Playing Musical Tiles

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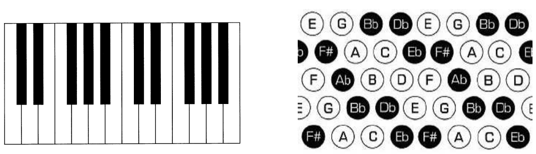
Abstract

In this survey paper, I describe three applications of tilings to music theory: the representation of tuning systems and chord relationships by lattices, modeling voice leading by tilings of n-dimensional space, and the classification of rhythmic tiling canons, which are essentially one-dimensional tilings.

1. Introduction

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4/4        |:...*...*...*...|...*...*...*...:|
doumbek     B   O mmB O   m B   O mmB O   m
variation   Bm mBm mBm mBm mBm mBm mBm mBm m
reqq        O   O mmm O   O O   O mmm O   O
variation   O mmO mmO mmO mmO mmO mmO mmO mm
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Figure 1: *Musical tilings.*

Figure 1 shows three "musical tilings": from left, a piano keyboard, a chromatic accordion keyboard, and a notation of the Egyptian *raqs sa'idi* rhythm [18]. Although, at least on the surface, it may not be clear that the tilings are related to each other, or indeed have any musical significance, in fact each example reveals deeper symmetries present in music, both in the domains of pitch and rhythm. These symmetries and their relationship to tilings are the subject of this article.

A *tiling* is a partition[1] of some space into congruent pieces, called tiles. There are many ways, both periodic and aperiodic, to tile the plane and higher-dimensional space. One-dimensional tilings, though less well known, are partitions of the real line into congruent collections of intervals. Tilings have long been of interest to visual artists and mathematicians alike. In addition, music theorists and mathematicians (going back to Euler) have discovered connections between tilings and musical structures. In this survey paper, I describe three applications of tilings to music theory: the representation of tuning systems and chord relationships by lattices, modeling voice leading by tilings of n-dimensional space, and the classification of rhythmic tiling canons, which are essentially one-dimensional tilings.

2. Euler and the Tonnetz

There are many ways to construct a scale. Solving the one-dimensional wave equation, which describes the behavior of string and wind instruments, produces a sequence of sinusoidal functions whose frequencies are the positive integer multiples of some constant. The scale is based on rationally related frequencies (octaves (2:1), fifths (3:2), and so on) or approximations to these frequencies. One approach builds upon a selection of the first few ratios among terms in the sequence. Fifths and octaves generate the Pythagorean scale; we

[1]Tiles are allowed to intersect in a set of measure zero—for example, tiles in two-dimensional space are allowed to share an edge, but not a two-dimensional area.

multiply the starting frequency by integer powers of two and three (including negative integer powers) to produce elements of the scale.[2] An alternate technique, an example of *just intonation*, is to generate a scale with fifths and major thirds (5:4). Although he did not originate just intonation, Euler [9] was the first to represent it as an infinite lattice, a portion of which appears in Figure 2 (left). Read left to right, rows are sequences of perfect fifths, and columns are sequences of perfect major thirds, read top to bottom—therefore, the lower left to upper right diagonals of the squares are perfect minor thirds (6:5). Since sequences of either fifths or thirds are geometric, we see that this lattice is drawn on a logarithmic scale. A similar lattice appeared in the late nineteenth century works of Oettingen [24] and Riemann[3] [25]. Riemann's lattice, called the *tonnetz*, depicts the major and minor third relationship more explicitly (Figure 2, center).[4] Since each triangle in the tonnetz represents a major or minor triad, vertices in its dual hexagonal lattice correspond to triads, and edges connect triads that have two notes in common. Figure 2 (right) labels the triads ("A" indicates A major and "am" indicates A minor), with major triads shaded.

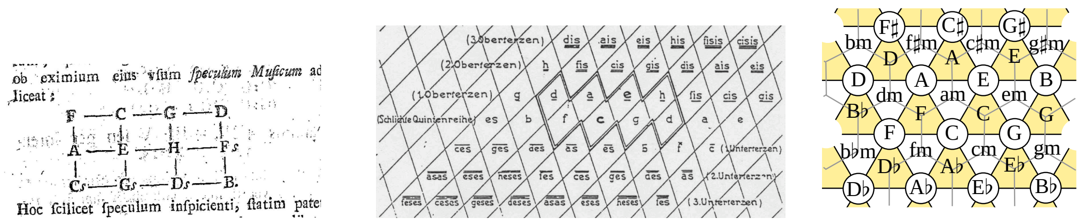


Figure 2: *Euler's Speculum Musicum [9, p. 350], Hugo Riemann's Tonnetz [25], and the tonnetz lattice with its dual.*

It is tempting to classify Figure 2 (right) as the tiling p3m1,[5] but one should be careful about what is actually represented. The symmetries of this lattice are transpositions by fifths, major thirds, or minor thirds (geometrically, translations along the lines of the lattice), so the tiling is p1, as is Euler's lattice. The fundamental region is a small rhombus. Inversion in the fifth, which exchanges major and minor thirds, introduces horizontal reflections, giving the tiling pm. The dual lattice, consisting of hexagons whose vertices are labeled by the alternating major and minor triads in Figure 2 (right), is p3m1 if we ignore everything except chord quality. Although this was not the original intention, the tonnetz can also represent notes in equal temperament. In this case, we have additional symmetries: enharmonic equivalence (for example, E♭ and D♯ share the same frequency), and, because twelve fifths equal seven octaves, octave equivalence. Using these two symmetries instead of transpositions by fifths and thirds gives a different p1 tiling whose fundamental region contains exactly one copy of each note in the twelve-tone scale.

3. Voice Leading and Continuous Transformations

We can describe relationships between chords in many ways—the circle of fifths is just the best-known example. If a sequence of chords is played by several voices, each sounding a single note, we can track the motion of individual voices in the progression from one chord to the next. This association is called *voice leading*. Although the conventions of voice leading have changed through the ages, some common principles persist. When leading between two chords, it is desirable that each voice move as short a distance (in pitch[6])

[2]Note that the octave will not "close up" in the familiar circle of fifths. That is, if we start with 440 Hz, no matter how many multiples we generate, this process never returns 440 Hz again. What is produced by repeating this method an infinite number of times is not the circle of fifths, which occurs only in equal temperament, but a dense subset of all frequencies.

[3]Hugo Riemann, not to be confused with the mathematician Bernhard Riemann.

[4]Interestingly, the tonnetz array forms the keyboard layout of a concertina patented in 1844 by the English physicist Wheatstone [17]. His instrument appears to be designed for equal temperament—though not all concertinas were—and he may have been motivated by the chord possibilities in the dual lattice. However, I have found no evidence that Wheatstone built this instrument, or that he had a role in the development of the tonnetz on the Continent.

[5]For an introduction to plane tilings, see `http://en.wikipedia.org/wiki/Wallpaper_group`.

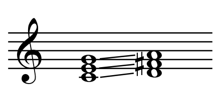
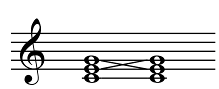
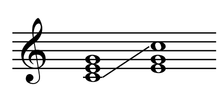
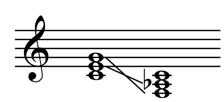
[6]Pitch is determined by the logarithm of frequency—precisely, if we arbitrarily decide that middle C is 0, then 440 Hz (the A above middle C) corresponds to note 9, and pitch $= 9 + 12 \log_2(\text{frequency}/440)$. In this system, integers correspond to notes in the chromatic scale of twelve-tone equal temperament.

as possible. In order to achieve this, voice-crossing—occurring when two voices change positions in the ordering of voices from low to high—is avoided. If we restrict ourselves to twelve-tone equal temperament, the closest distinct chords are those that differ by a semitone in one voice only (note that the chords in the dual tonnetz differ by either one or two semitones in one voice). This notion of closeness gives a structure to the space of chords of n voices. In fact, if we consider a "chord" to be an *ordered* multiset[7] of integer pitches, with each coordinate representing the pitch in one of the voices, we can map n-voice chords to the lattice $\mathbb{Z}^n$. The closest distinct chords are those that differ by one semitone in exactly one voice.

Our perception (and musical practice) gives this lattice of n-voice chords many symmetries: if two voices exchange pitches, if one voice shifts by an octave, or if all voices shift by the same amount, the respective resulting chords will sound quite similar to the original. How can we model voice leading in a way that respects these relationships? Quite a few music theorists have described voice leading using lattices or graphs: see Roeder [26], Douthett and Steinbach [8], Straus [27], Cohn [5], and Tymoczko [30]. The innovation that Callender [3], Quinn, and Tymoczko [29, 28] (henceforth, CQT) introduced is to embed the lattice of n-voice chords into *continuous* n-dimensional space (since pitch is continuous, not discrete) and study the effects of musically relevant symmetries. If we identify points in $\mathbb{R}^n$ that represent "similar" chords, what shape is the resulting space? Of course, the answer depends on which similarities we consider. CQT describe families of *chord spaces*, all of which are quotients of $\mathbb{R}^n$ under various isometries or combinations of isometries. Many discrete models of voice-leadings relationships embed nicely into these spaces. We will consider an example that Callender develops in detail in [3], and then touch on its relationship to CQT's general construction of chord spaces.

3.1. Representation in $\mathbb{R}^n$. In the discussion that follows, an "n-voice chord" means a vector in $\mathbb{R}^n$. We now represent operations on chords as rigid transformations of $\mathbb{R}^n$: transposition moves each voice by k pitches; permutation exchanges the pitches in two voices; octave shift moves one voice by some integer number of octaves; and inversion sends each voice to its additive inverse. Each of these operations describes a musical similarity of some sort. For example, all major triads in root position are equivalent under transposition. We call the set of vectors equivalent to $\mathbf{v}$ under all combinations of the four operations the *multiset class* of $\mathbf{v}$.

Using $\mathbf{e}_i$ to represent ith standard basis vector of $\mathbb{R}^n$, $\mathbf{1}$ to represent $\langle 1, 1, \ldots, 1\rangle$, and P_{ij} to represent the exchange of pitches in voices i and j given by $P_{ij} : \langle \ldots, v_i, \ldots, v_j, \ldots\rangle \to \langle \ldots, v_j, \ldots, v_i, \ldots\rangle$, we can write the operations as below. The CQT notation for these operations is $\mathbf{T}$, $\mathbf{P}$, $\mathbf{O}$, and $\mathbf{I}$.

Transposition	Permutation	Octave Shift	Inversion
$\mathbf{T} : \mathbf{v} \to \mathbf{v} + k\mathbf{1}, k \in \mathbb{R}$	$\mathbf{P} : \mathbf{v} \to P_{ij}(\mathbf{v})$	$\mathbf{O} : \mathbf{v} \to \mathbf{v} + 12n\mathbf{e}_i, n \in \mathbb{Z}$	$\mathbf{I} : \mathbf{v} \to -\mathbf{v}$
			

3.2. Continuous Transformations and Callender's T-class Space. Although Callender's construction of $\mathbf{T}$-class space in [3] does not explicitly discuss voice leading, it is consistent with the CQT model.[8] He begins with the composer Kaija Saariaho's *Vers le blanc* (Figure 3). This piece abandons the idea of pitch as discrete altogether; it consists of a *continuous* transformation from the chord C-A-B to the chord D-E-F over the course of fifteen minutes. Lines on the score indicates the position of the voices—note that the bottom two voices are briefly in unison towards the end of the piece.

[7]I'm glossing over some important issues here—for one, chords are usually considered to be sets, not multisets. See [28] for a full explanation.

[8]In the general literature, voice leadings are represented by associations between sets, rather than multisets, of pitch classes (pitches modulo 12). Callender's construction actually gives us equivalence classes of multiset voice leadings modulo transposition.

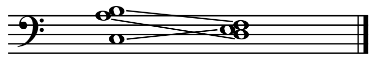


Figure 3: *Saariaho's Vers le blanc.*

Callender's model of continuous transformations is as follows.[9] As above, an n-voice chord is a vector of real numbers $\langle v_1, v_2, \ldots, v_n\rangle$, where v_i represents the pitch of the ith voice. For example, Saariaho's composition is a continuous interpolation from $\langle -12, -3, -1\rangle$ to $\langle -8, -10, -7\rangle$; it can be written as $\langle -12, -3, -1\rangle + (t/15)\langle 4, -7, -6\rangle$, where t is time in minutes and $0 \leq t \leq 15$. Callender begins by mapping the n-dimensional space of chords onto $(n-1)$-dimensional "$\mathbf{T}$-class space." The space of three-voice chords is a convenient example. Mapping each chord $\mathbf{v}$ to its transposition equivalence class ($\mathbf{T}$-equivalence class, or $\mathbf{T}$-class for short), defined to be $\{\mathbf{v} + k\mathbf{1} | k \in \mathbb{R}\}$, can be visualized as orthogonal projection onto the plane $\{\langle v_1, v_2, v_3\rangle | v_1 + v_2 + v_3 = 0\}$. For example, $\mathbf{e}_1$ maps to $\langle 2/3, -1/3, -1/3\rangle$. Note that the images of $\mathbf{e}_1$ and $\mathbf{e}_2$ form a basis for $\mathbf{T}$-class space; we will call them $\mathbf{a} = \langle 2/3, -1/3, -1/3\rangle$ and $\mathbf{b} = \langle -1/3, 2/3, -1/3\rangle$. The image of $\mathbf{e}_3$ is $\mathbf{c} = -\mathbf{a} - \mathbf{b}$. Thus, the projection of Saariaho's composition onto the plane is $-12\mathbf{a} - 3\mathbf{b} - \mathbf{c} + (t/15)(4\mathbf{a} - 7\mathbf{b} - 6\mathbf{c}) = -11\mathbf{a} - 2\mathbf{b} + (t/15)(10\mathbf{a} - \mathbf{b})$.

Now let's consider the effect of permutation of voices. Exchanging two voices corresponds to reflection in the planes $v_i = v_j$; in the plane $v_1 + v_2 + v_3 = 0$ this becomes reflection in one of the lines containing $\mathbf{a}$, $\mathbf{b}$, or $\mathbf{c}$ (that is, the projections of the coordinate axes onto $\mathbf{T}$-class space). These lines intersect at the origin at 60°, as shown in Figure 4. Each equivalence class under permutation and transposition ($\mathbf{PT}$-class) has a unique representative in the shaded sector, which is the projection of the vectors $\mathbf{v}$ where $v_1 \leq v_2 \leq v_3$. The symbols •, □, ○, and △ indicate, respectively, the $\mathbf{T}$-classes of the major triad $\langle 0, 4, 7\rangle$, minor triad $\langle 0, 3, 7\rangle$, diminished triad $\langle 0, 3, 6\rangle$, and augmented triad $\langle 0, 4, 8\rangle$ and their equivalents under permutation. In addition, the long arrow indicates the projection of *Vers le blanc*. Note that this projection crosses the line $r\mathbf{c}$ at the moment the bottom two parts are in unison.

We now consider the effect of octave shift. One generally perceives a C major chord played with the C in the highest voice as similar to one with the C in the lowest voice (root position). So, we identify all chords that are equivalent under octave shift; that is, $\mathbf{v} \equiv \mathbf{w}$ if and only if $\mathbf{v} - \mathbf{w} \equiv \mathbf{0} \pmod{12}$. The projection of the planes $v_i = 12n$ ($n \in \mathbb{Z}$) are shown in Figure 5 (left); octave equivalence introduces glide reflections in $\mathbf{T}$-class space. At this point, we have the tiling known as p31m, with the shaded kite-shaped fundamental region. We draw "mirror compositions" that lie in the same $\mathbf{OPT}$-equivalence class as *Vers le blanc*. It is evident that the composition begins and ends in the same $\mathbf{OPT}$-class!

Inversion is the last transformation to consider. The map $\mathbf{v} \to -\mathbf{v}$ exchanges minor triads and major triads; in $\mathbf{T}$-class space, reflection in the line $\mathbf{a} = \mathbf{b}$ is an inversion. Figure 5 (right) shows the tiling (p6m) of $\mathbf{T}$-class space for three-voice chords produced by permutation, octave shift, and inversion—note that there is now one symbol for minor and major triads. The fundamental region, originally depicted in Callender [3, Fig. 10, p. 12], contains exactly one representative of each multiset class (that is, $\mathbf{OPTI}$-equivalence class). The lattice points in the fundamental region correspond to multiset classes in twelve-tone equal temperament.

We now desire a *metric* on $\mathbf{T}$-class space—that is, a natural notion of the distance between two transposition classes. Callender defines the distance between $\mathbf{T}$-classes to be the distance between their projections into $\mathbf{T}$-class space, and sets $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$. There are many possible metrics with this property—Callender's preferred candidate is the Euclidean distance in $\mathbf{T}$-class space. Multiset classes now inherit a metric from $\mathbf{T}$-class space: the distance between multiset classes is the distance between their unique representatives in the triangular fundamental region for $\mathbf{OPTI}$-classes. Since the distance between any two points in $\mathbf{T}$-class space is never less than the distance between the respective members of their multiset

[9] I have changed his notation somewhat, but the essentials remain the same.

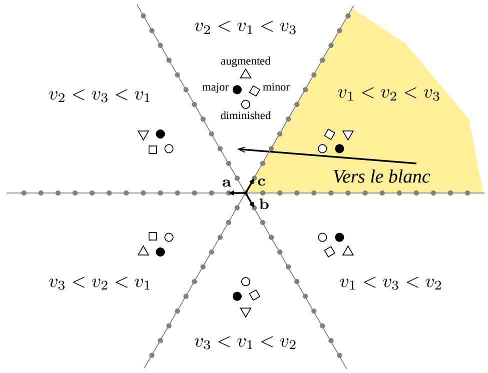


Figure 4: $\mathbf{T}$-*class space for three-voice chords, with permutation symmetries.*

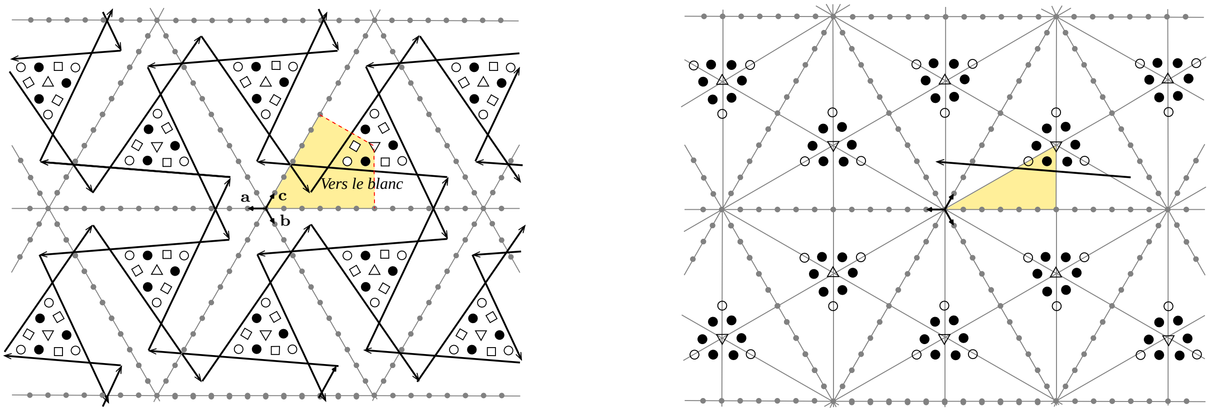


Figure 5: $\mathbf{T}$-*class space, with permutation and octave shift (left), and the addition of inversion (right).*

class in this fundamental region, his metric on multiset classes is well-defined. Callender uses this metric to investigate the shifts in our perception of chords in *Vers le blanc* as the voices draw near, but do not intersect, a number of lattice points in the fundamental region—see his article for more details [3].

Although not explored in [3], an important issue arises when we try to define the distance between $\mathbf{OPT}$-equivalence classes. In this case, the $\mathbf{OPT}$-classes on the dotted boundary of the kite-shaped fundamental region in Figure 5 (left) have *two* representatives on the boundary. In order to get rid of the doubles, we must identify the fundamental region's dotted edges and compute the minimal distance in the resulting cone. This cone is the quotient space of $\mathbf{T}$-class space modulo permutation and octave equivalence; in other words, it is the quotient of $\mathbb{R}^3$ under transposition, permutation, and octave equivalence. As such, it is an example of the more general quotient space construction developed by CQT and explored in the next section.

3.3. Generalized Chord Spaces and Orbifolds. Tymoczko [28] recognized that chord spaces are properly *orbifolds*, meaning quotients of $\mathbb{R}^n$ under the action of a finite group of isometries. For example, $\mathbf{T}$-class space is the orbifold $\mathbb{R}^n/\mathbf{T} \simeq \mathbb{R}^{n-1}$. The fundamental region of any tiling of $\mathbb{R}^n$, with the edge identification dictated by the tiling's symmetry group, is an orbifold, and that orbifold completely determines the tiling. Callender's kite-shaped fundamental region for $\mathbf{OPT}$-classes, with edge identifications, is the orbifold $\mathbb{R}^3/\mathbf{OPT}$ (**3*3** in Conway's orbifold notation); the fundamental region for $\mathbf{OPTI}$-classes is $\mathbb{R}^3/\mathbf{OPTI}$ (the orbifold ***632**). Other nice examples (from [28]) are the spaces of two-note chords modulo octave equivalence ($\mathbb{R}^2/\mathbf{O}$, a torus corresponding to the tiling p1 and orbifold **o**) and octave equivalence plus permutation ($\mathbb{R}^2/\mathbf{OP}$, a Möbius strip corresponding to the tiling cm and orbifold ***x**). Equivalence classes of voice leadings correspond to lines in these orbifolds. In their forthcoming paper [4], CQT further develop universal models for voice-leading—that is, a set of “parent spaces” in which many of the lattice models proposed by other music theorists embed. These parent spaces are orbifolds that are quotients of $\mathbb{R}^n$ under the action of $\mathbf{O}$, $\mathbf{P}$, $\mathbf{T}$, and $\mathbf{I}$, and combinations of these. Tymoczko’s interactive program ChordGeometries, available at `http://www.princeton.edu/~dmitri`, will help the reader visualize and explore some of the possibilities.

3.4. Open Problems. The geometric representation and exploration of chord relationships and voice leadings is an active area of research. There is much interest in measures of voice-leading size and in developing efficient algorithms to find minimal voice leadings (one such algorithm is described in [28]). CQT have thoroughly described the orbifolds $\mathbb{R}^n$ modulo various combinations of $\mathbf{O}$, $\mathbf{P}$, $\mathbf{T}$, and $\mathbf{I}$ for $n \leq 4$, and in higher dimensions in some cases [4]. However, there is more work to be done. In addition, Tymoczko posed the question of finding coordinate systems for these orbifolds that have some natural musical interpretation. He proposes using products of so-called “deep scales,” but there are other possibilities [32]. There are also issues dealing with the conflict between the CQT representation of chords as multisets and their more common representation as sets (no duplication allowed) that have yet to be resolved geometrically.

4. Tiling Canons

Vuza showed that, upon mapping beats to integers, a rhythm forms a tiling canon if and only if its inner rhythm and outer rhythm correspond to sets A and B forming a tiling of the integers [33]. I will summarize the literature on tiling canons and integer tilings and state some open problems.

4.1. Rhythmic Tiling Canons. A *canon* is a musical figure produced when two or more voices play the same melody, with each voice starting at a different time. Canons appear in the works of J.S. Bach and others. *Rhythmic canons* are canons in which rhythms, and not necessarily melodies, are duplicated by each voice. The composer Olivier Messiaen (1908–1992), who coined the term “rhythmic canon,” used rhythmic canons in his work (*Harawi*, “Adieu,” and others). He describes the sound of a rhythmic canon as a sort of “organized chaos” [21, p. 46]. Using the symbols `x` to represent a note onset and `.` a rest, the canon in *Harawi* looks like this:

```
Voice 1:  x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
Voice 2:    x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
Voice 3:      x..x....x.......x....x..x...x..x......x..x...x.x.x..x....x..
                                      *                * * * *
```

In this notation, simultaneous events are in vertical alignment. A rhythmic canon is *complementary* if, on each beat, no more than one voice has a note onset. For the most part, Messiaen’s canon is complementary; asterisks mark deviations from this rule. A *tiling canon* is a complementary canon of periodic rhythms that has exactly one note onset (in some voice) per unit beat. Each voice plays a rhythm pattern, called the *inner rhythm*, and the voices are offset by amounts determined by a second pattern called the *outer rhythm*. For example, the inner rhythm `|: x.x..... :|` and outer rhythm `|: xx..xx.. :|` form a tiling canon. Rhythmic canons and tiling canons were first recognized as integer tilings and studied mathematically by Vuza [33] (see [2, 10] for further background); all these articles consider canons of periodic rhythms.

4.2. Integer Tilings. A *tiling of the integers* consists of a finite set A of integers (the *tile*) together with an infinite set of integer translations B such that every integer may be written in a unique way as an element of A plus an element of B. If the pair (A, B) forms a tiling of the integers, we write $A \oplus B = \mathbb{Z}$, where $\mathbb{Z}$ denotes the integers. The example

$$A = \{0, 2\} \text{ and } B = \{\ldots -4, -3, 0, 1, 4, 5, \ldots\} = \{0, 1\} \oplus 4\mathbb{Z}$$

corresponds to the tiling canon in the previous section. Tilings of the integers were first studied in 1950 by Hajós [14] and de Bruijn [7] in connection with factorizations of abelian groups. Newman (1977) and others showed that all integer tilings are periodic [23]—which is not the case in higher dimensions. One-dimensional aperiodic tilings are possible only if we allow reflections (a *monohedral* tiling). Restricting one's attention to the integers, rather than the real numbers, may appear to be an oversimplification of the one-dimensional tiling problem. However, Lagarias and Wang [20] showed in 1996 that all tilings of the real number line by finite sets of intervals may be reduced to tilings of the integers.

Although many have studied this problem, the complete classification of such tilings is an open question; indeed, for a given finite set of integers A, it is not known whether A is a tile in a tiling (that is, whether there exists a B such that (A, B) defines a tiling), although there are results in some special cases. If the number of elements of A is a prime power, there is a simple criterion for determining whether A tiles (see Newman [23]). In 1999, Coven and Meyerowitz answered the question for sets A whose cardinality has at most two prime factors, and, in their 2005 article, Granville, Laba, and Wang [13] solved the problem for sets A whose cardinality has three prime factors. Klingsberg and I approached the problem from a different angle. Instead of specifying the number of elements in the tile set A, we started with the period N of the tiling, and counted the number of tilings of $\mathbb{Z}_N$, the integers mod N, by equally-spaced tiles. These are the tilings of the form $A \oplus n\mathbb{Z}$, where n divides N. We proved that a periodic rhythmic canon of ℓ voices, each spaced by n notes from the previous, is complementary if and only if its inner rhythm is ℓ-asymmetric, as defined in our articles [16, 15]; if, in addition, the inner rhythm has n notes, then it is a tiling canon. Our formulas in [15] give the number of such tilings for each N; we have since extended our results to enumerate tilings by symmetric tiles (that is, $A = -A$). The ℓ-asymmetry condition was originally defined to classify certain African rhythms.

4.3. Open Problems. As mentioned before, the complete classification of integer tilings is an open question—in particular, if the cardinality of A is divisible by more than three primes, it is not known whether A tiles, except in special cases. Laba [19] proved that solving the one-dimensional tiling problem is equivalent to proving (or disproving) Fuglede's Conjecture [12] in dimension one—a question posed in 1974 that is still unsolved. Another area of study concerns enumerating all tilings of a given period. The requirement in our articles [16, 15] that the tiles be equally spaced greatly simplifies the problem of enumerating them. Fripertinger [11] has enumerated all tilings up to period forty. A special class of tilings occurs when both the inner and outer rhythms of a tiling canon are primitive,[10] producing a *tiling canon of maximal category*. Vuza proved that no nontrivial tiling canons of maximal category exist for period of less than seventy-two [33, Theorem 2.2, part one, p. 33]; this result was proved independently by Hajós [14]. There is no known formula for the number of tiling canons of maximal category of a given period.

The *inversion* of a rhythm pattern is that pattern played backwards. Beethoven used a modified tiling canon, in which the rhythm patterns are inversions of each other, in his string quartet Op. 59, no. 2. This type of tiling canon corresponds to a monohedral tiling of the integers. The problem of finding tiling canons using one rhythm and its inversion is equivalent to a mathematical problem considered by Meyerowitz [22]. He proved that any set of three integers forms a monohedral tiling. The general question of which sets can form monohedral tilings remains open. Incidentally, monohedral tilings can be aperiodic, creating interesting possibilities for composers. In the pitch domain, certain *tone rows*—orderings of the twelve-tone scale—called *derived rows* are based on monohedral tilings of period twelve. Such tilings appear in the work of Schoenberg, Babbitt, and others. The idea is to start with a generator of n pitches, where n divides 12, and tile $\mathbb{Z}_{12}$ with the generator and transpositions of its retrograde (mirror image in the time domain), inversion (mirror image modulo 12 in the pitch domain), and retrograde inversion. Given an arbitrary period N, how many derived rows are possible, and how does the tiling determine the symmetries of the derived row?

5. Conclusion

Tilings are a locus of cross-fertilization of mathematics and the visual arts. Regular tilings of the plane were known to artists long before they were classified by mathematicians. Aperiodic tilings, first discov-

[10] A canon of period N is primitive if N is its smallest period.

ered by mathematicians, have now been used in art and architecture. I hope that investigation of tilings in music theory will inspire composers, and interest in the musical applications of tilings will lead to further investigation by mathematicians.

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