. Zome and vZome have allowed us to see some fantastic properties of these objects.

2. Octahedral Serendipity

The journey began with David's mention of the fact that one may inscribe five copies of the 600-cell in the vertices of the 120-cell, [1]. One may see this in the usual three-dimensional Zome projection of the 120-cell: Starting with any one of the projected dodecahedra, one selects any one of the ten regular tetrahedra inscribed in its vertices. The faces of this tetrahedron are shared by four more tetrahedra, and one can proceed in this way to construct a projection of all the tetrahedra in the 600-cell. Since the original 120-cell has a dodecahedral cell at the center, rather than a face, edge, or vertex, it is a "cell-first" projection, and hence the new 600-cell projection is necessarily also a cell-first projection. This projection of the 600-cell had not been seen by either of us, but is certainly not unknown, [1, 7]. The figure below shows a "4D-cutaway" view; it is halved in four dimensions to omit the central involution, then almost-halved again in three dimensions. The left-hand pair is a parallel binocular stereo view, and the right-hand pair is a cross-eyed binocular stereo view.

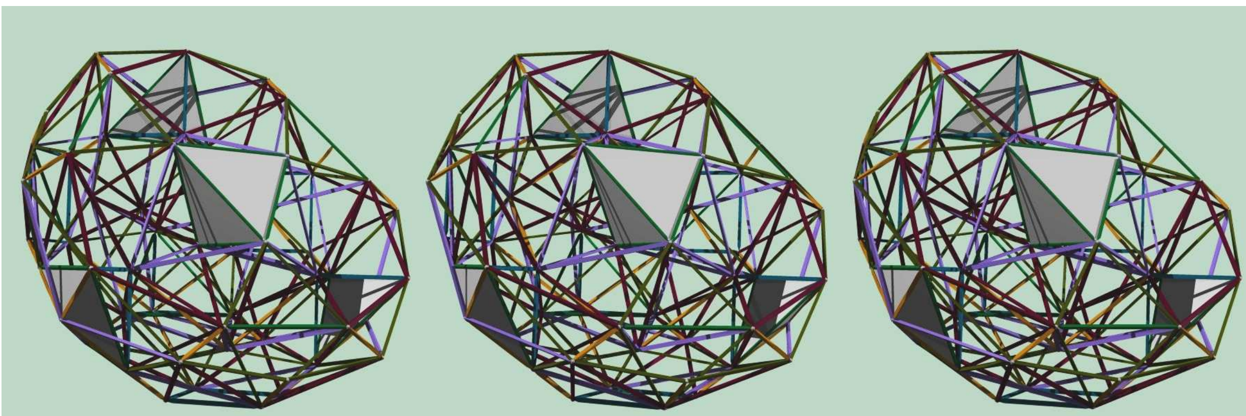

Figure 1. *Octahedral Projections of the 600-Cell, Quartered.*

It turns out that most of the edges of these tetrahedra do not correspond to any existing Zome struts. The vZome program, however, can construct a virtual strut between any two connector balls. When constructing such an "unknown" strut, vZome assigns a color in the same pattern as real Zome: All directions ("zones") that are equivalent under icosahedral symmetry are assigned the same color. This has the effect of highlighting the symmetry (or asymmetry) of models. After performing the construction in vZome, when we finally cleared away all the 120-cell "scaffolding" to see the 600-cell, we realized that the model used some of the blue, yellow, and green zones from the original Zome System, and three new zones. We have since christened these new zones with the colors maroon, olive, and lavender.

Although our original construction appeared to have tetrahedral symmetry overall, the full projection actually has "pyritohedral" symmetry, with two tetrahedra combined as in Kepler's stella octangula at the center. The pyritohedral symmetry group is the symmetry group of an idealized pyrite crystal. Geometrically, this is the direct product of the group of 12 orientation-preserving symmetries of the tetrahedron with the 2-element group generated by the "central involution" or "antipode" map

$$(x, y, z) \mapsto (-x, -y, -z).$$

This group arises in the context of Zome when one attempts to build a model with octahedral symmetry; it is actually impossible to build a Zome model with ideal octahedral symmetry (when one includes the symmetry of the Zome parts themselves rather than treating them as idealized points and line segments). The symmetry group of every Zome model is necessarily a subgroup of the full group of symmetries of the icosahedron, but no subgroup of the icosahedral group is isomorphic to the octahedral group. The largest subgroup of the icosahedral group which is also a subgroup of the octahedral group is the pyritohedral group.

Examination of the cell-first 600-cell model reveals that it actually has full octahedral symmetry in every sense except in the precise geometry of the real Zome balls and struts, for the reason explained above. Observing this and the fact that only three new directions were necessary for the whole model, we realized that we had stumbled on a novel Zome-like system with octahedral symmetry rather than icosahedral symmetry. This system shares all the other characteristics with the original Zome system: the ability to scale by powers of the golden ratio τ, strut coloring by symmetry group equivalence, and a surprising variety of constructible triangles and tetrahedra from a small set of zones. Although we have not yet proposed a formal mathematical definition, we consider this new Zome-like system as an example of a "zoning system". Such "zoning systems", generally, should possess properties like those outlined above, and indeed we will show that there are many systems that fit within such a framework.

With our discovery of the octahedral system, Scott immediately set to work to make vZome support it, both in the rendering of the virtual parts, and in the available symmetry operations. Scott soon realized that he could construct a 120-cell projection with the same symmetry by constructing a vertex at the center of each tetrahedron in the 600-cell projection, then connecting each vertex to its four nearest neighbors. After adding a centroid command to vZome, we quickly produced a vertex-first, octahedrally symmetric projection of the 120-cell. Although a black-and-white figure cannot readily convey it, this projection is entirely constructible within our new octahedral Zome-like system. As above, this figure shows a 4D-cutaway view in two stereo pairs.

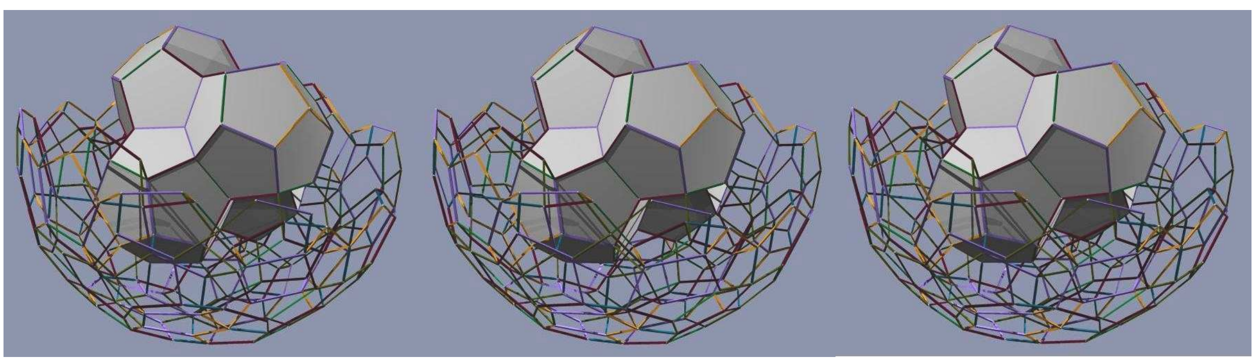

Figure 2. *The Octahedral Projection of the 120-Cell.*

3. What Color Is Your Quaternion?

Having discovered one new zoning system and being interested in finding more, we set out to explore how and why the new system worked. Although we constructed the octahedral 600-cell projection in three dimensions, David pointed out that as a faithful projection of the 4-dimensional object, it could equally well be produced by rotating the 600-cell in 4-dimensional space before projecting to three dimensions, and that such rotation could be accomplished by quaternion multiplication. Hamilton's quaternions $\mathbb{H}$ are intimately related to 4-dimensional geometry [3]. Furthermore, the original blue-yellow-red Zome System of Zometool is intimately related to the binary icosahedral group I and its embedding in the group S^3 of unit quaternions.

There are infinitely many different ways to embed the group $I \hookrightarrow S^3$, but one in particular leads to a set of easily expressed vectors that correspond to the Zome System. First, denote $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\sigma = \frac{1}{2}(1-\sqrt{5})$, so that τ is the golden ratio and σ is its conjugate. One can list all 120 vectors quickly after noticing that they only take on three different "shapes". These shapes are representatives of orbits in the 192-element group which contains arbitary sign changes and all even permutations on the four coordinates. The shapes of vectors in this embedding of I are

$$(1,0,0,0),\ \frac{1}{2}(1,1,1,1),\ \text{and}\ \frac{1}{2}(0,\sigma,1,\tau),$$

of which there are respectively 8, 16, and 96 vectors. (We often use the two notations $w+ix+jy+kz$ and (w,x,y,z) for quaternions interchangebly. As a rule, we use (w,x,y,z) when we want to denote points or vectors and $w+ix+jy+kz$ when we want to denote a multiplication operator.) Notice that, although here we have presented three different shapes of vectors, this set of 120 vectors constitutes a single orbit under the H_4 symmetry group, of which this 192-element group is a subgroup.

One obtains the Zome System by considering $\pi(2I)$, where π is the projection from 4-dimensional to 3-dimensional space that maps according according to the formula

$$\pi : (w,x,y,z) \mapsto (x,y,z).$$

(We use $2I$ to indicate that we pre-multiply each element of I to avoid dealing with $\frac{1}{2}$ factors.) One assigns a color to each vector in $\pi(2I)$ according to its shape. Blue vectors have the shape $(2,0,0)$ or $(\sigma,1,\tau)$, yellow vectors have the shape $(1,1,1)$ or $(\tau,0,\sigma)$, and red vectors have the shape $(1,\sigma,0)$ or $(0,\tau,1)$. Here "shape" refers to orbits on 3-space under the pyritohedral group, as described earlier. Naturally, Zome also allows that every multiple of these vectors by a power of the golden ratio τ be a standard strut. Indeed, one should notice, for example, that $\tau(1,\sigma,0) = (\tau,-1,0)$, and this is the longer of the two types of red struts described above. Every blue strut has length $\tau^n \cdot 2$, every yellow strut has length $\tau^n \cdot \sqrt{3}$, and every red strut has length $\tau^n \cdot \sqrt{1+\sigma^2}$.

Recall that these are all projections of a single orbit of vectors equivalent under H_4 symmetry. If one imagines a four-dimensional analogue of Zome, the actual 600-cell and 120-cell could be entirely constructed using "blue hyper-struts" based on the vectors in I. In fact, all 15 convex uniform polychora with H_4 symmetry could be constructed with them, and therefore 3-dimensional π-projections of them can be constructed with the Zome system. The current existence of the Zome system itself is directly related to Marc Pelletier's recognition of this fact (when he was just 17).

If the set I can be considered the "blue hyper-struts" in four dimensions, the set $(1+i)I$ can be considered the "green hyper-struts". Note that the two sets are related to each other by a quaternion multiplication that composes a dilation by $\sqrt{2}$ with a 4-dimensional "rotation", and an object with H_4 symmetry can be constructed with either set alone. However, if the two constructions are both projected to a hyperplane perpendicular to a "blue" vector, the resulting three-dimensional projections are very different, as one might expect. Remarkably though, that difference is manifested in the fact that one projection has icosahedral symmetry and the other has octahedral symmetry.

We obtain our 6-color octahedral zoning system by considering set $\pi(2(1+i)I)$. As is the case with the elements of I, one notices that the vectors in the set $2(1+i)I$ also take on a small number of shapes. However, for reasons which one will find in [2], the group that we need is slightly restricted. Instead of allowing arbitrary sign changes, we only allow an even number of sign changes of the 4 coordinates. Since we still allow even permutations of the coordinates, we have a group of order 96 that preserves the shape of the vectors. Explicitly, there are four shapes

$$(-\tau^2, \sigma, \sigma, \sigma),\ (\sigma - \tau, 1, 1, 1),\ (\sigma^2, \tau, \tau, \tau),\ \text{and } (2, 2, 0, 0),$$

with corresponding orbit sizes 32, 32, 32, and 24. Projecting down to 3 dimensions, we notice that the number of signs is no longer relevant. For example, notice that (w, x, y, z) and $(-w, -x, y, z)$ agree except for an even number of sign changes, whereas their images $\pi(w, x, y, z) = (x, y, z)$ and $\pi(-w, -x, y, z) = (-x, y, z)$ agree except for an odd number of sign changes. Indeed, again we use the pyritohedral group to obtain the entire orbit of directions, and one may quickly tabulate the 6 shapes of vectors.

Shape	Number of Zones	Color	Shape	Number of Zones	Color
$(0, 0, 2)$	3	blue	$(1, 1, \tau - \sigma)$	12	maroon
$(2, 2, 0)$	6	green	(σ, σ, τ^2)	12	lavender
$(1, 1, 1)$	4	yellow	(τ, τ, σ^2)	12	olive

Table 1. *The Octahedral Zoning System.*

Note that for any vector in this zoning system, at least two of the coordinates are equal in magnitude. This is equivalent to saying that all these struts lie in a plane orthogonal to some green strut. Also notice that a subset of the original blue, yellow, and green zones appear naturally in this system. However, if we want this new zoning system to accommodate the 600-cell, we must include the new maroon, lavender, and olive zones as well.

Armed with some of this knowledge, Scott was able to quickly implement quaternion multiplication in vZome when importing a four-dimensional data set or when generating one of the H_4 polychora. In vZome, one specifies a quaternion by selecting an existing strut. Although the strut is specified by a vector (x, y, z) with only 3 coordinates, the above analysis shows that the fourth coordinate w is determined up to a sign by the zone in which it lies. Selecting any green strut results in a quaternion having the same shape as $1 + i = (1, 1, 0, 0)$, and all such quaternions have effect equivalent to mapping $2I \to 2(1 + i)I$. In short, achieving the octahedral projections of the 120-cell or 600-cell is now as easy as three clicks, a vast improvement over our original manual derivation.

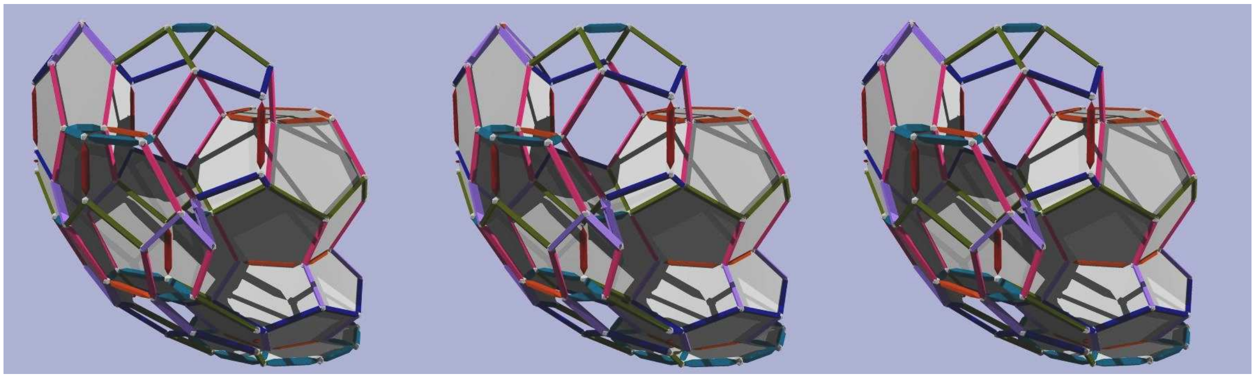

Figure 3. *Pentagon-First Projection of the 120-Cell.*

Availability of quaternion multiplication in vZome immediately begged the question: What is the effect of using the various "colors" of quaternions? First, since blue struts are all in the orbit of the quaternion $(2, 0, 0, 0)$, which is an axis of symmetry in H_4, using a blue quaternion has no effect modulo a dilation. For other colors, the answer proved serendipitous in the way we have come to expect of Zome. Applying a "red" quaternion to an object with H_4 symmetry yields a 3-dimensional object with the symmetry of a pentagonal antiprism; the projection is symmetric around the red strut used as the quaternion. This results in a face-first projection of the 120-cell with two overlapping pentagons in the center, and an edge-first projection of the 600-cell. The figure above is a cutaway that shows one example of each of the 9 different dodecahedral cell shapes in the former model, with some faces present for clarity.

Naturally, applying a "yellow" quaternion produces projections symmetric around that yellow strut. This yields a face-first 600-cell, centered on overlapping triangles, and an edge-first 120-cell. Both red and yellow quaternion multiplications yield new zoning systems analogous to the octahedral system we have described, but with different symmetries. In both cases, the vectors of I are mapped to a small number of 3-dimensional shapes, although more shapes (thus colors) are required than for the octahedral system. Indeed, any quaternion multiplication of I yields a zoning system capable of rendering orthogonal projections of H_4 polychora.

4. Tenacious Symmetry

The group H_4, with 14,400 elements, is moderately large and exotic, compared to the sizes and variety of the finite groups which act in 4-dimensional space. Having so much symmetry, some unusual traces of this symmetry remain when these objects are projected down to three and two dimensions. We refer to this as "tenacious symmetry": Much symmetry is necessarily lost in a projection to lower dimension, but it somehow refuses to be totally eradicated. Moreover, due to the observation in [8], we see that the original Zome System is directly related to the famous E_8 "Gosset" lattice, whose point symmetries comprise a group with nearly 700 million elements. Certainly we will see a wide variety of highly-symmetrical objects by considering different views of this amazing object.

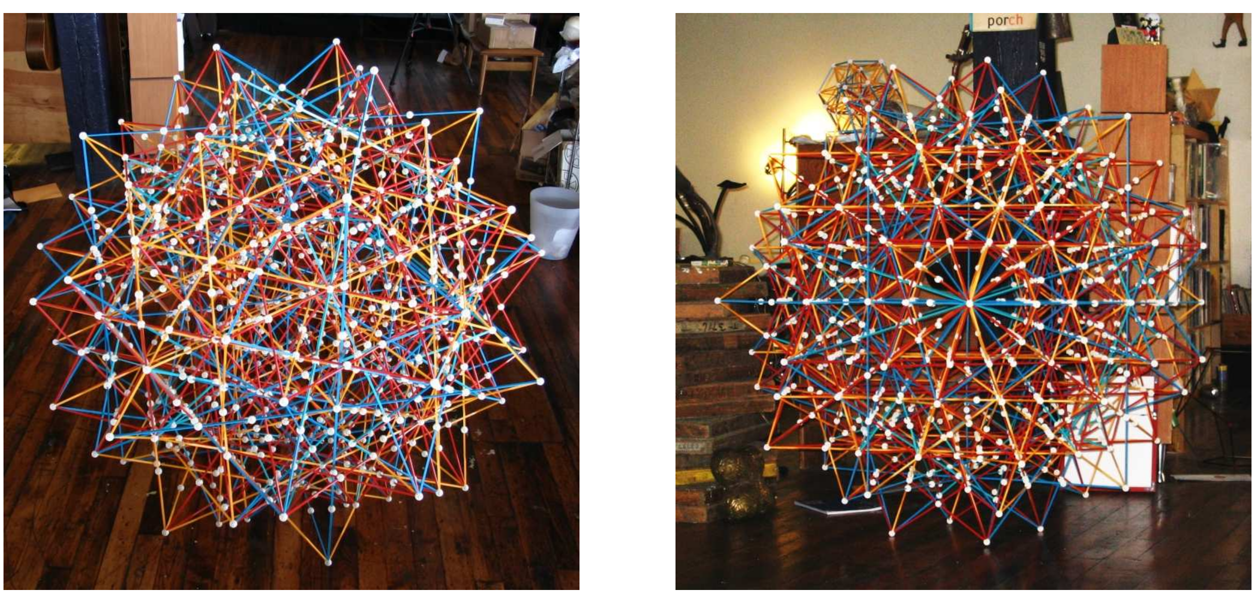

Figure 4. *Two Views of the Compound of Fifteen 16-Cells.*

An easily-accessed example of this phenomenon is provided by the Van Oss projection of the 600-cell, which appears as the frontispiece in [1]. The symmetry of this figure is the dihedral group with 60-elements, i.e., the symmetry group of a regular 30-sided polygon. This projection reveals the fact that this dihedral group, while it may not be isomorphic to any subgroup of H_4, is still closely related; indeed, H_4 contains elements of order 30.

Anyone who has worked on a Zome model of an H_4 polychoron is familiar with the obvious symmetries visible when looking through such a model. Few are surprised, moreover, that an object with icosahedral symmetry can exhibit 10-fold rotational symmetry in a parallel 2-dimensional projection. When David undertook to build a Zome model of the regular compound of fifteen 16-cells, using vZome as an aid for visualization, he began to see more surprising symmetries. Scott and other Zome enthusiasts experienced this firsthand when David led construction of the latter model, pictured above, at a meeting in Chicago in September 2005. This model has pyritohedral symmetry, and, in particular, its symmetry group has no elements of order 5. Nevertheless, as the photo on the right shows, the projection of the model onto a plane perpendicular to a red zone has pentagonal symmetry. This is a remnant of the symmetry group of the corresponding 4-dimensional polytope, which does have elements of order 5.

The vZome program is ideally suited to exploring this phenomenon. Consider the red-quaternion projection of the 120-cell; that projection has the same symmetry as a pentagonal antiprism. Although that symmetry group has no elements of order three, there are ten distinct orthographic projections of this object to two dimensions that exhibit six-fold rotational symmetry. Five of those projections produce a figure as shown below in the first two images, first showing the strut colors and second in a simple wireframe. The other five produce a figure as shown in the third image.

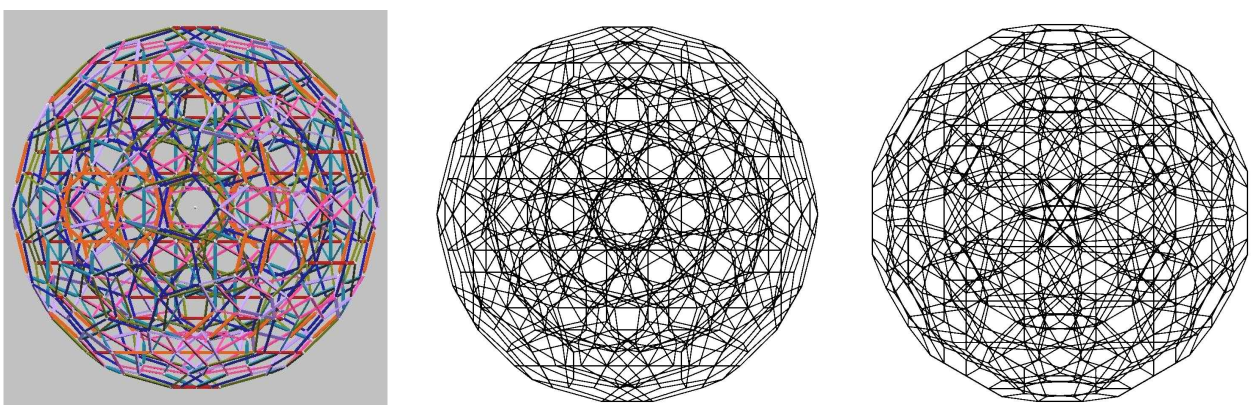

Figure 5. *Red-Quaternion Projections of the 120-Cell.*

Tenacious symmetry is in fact displayed in all Zome-axis projections of any H_4 polychoron, and similarly if a yellow quaternion is applied rather than a red one. In other words, although most of the rich symmetry of H_4 is inevitably lost in these 3-dimensional projections, that symmetry is too “tenacious” to be completely eradicated – traces of it remain when one projects again down to two dimensions along particular axes.

5. Conclusion

To reiterate, there are in fact an infinity of Zome-like systems based on the binary icosahedral group, one for each unit quaternion. However, they get progressively less interesting as the elements of I are mapped to more generic elements with respect to the group H_4, and less and less symmetry is preserved in the three-dimensional projection. With icosahedral symmetry, the original blue-yellow-red Zome System is clearly the most symmetric of this infinite set. As in the case of the octahedral system based on green-quaternion multiplication, the red and yellow quaternions similarly generate Zome-like systems with a small set of vector shapes. All of these systems share the useful property of arbitrary scalability by powers of τ, without requiring additional, longer strut lengths. These scalability and "small inventory" properties make all such systems potentially interesting to artists and engineers alike.

We are proposing to generalize and study a wide class of such "zoning systems", such as those that have been discussed here. Here is a review some of the critical properties shared by all of the zoning systems we have seen. First, all of these systems are based on connector balls and struts. Second, there is a group G which acts on the ambient space and for which the symmetry group of every model is necessarily a subgroup of G; the symmetry of the connector ball is G. Third, the original 4-dimensional orbit of 120 elements in I maps to a small set of orbits under G, with the size of that set dependent on the quaternion q used to map I to $\pi((2q)I)$.

As has been observed in [8], another characteristic of the Zome System is that it is closely related to a genuine lattice in 8 dimensions. One may describe this lattice quickly as a subset of $\mathbb{R}^8$, but is is also embedded naturally in $\mathbb{R}^4$ in such a way that the lattice points may be positioned arbitrarily close to each other. More precisely, the lattice points comprise a dense subset of $\mathbb{R}^4$ under the usual norm. This property leads to a fourth characteristic of all these zoning systems: The idealized locations of the connector balls comprise a dense subset of the ambient space. From the perspective of one who wishes to create an interesting model, we regard this property as critical; in a theoretical sense, it provides the user with the liberty to place objects in virtually any location s/he desires.

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