Models of cubic surfaces in polyester

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Abstract

Historically, there are many examples of model building of mathematical surfaces. In particular, models of a very special cubic surface called the Clebsch diagonal have been built in plaster and clay since the 19th century. The sculptor Cayetano Ramírez has succeeded in building this surface using polyester. With this material, the resulting sculpture shows all the mathematical properties of the surface. We first give a short mathematical introduction and an overview of the models that have been built in the past to respresent it. Next, we proceed to describe the work of Cayetano, explaining the techniques used by him in the whole procedure.

1. Introduction. Real Cubic Surfaces

In the second half of the nineteenth century, the interest of mathematicians in algebraic geometry grew enormously. This interest began after the fascinating discovery of Salmon and Cayley in 1849:

Any smooth cubic surface contains precisely 27 lines.

By a smooth cubic surface it is meant the set of roots of a polynomial of degree 3 in projective space and containing no singularities. The cubic surface given by the equations

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$
$$x_0 + x_1 + x_2 + x_3 + x_4 = 0$$

is known as the *Clebsch diagonal surface*. It was defined by Clebsch in 1871 in [2], and it is one of the most famous cubic surfaces. The reason is its special property that all the complex 27 lines are real. It is also the only cubic surface with 10 Eckard points, i.e., points on the surface where three of the 27 lines intersect.

In 1863 (see [12]), Schläfli had classified the projective real cubic surfaces with respect to their number of real lines. Moreover, Cayley [1] and Segre [13] worked on cubic surfaces with singularities.

More recently [8], Knörrer and Miller classified the real cubic surfaces in 45 types, according to their topology.

In the last years, thanks to the development of visualization software that allows us to produce high quality raytraced images, it has been possible to easily visualize surfaces. See for example the webpage [14] that contains some movies and images. Surfex uses S. Endrab's Surf (see [5]) to produce high quality raytraced images of the surfaces. In the article [10] we can find how to obtain a nice affine equation to visualize the 45 topological types of real cubic surfaces of [8].

2. Artistic Models of Real Cubic Surfaces Through History

Due to the great development of algebraic geometry at the end of the 19th century, mathematical models were built in order to illustrate geometrical properties of some surfaces. This interest in model building was specially popular in Germany, Felix Klein being one of its main developers. According to him [7], in Whitsun 1868, Christian Wiener for the first time built a model of a surface of degree 3 with 27 real lines. However, this model was still very asymmetric. In the summer of 1872, and under the guidance of Alfred Clebsch, the mathematician and sculptor Adolf Weiler built a model of the Clebsch diagonal surface (see [3]). In the Proceedings of the Royal Society of Sciences in Göttingen [7], Clebsch and Klein reported that Clebsch submitted two models built by Weiler: one model of the diagonal surface, and one model representing its 27 lines.

In 1879, Carl Rodenberg (see [11]), a student of Felix Klein, published his thesis that characterized the cubic surfaces with singularities. Together with it, he presented a series of 27 plaster models of cubic surfaces, among which the diagonal surface was present.

Mathematicians such as Felix Klein or Eduard Kummer starting building models of some surfaces. These models were mostly made of plaster and string. Many of the models built were reproduced and sold by the German companies L. Brill (and later M. Schilling) from 1888 to universities and museums around the world. By the turn of the century there were a large number of models of surfaces available

In 1986, Gerd Fischer [4], published a two volume book with a compilation of these German models, together with mathematical explanations and photographs. In particular, The University of Groningen contains a numerous collection of German models. The webpage [15] shows pictures and mathematical description of this collection.

We find the next evidence of modeling the Clebsch diagonal surface in 1999, when the sculptors Claudia Carola Weber and Ulrich Forster built the surface in clay (see [6]). This big sculpture (1,40 m wide and 2,50 m high) can be found at the Cafeteria of the University Heinrich Heine, in Düsseldorf. Lately, 3D printers have also been used in order to rebuild Rodenberg series, again in plaster.
During the spring of 2005, the sculptor Cayetano Ramírez built again the diagonal surface in plaster with the 27 lines on it. The model can be found in the Department of Mathematics at the University of Groningen (see figure 1).

Figure 1: *Plaster model of the Clebsch surface, by Cayetano Ramírez, May 2005*

3. New Artistic Models of Polyester of Real Cubic Surfaces

The building of mathematical models can have educational or research purposes: the visualization of the surface helps much in the understanding of its geometrical properties, as in the case for the German mathematical models of the end of the 19th century. Seeing is of basic importance also in art: mathematical surfaces have been used as ideas to generate art forms, like in the case of the clay model in Düsseldorf.

Computer representation often enable us to visualize the exact mathematical surface (see figure 2).

For example, the Clebsch diagonal surface has the following properties:
1. It contains 27 lines.
2. It contains 10 Eckard points (i.e., points where three of the lines meet)
3. It contains 7 passages (or "holes").

This surface can be represented in the computer showing these 3 properties. When it comes to plaster or clay models, the interior of the surface has to be filled in order to make the result solid (see figure 1). In this way, some of the properties can be lost, as it is the case for the seven passages of the Clebsch diagonal surface: only three of them are really visible in the models (compare figure 1 and figure 2).

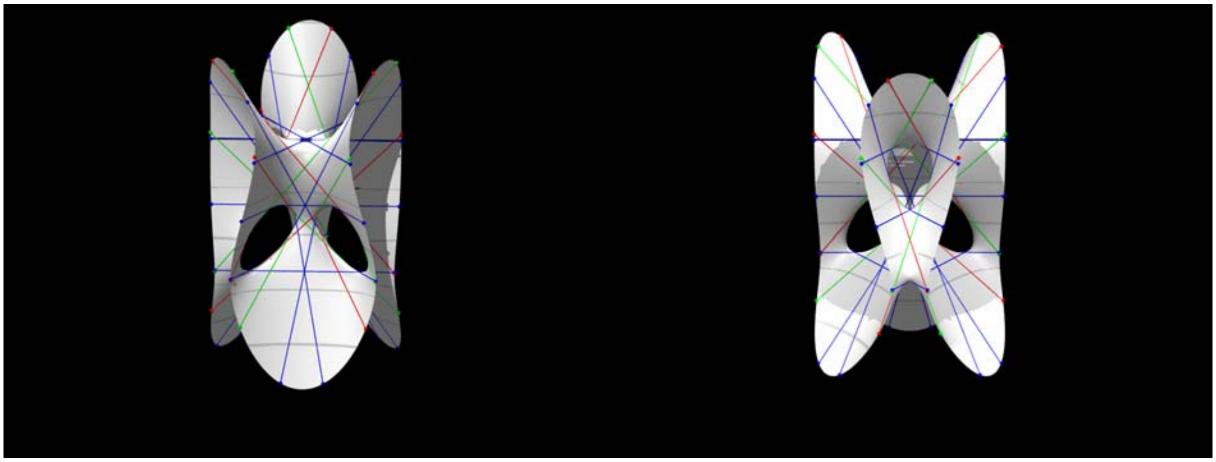

Figure 2: Clebsch diagonal cubic surface with the 27 lines. (Representation realized by the authors in POV-Ray 3.1)

Our present project consists of building the Clebsch diagonal surface in polyester. With this new construction, the model of polyester provides us with both the computer and the model advantages: it shows the precise mathematical surface with the 27 lines, the 10 Eckard points, and the seven passages. Moreover, we obtain a new artistic form made of a thin and transparent surface, that enable us to see the inside of the figure.

Before the final sculpture was built, several test figures where constructed as preliminary tests, obtaining imperfect Clebsch that showed the desired properties (see figure 3).

Figure 3: Sketch of the Clebsch diagonal cubic surface in polyester

For the final result, special care was take to assure an almost-perfect mathematical surface. The process starts by creating a perfect plaster model which would later be covered by a thin layer of liquid resin, and polyester fibre. However, in order to obtain a perfect surface, instead of constructing the whole model in plaster, it is more practical to build it first in polythene cork, and to cover it later with a thin layer of plaster.

For this purpose, we used computer images of different sections of the model (see figure 4), as well as, the outline of the Clebsch model intersected with a cylinder (see figure 5).

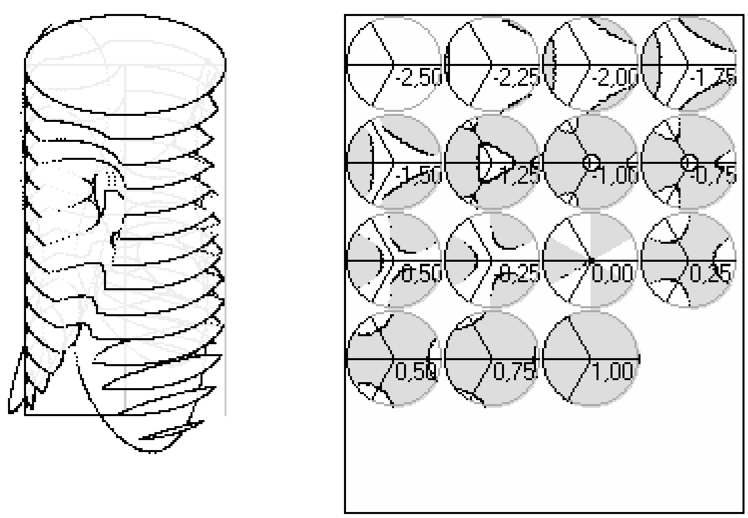

Figure 4: Transversal sections of the Clebsch diagonal cubic surface

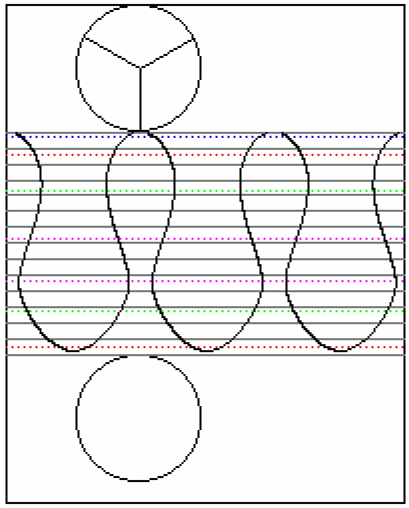

Figure 5: Outline of the Clebsch model intersected with a cylinder

Using the plans of figures 3,4, and 5, we construct a wood pattern to make sure that the model is correct (see figure 6 and 7)

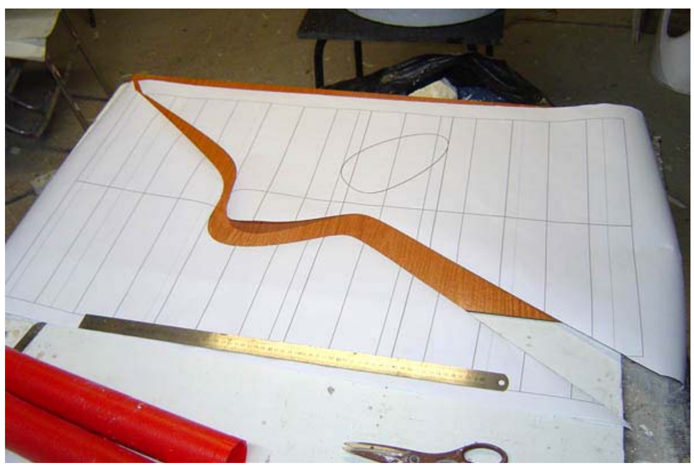

Figure 6: Pattern of the Clebsch model

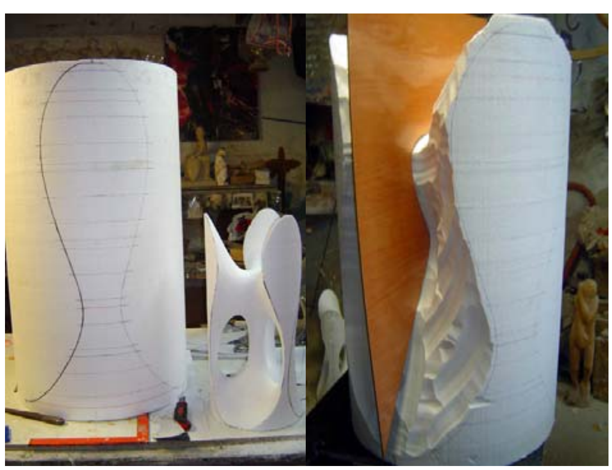

Figure 7: Process of construction

Using a similar procedure as the one described for the model of the Clebsch diagonal surface, we construct other cubic surfaces like the ones in figure 8.

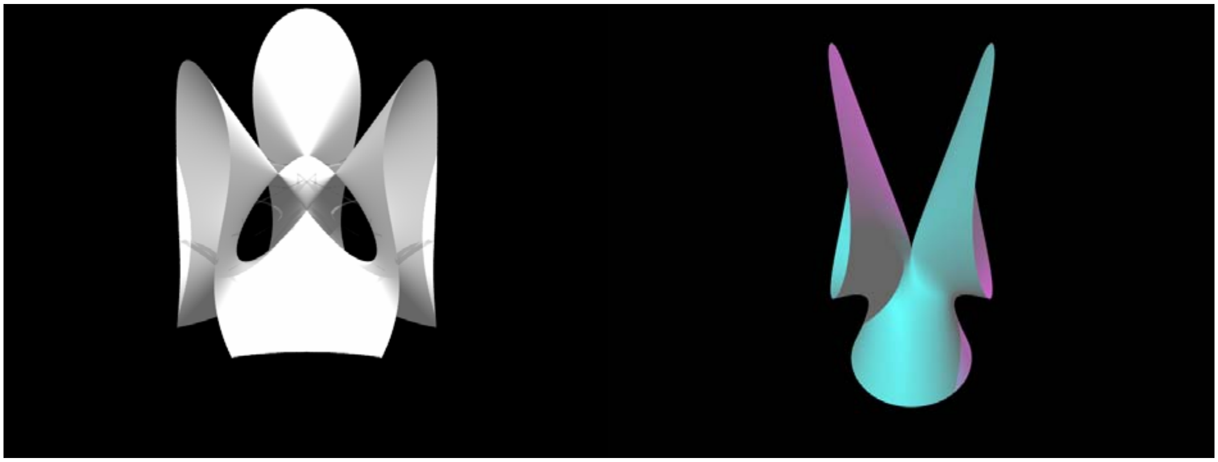

Figure 8: Real cubic surfaces represented by the authors with Pov-Ray 3.1

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