Patterns on the Genus-3 Klein Quartic

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Abstract

Projections of Klein's quartic surface of genus 3 into 3D space are used as canvases on which we present regular tessellations, Escher tilings, knot- and graph-embedding problems, Hamiltonian cycles, Petrie polygons and equatorial weaves derived from them. Many of the solutions found have also been realized as small physical models made on rapid-prototyping machines.

Figure 1: *Quilt made by Eveline Séquin showing regular tiling with 24 heptagons (a); virtual tetrus shape with cracked ceramic glazing programmed by Hayley Iben (b); and dual tiling with 56 triangles (c).*

1. Introduction

The Klein quartic, discovered in 1878 [5] has been called one of the most important mathematical structures [6]. It emerges from the equation $x^3y + y^3 + x = 0$, if the variables are given complex values and the result is interpreted in 4-dimensional space. This structure has 168 automorphisms, where, with suitable variable substitution, the structure maps back onto itself. To make this more visible, we can cover the 4D surface with 24 heptagons. Every automorphism then maps a particular heptagon onto one of the 24 instances, in any one of 7 rotational positions. This means that all 24 heptagons, all 56 vertices, and all 84 edges are equivalent to each other. In 4D this is a completely regular structure in the same sense that the Platonic solids are completely regular polyhedral meshes.

If we try to embed this construct in 3D so that we can make a physical model of it, we lose most of its metric symmetries, the regular heptagons get distorted, and only the symmetries of a regular tetrahedron are maintained. However, the topological symmetries are fully preserved. Helaman Ferguson's sculpture *The Eight-Fold Way* at MSRI in Berkeley celebrates this shape. A template for one of the 24 heptagons obtained from Bill Thurston allowed my daughter Eveline to stitch together the quilt shown in Figure 1a. The underlying shape is a rounded tetrahedral frame; we will call such a smooth, symmetrical genus-3 torus with full tetrahedral symmetry a *tetrus* (Fig.1b). Polyhedral approximations of this shape, such as the structure exhibiting the dual tiling of Figure 1a, are called *tetrads* (Fig. 1c).

In this paper, we use the tetrus shape as a canvas to study knot-, link-, and graph-embedding problems, as well as patterns derived from the underlying regular heptagonal tiling, and explore what artistically satisfying structures may result. In particular, I will subject the Klein surface to some of the same analyses and embellishments that I have applied to the 3 and 4 dimensional regular solids in the past. We will look for symmetrical colorings, Escher-like tilings, Hamiltonian cycles on the edges of that tiling, and interwoven ribbons forming orderly tangles [3] on that surface. Some of these patterns may lead to rather attractive abstract geometrical sculptures.

2. Regular Two-Manifold Meshes

Geometrically, Klein's highly symmetrical quartic can bee seen as a hyperbolic "Platonic" solid of genus 3. It is a completely regular 2-manifold composed of 24 heptagons, 84 edges, and 56 valence-3 vertices. Embedded in 4-dimensional space it exhibits 168 automorphisms and 168 anti-automorphisms (mirrored mappings). This is the maximal number of symmetries for a compact Riemann surface of genus 3. (The next genus, for which this maximal limit of 84*(g–1) automorphisms can be reached, is genus 7, leading to a group with 504 automorphisms [4].)

Helaman Ferguson's marble sculpture carries its name *The Eight-fold Way* because of an intriguing property: You can start at any vertex and move along consecutive edges, while alternately taking the left or the right branch at each subsequent vertex, and you will then end up where you started after exactly eight moves. These particular paths, called *Petrie polygons*, which always hug any face for exactly two consecutive edges before moving away from it, can also be drawn on the Platonic solids. On the tetrahedron the Petrie polygon has only four segments; on the cube and octahedron we obtain a "6-fold way" (Fig.2a), and on the dodecahedron and icosahedron we obtain zig-zag loops with 10 edges (Fig.2b). On the Klein quartic, the six Petrie polygons wrapping individually around each arm are easy to follow (yellow in Fig.2c), the other ones are trickier to trace without making a mistake. Looking for these Petrie polygons is a good way to check connectivity, when making a polyhedral model of the Klein quartic (Fig.1a); because it is very easy to join the four tripodal hubs with the wrong amount of twist.

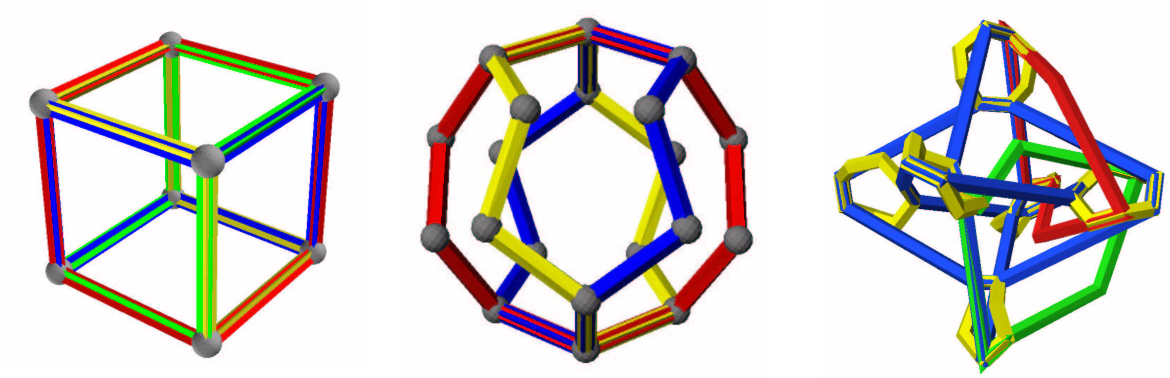

Figure 2: *Petrie polygons on the cube (a), on the dodecahedron (b), and on the Klein quartic (c).*

3. Equatorial Weaves

These Petrie polygons serve as a starting point for other intriguing constructions. If we connect subsequent midpoints of the edges in each Petrie polygon, we obtain 21 geodesically smoother cycles on the Klein quartic. This network can now be turned into a "woven" structure by having each strand alternatingly passing over and under subsequent strands that it crosses. Applying this technique to the Petrie cycles of the dodecahedron, and realizing each strand with a tube of finite thickness, one obtains a very pleasing and intricate structure, that is being sold in many museum stores (Fig.3a). Notice that all six star-shaped equatorial orbits are perfectly planar.

The question now arises, whether we can construct a similar assembly for the Klein quartic. Each loop comprises 8 vertices. If we restrict the equatorial loops around the 6 arms of the tetroid frame to be regular octagons (yellow in Fig.3b), then the overall figure is defined in its gross shape by two main parameters; the diameter of these arm loops and their distances from each other. Among the 21 Petrie cycles, there are: the six yellow "arm loops", three "major loops" in a Borromean configuration (white in Fig.3b,c), and 12 "shoulder loops," each running over one of the three shoulders of the four tripodal hubs, and then diving inside the structure and winding around exactly one arm of the tetrus. Our first goal is too keep all of these loops planar. The arm loops are planar by construction. The major loops share 4 points with 4 arm loops that all lie in the same coordinate plane; all we thus need to do, is to keep the 4 intermediate points also in this plane. The shoulder loops cross only three arm loops. These three points are just sufficient to define the planes for these shoulder loops. The intersection points between the shoulder loops and the major loops can be calculated in closed form (with some non-trivial expressions) from the intersection of the two loop planes involved and from some symmetry constraints. The intersections between pairs of shoulder loops leave 2 free parameters, which determine the height of the inner and outer tetrahedral hubs. Figure 3b shows the resulting network of 21 planar octagons.

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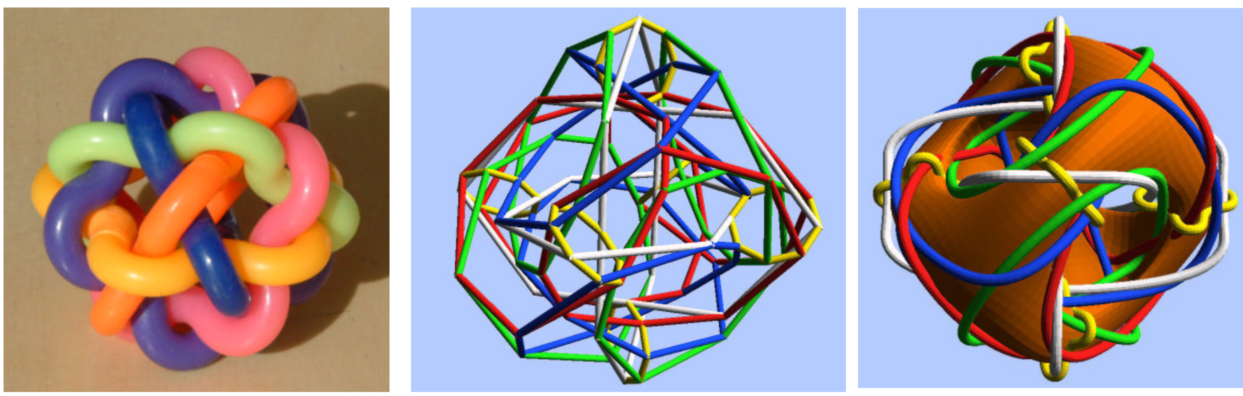

Figure 3: *Equatorial weave on a dodecahedron (a) and on the Klein quartic; network of Petrie cycles (b) and corresponding {over-under-}weaving loops on a tetrus surface (c).*

Starting from this basis, we now apply suitable in-plane distortions to these Petrie cycles so as to achieve the desired {over-under-} weave for all paths. We turn the paths into smooth B-splines, and sweep circles along them to form tubes of a finite cross section. The result of this equatorial weave, superposed onto a smooth tetrus body, can be seen in Figure 3c.

4. Knot and Link Embedding

Figure 4a singles out the three white Borromean loops found in the equatorial weave of Figure 3c. When drawn in the plane, this would be a 6-crossing configuration. In the typical knot tables [1], this link is identified as 6^3_2. On the tetrus (or on any other genus-3 surface) it can be embedded crossing free.

Among all the knots, only the unknot (0_1) can be drawn without crossings in the plane. On the torus, all the torus knots and links can be drawn crossing-free. Among those the simplest knot is the trefoil knot (3_1), and the simplest link is 2^2_1, composed of two interlocking unknots. The latter partitions the torus surface into two regions that can be colored differently.

It is thus natural to ask: Which is the simplest knot (lowest index) in the Knot Book [1] that can be drawn crossing-free on the Klein surface, but which cannot be embedded in a surface of lower genus. Progressing through the table of knots, we find that we can embed all knots up to 6_2 on either a 1- or 2-hole torus. Knot 6_3 (Fig.4b) is the first one that needs a genus-3 surface to be embedded crossing-free (Fig.4c).

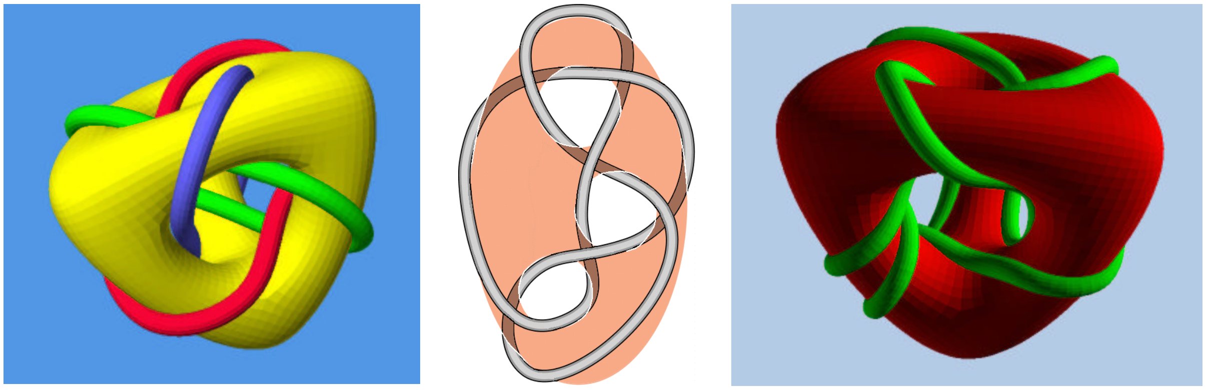

Figure 4: *Borromean link (a), and knot 6_3 on a 3-hole disk (b) and on the Klein quartic (c).*

These findings were established by looking at the graphs of the various knots and trying to see them embedded in the surface of a disk with holes. In an alternating {over-under-} knot, every single segment of the graph between two subsequent crossings makes a transition from the front of the disk to its back, or vice versa. Thus all these knot segments need to go around an outer border of the disk or through one of its holes to make such a transition. We thus draw the necessary border-loops into the knot diagrams as follows: Draw one outer border-loop around the whole knot figure, touching each of the outermost knot segments exactly once. Also draw similar border-loops into each region of the knot diagram that has an even winding number; expanding those loops so that they touch exactly once each of the knot segments forming that region. The number of these inner border loops then tells us the genus of the surface that is sufficient for a crossing-free embedding of that knot. (But this may not be the lowest possible genus!) Figure 5 shows representative examples of this construction for some genus-1 and genus-2 cases.

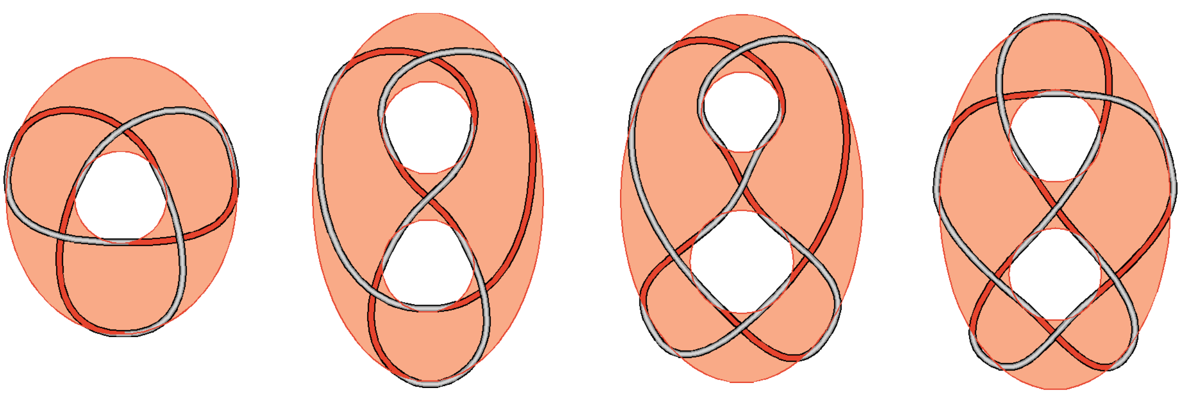

Figure 5: *Embedding analysis for knots 3_1 (trefoil), 4_1 (figure-8 knot), knot 5_2, and knot 6_2.*

The knot 6_3 does not partition the Klein surface into separate regions (Fig.4c). The simplest knot/link that can achieve this is the link 6^2_2. Its embedding diagram on a disk is shown in Figure 6a, and a smooth geodesic embedding on the Klein surface appears in Figure 6b. This figure exhibits C_2 symmetry; the intersections of the symmetry axis with the two links are marked with small bright circles. Figure 6c shows the resulting partitioning of the Klein surface into two differently colored domains.

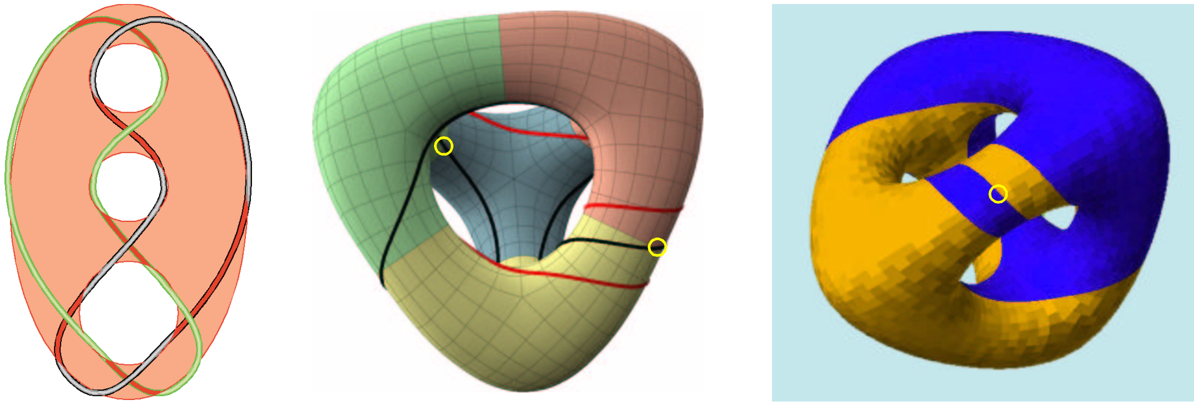

Figure 6: *Link 6^2_2; embedding analysis (a), geodesic embedding on the Klein surface (b), and the resulting partitioning of the tetrus surface (c). Circles indicate C_2 symmetry points.*

5. Hamiltonian Cycles

Given the edge graph of the Klein quartic with 24 heptagons, it is natural to ask whether this graph admits a Hamiltonian cycle. We expected the answer to be positive, and thus, in addition, we aim to find as symmetrical a path as possible. On all the Platonic solids we can find such a cycle as the perimeter of a strip of simply connected faces, using exactly half of them; and for all 5 solids such a cycle exists that dissects the surface into two disjoint congruent regions. On the tetrus tiled with heptagons this is not possible. I conjecture that on a surface of odd genus no simple knot, that requires that genus for crossing-free embedding, can partition the surface into disjoint regions. I started a manual search on the 4-fold symmetrical parallel projection of the 84-member edge graph (Fig.7a) and boldly searched for a path with bilateral symmetry across both the x- and the y- axis. On the fifth try I found one with full D_2 symmetry (Fig.7a,b,c).

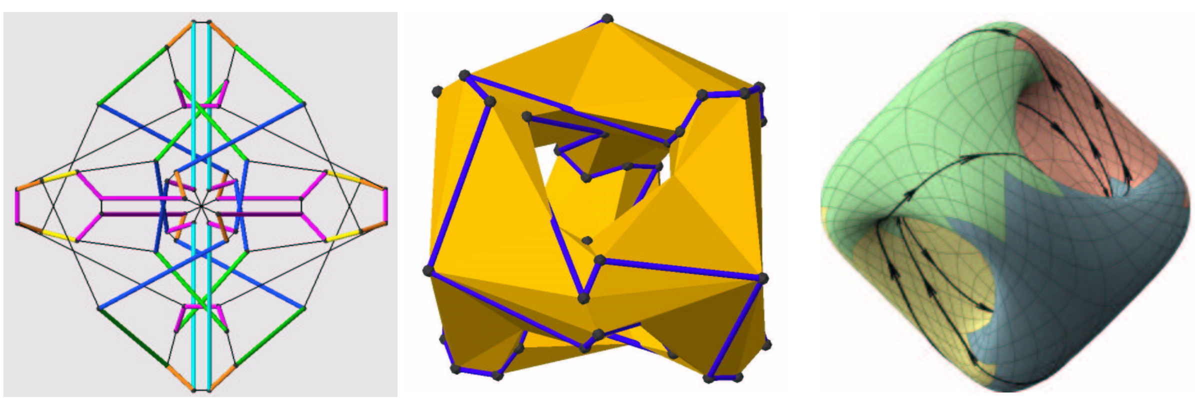

Figure 7: *Hamiltonian cycle on the Klein quartic with 24 heptagons: Path found on the projected edge graph (a), path shown on tetroid (b), and geodesically minimized path on a smooth tetrus shape (c).*

It is now interesting to explore what kind of knot this cycle forms. It turns out to be simply the unknot, but with a non-reducible embedding on the tetrus surface. And – no – it does not partition the surface!

6. "Viae Tetrus" – Circumnavigating the Klein Quartic

Tracing along the above Hamiltonian cycle with a tube or ribbon does not yield a good representation of the underlying shape in the same way that in Escher's "Bond of Union" (1956) a single ribbon defines the shape of the two heads that it is wrapped around. Of course, in Escher's drawing it helps that the ribbon also carries some of the salient features of the two faces, i.e., eyes and lips, as decorations. More to the point, most of my "Viae Globi" sculptures of a few years ago (Fig.8a) [7] make the underlying sphere readily apparent. We now look for ways to describe the tetrus shape in an artistically pleasing way by winding a ribbon over its surface. Ideally, I would like to find a solution that preserves the tetrahedral symmetry of the tetrus shape. However, I have come to the conclusion that this is not possible; deep down it has something to do with the fact that there exists no Eulerian cycle on the tetrahedron edges. Figure 8b shows a "multi-Eulerian" cycle on the tetrahedron, which visits every arm exactly twice. Such cycles are not difficult to find, if we use an even number of visits on each edge. However, this cycle has only 3-fold symmetry and does not preserve the full tetrahedral symmetry.

To obtain a good representation of the tetrus shape it is preferable to use **four** passes through each arm. We thus start with four intertwined helical paths. This leads to a good surface definition, since every cross section through an arm encounters four ribbon branches. The one parameter we can adjust most easily to obtain the desired connectivity, is the amount of twist we give to each of these cork-screws. It turns out that we can achieve the desired connectivity, that links everything into a single cycle, by using only two different twist numbers: On three of the arms, the helical paths twist through an angle of 180^o and on the other three arms through 270^o degrees. The result is shown in Figure 8c.

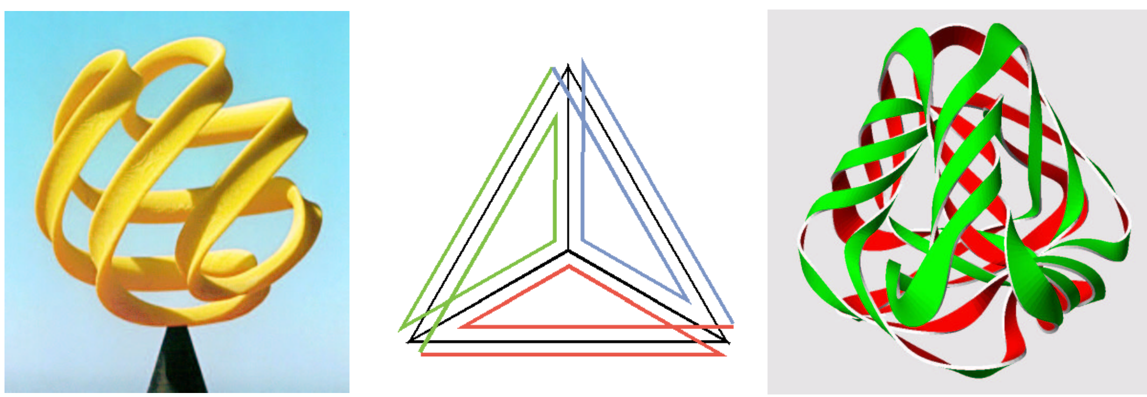

Figure 8: *'Altamont' from the Viae Globi series (a). Multi-Eulerian cycle on the tetrahedron (b). A single ribbon wound around a tetrus (c).*

7. Symmetrical Graph Embedding

The tetrus shape is a convenient canvas for the embedding of more complex graphs, in particular for those that have an inherent tetrahedral symmetry. One such graph is the tripartite graph $K_{4,4,4}$, the dual of Dyck's graph [8]. Each of its twelve nodes is connected to eight other ones in a 4-fold symmetrical manner (Fig.9a). It can be embedded in an orientable genus-3 surface forming a regular map, and the corresponding triangulated 2-manifold has 48 edges and 32 three-sided facets. The main challenge is to find good locations on the tetrus for the twelve nodes of this graph. A key issue is how to place the sets of nodes that are **not** connected to one another. For every node there are three others to which there are no direct connections. After some study, it becomes quite plausible to place each set of four such nodes onto one of the three D_2 symmetry axes of the surface. We believe that this results in an optimal overall embedding of the graph. Thus each tetrus arm carries two vertices, one on the inside and one on the outside. A physical model of the genus-3 Klein surface has been built on a rapid prototyping machine. Nodes, edges, and facet colorings have then been painted by hand onto this model (Fig.9b).

A virtual rendering of that same structure, using translucent surface panels and four internal light bulbs, thus simulating a genus-3 Tiffany lamp [8], is shown in Figure 9c.

Figure 9: *Graph embedding: The graph $K_{4,4,4}$ (a), painted onto a tetrus (b), and a virtual rendering of a corresponding "Tiffany lamp" (c).*

8. Regular Tilings and Colorings

After studying 1-manifolds embedded on this highly regular genus-3 canvas, we now look at various tilings of this surface, in particular, the tiling patterns implied by the symmetry group associated with Klein's quartic. The tilings that makes this surface such an important mathematical object are the topologically completely regular tessellation into 24 heptagons (Fig.10b), and its dual consisting of 56 triangles joined in 24 valence-7 vertices (Fig.1c). To maintain as much symmetry as possible, the points of 3-fold symmetry in these tiling patterns should be aligned with the four tripodal poles on a genus-3 surface with tetrahedral symmetry. Figure 10b shows the adjacencies of these heptagons on the tetrus. The two underlying dual tiling patterns, {7,3} and {3,7}, can also be embedded in the Poincaré disk, where infinitely many tiles can be accommodated within the circle limit; but only 24 heptagons are needed to cover the Klein quartic, after that, the same pattern (tiles #1– #24) repeats itself (Fig.10c).

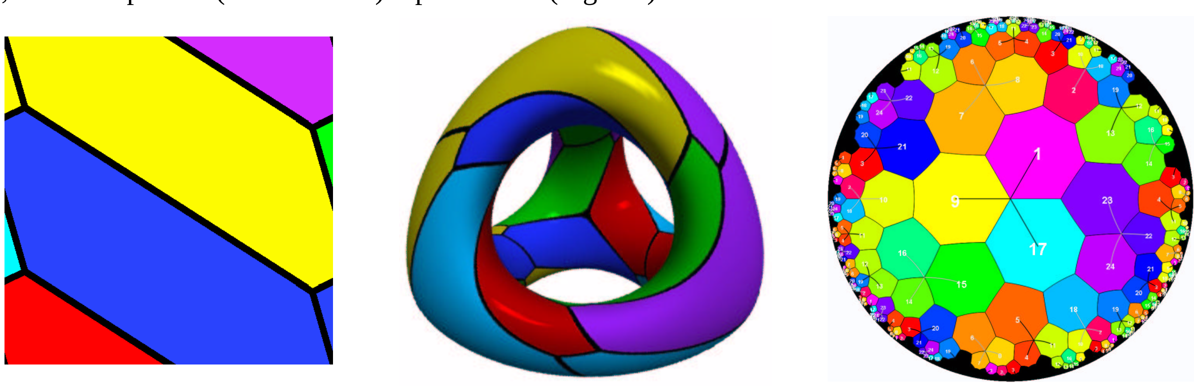

Figure 10: The basic tile texture (a) to produce Klein's heptagonal tiling on a Catmull-Clark subdivision tetrus with 48 quadrilaterals (b), and the {7,3} tiling of the Poincaré disk (c).

What is the smallest number of colors needed, so that no two tiles of the same color are adjacent? How many different color patterns are there that respect the symmetry of the tetrus? Clearly, that number must be an integer divisor of the number of tiles, and the occurrences of tiles of the same color should be distributed over the tetrus structure as evenly as possible. For the group with 24 heptagons a most appropriate solution is to use 8 colors. Each heptagon can then be surrounded by all the other seven colors. Along the two sides of any Petrie polygon we also find all 8 colors. At the 56 vertices we will see all possible color triplets; and every possible color pair shows up exactly 3 times at the 84 edges. Figure 11a shows this coloring scheme spread out in a {7,3} hyperbolic Poincaré disk. There are three groups of 8 colors; each group is marked with a different central dot. Heptagons with the same color combinations map onto the same face, when this hyperbolic tiling is wrapped around the Klein quartic. Taking the special tetrahedral embedding in 3D space into consideration, this color mapping places 4 colors entirely on the “outside” of the tetrus structure and the other 4 colors on its “inside” (Fig.11b). If we don’t like this color separation, we also find a highly symmetrical arrangement with only 6 colors. In this case, each color appears on two outer and two inner heptagons, and each color appears in every one of the four tripodal hubs, either on an outer heptagon or on an inner one (Fig.11c).

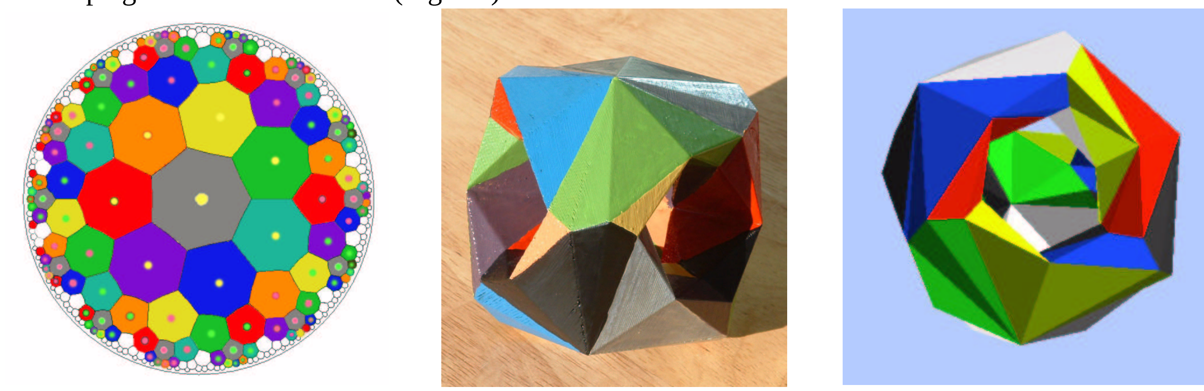

Figure 11: 8-color pattern on the Poincaré disk (a) and on a painted tetroid (b); tetroid coloring with tiles of 6 different colors (c).

Finally, even though 4 colors are not sufficient to color all heptagons so that no two adjacent tiles have the same color, this choice still offers an intriguing possibility. Adjacent pairs of equal colors (Fig.12a) can be chosen in such a way that we always combine an inner tile with an adjacent outer tile to form twelve identical double tiles. This can be done in four different ways. An extreme case where an inner and an outer tile are just joined by a short edge between them is shown in Figure 12b. A partial assembly of a tetroid built from such tiles is shown in Figure 12c.

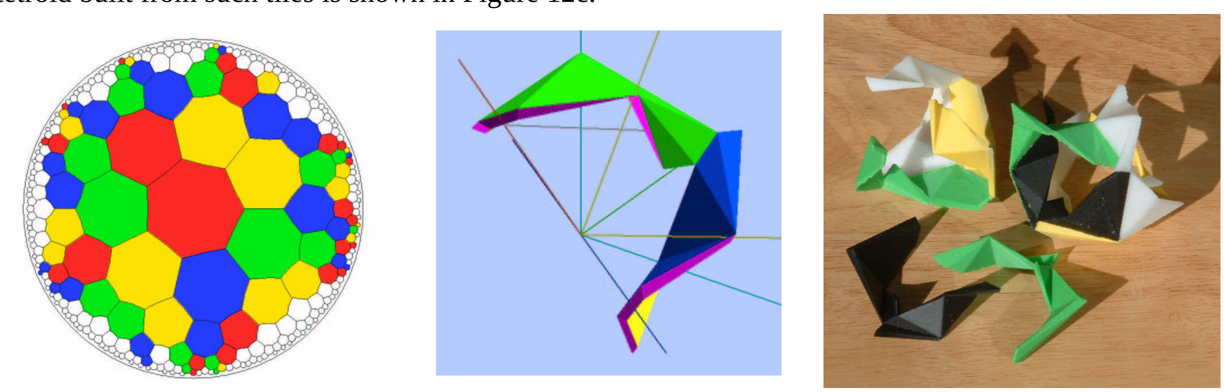

Figure 12: 4-color pattern on Poincaré disk (a); corresponding double tile on Klein’s quartic (b), and a partially assembled tetroid from such double tiles (c).

Similarly, there are several options for pleasing regular or semi-regular color patterns for the dual tiling with 56 triangles. The possible contenders are, 4, 7, and 8 colors (Fig.1c). Four colors allow a coloring so that no two adjacent triangles are of the same color. Seven colors allow a mapping in which all 7 colors show up at each vertex. The possible tilings for the assembly of 84 quadrilaterals are being investigated.

9. Escher-like Tilings

Inspired by Douglas Dunham’s pattern of *168 on a Polyhedron of Genus 3* [2], we now explore the ways in which the above regular tilings can be deformed to form Escher-like patterns on the tetrus shape. Ideally, we would like to have a tool such as the Escher-sphere editor created by Jane Yen [10]. The principle of generating an Escher tile is the same as in the plane: We distort the edges of a fundamental region (a heptagon) on the Klein quartic in such a way as to maintain all its symmetries, i.e., C_2 symmetry around the edge midpoint, C_7 symmetry around the tile center, and C_3 symmetry around the vertices. The so distorted tiles will then fit together again and seamlessly cover the whole surface. We just need to find a good way to map these tiles onto the tetrus surface. A difficulty arises from the fact that not all tiles have the same shape, even though they are topologically equivalent; the mapping onto the tetrus shape distorts the geometry of each heptagon in different ways, depending on where it lies. Fortunately, this distortion is not totally random; among the 24 heptagons there are only two different types of tiles: *outer* and *inner* heptagons. In the dual structure with 56 triangles we find 6 different tile shapes: 2 at the inner and outer poles, and 4 types around each arm. For the tessellation with 84 quadrilaterals resulting from the diamonds straddling the original heptagon edges, we find 8 different tile shapes, straddling respectively: the outer and inner pole edges, the long seams between inner and outer heptagons, and five different edges around each tetrus arm. These differently shaped tiles represent additional challenges when designing a decorative motif: We have to make sure that it works well for all occurring distortions of the tile. Fortunately, Escher's "creature tiles" are very deformable.

Figure 13: Texturing the tetrus. A polyhedral tetroid suitable for Catmull Clark subdivision (a); a quadrilateral tile with two "newts" (b), and the result of texture-mapping it onto a tetrus (c).

There are also some implementation and rendering difficulties. To obtain a smooth subdivision surface one needs nice, simple tiles, – not some concave animal-shape. Even just using the basic heptagonal regions shown in Figure 11b as a control polygon, leads to a badly wrinkled subdivision limit surface. For a Catmull Clark subdivision surface, we prefer to start with simple quadrilateral tiles lined up with the symmetry planes of the tetrus and/or with its principal directions (Fig.13a). But in that case the basic heptagon, and any Escher tile derived from it, will overlap several of the quadrilateral facets in odd ways (Fig. 10b), and designing suitable textures becomes more challenging.

Figure 13b shows a quadrilateral tile inspired by a combination of Escher's notebook patterns #25 and #35. It has been designed to fit directly onto a description of the tetrus structure using 48 quadrilaterals, covering a fundamental domain composed of four of them. For this case, texture mapping was easy, but the resulting pattern (Fig.13c) does not match the topological structure of Klein's symmetry group.

To capture the full topological symmetry with 168 automorphisms, we create a heptagonal tile with seven replicas of the fundamental tile (Fig.14a) – a fantasy fish inspired by Escher's pattern #55. The assembly of this tile in the {7,3} tessellation of the Poincaré disk (Fig.14b) has the same symmetry as Dunham's pattern of butterflies [2]. Since this display was created by texture mapping and I wanted to use only a single tile, I had to select a very special coloring pattern and carefully assign the orientation for each tile, in order to have all colors properly match up at the tile borders. Figure 14c shows the result of mapping 24 of these tiles around the tetrahedral genus-3 Klein quartic. Note that all yellow fish point towards the inner and outer tripodal poles of the tetroid. Other tiles and mappings can be found on my web site [9].

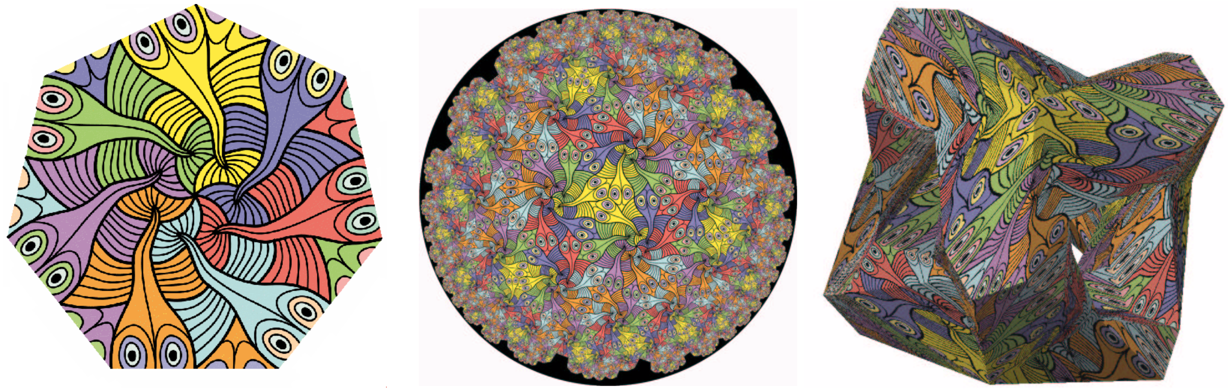

Figure 14: The {7,3} fish tessellation: the heptagonal tile (a), the corresponding Poincaré disk (b), and a symmetrical mapping of this pattern onto the tetrus (c).

10. Work in Progress and Conclusions

Klein's quartic and the genus-3 tetrus structure provide a rich domain for experimentation with geometrical operators and artistic effects. Beyond the examples presented in this paper, there are many more ways in which the Klein quartic can be celebrated. Work in progress includes a tangle of 24 knots, where each knot corresponds to one of the heptagonal tiles and links with all adjacent ones. A special knot has been designed to gracefully match the distorted shapes of these tiles on the tetrus surface. Other work concerns the development of a dissection puzzle for the tetrus shape into interlocking snap-together parts (Fig.12c).

Acknowledgements

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