The Integrated Scale Desirability Function:

A Musical Scale Consonance Measure Based on Perception Data

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1 Introduction

Based on previous work by Krantz and Douthett [1, 2, 3] and Sethares [7, 8], briefly reviewed in Section 2, a universal measure of the equal-tempered musical scale consonance is developed in Section 3. Preliminary results applying this, so-called, Integrated Scale Desirability Function for scales made up of complex tones show that higher frequency components of complex tones are necessary to explain the emergence of our usual 12-tone equal-tempered musical scale. These results are discussed in Section 4.

The formalism is also applied to the well-known Bohlen-Pierce scale. Here, too, it appears that higher frequency components are necessary for the emergence of the historically important 13-note to the "tritave" Bohlen-Pierce scale. These results are also discussed in Section 4.

Subsequently, in Subsection 4.4, we show that complex interval spectra can be "sculpted" so that non-standard equal-tempered musical scales turn out to be consonant as measured by the Integrated Scale Desirability Function.

2 Background

2.1 The Generalized Desirability Function: Previously, Krantz and Douthett [1, 2] articulated an approach for comparing the reasonableness of c-tone equal-tempered musical scales that simultaneously approximate multiple just musical intervals. This so-called Desirability Function is:

$$D(c,N) = 10 - 20\sum_{i=1}^{N}\left|\left\{c\ \log_2(R_i) + 0.5\right\} - 0.5\right|, \tag{1}$$

where c is the chromatic cardinality of the equal-tempered scale (the number of notes to the octave), N is the number of target intervals to be approximated, the R_i*'s* are the frequency ratios of the target intervals, $\{x\}$ is the fractional part of x, and $|x|$ is the absolute value of x.

This expression was used to determine which c-tone equal-tempered scales with octave closure best approximated the musically important intervals of the pure fifth, major third, and minor third. Subsequently, the Desirability Function was generalized to measure which c-tone equal-tempered scales with non-octave closure best approximated appropriately weighted multiple-intervals [3]. This Generalized Desirability Function is:

$$D_b(c,N) = 10 - 20\sum_{i=1}^{N} p_i\left|\left\{c\ \log_b(R_i) + 0.5\right\} - 0.5\right|, \tag{2}$$

where b, the base of the logarithm represents the interval of closure and the p_i are the respective normalized weights of the R_i*'s.*

This formalism was then used to describe and compare the desirability of non-traditional scales such as the Bohlen-Pierce scale [4, 5] and Balzano's 20-fold scale [6].

2.2 Dissonance Function, Perception, and Normalized Weights: In 1993 W. A. Sethares [7] developed a Dissonance Function based on the data of Plomp and Levelt [9]. Sethares fit the consonance/dissonance data of Plomp and Levelt to develop a standard consonance/dissonance curve, the so-called "roughness curve" which described the perceived tonal dissonance of two pure tones played simultaneously. Figure 1 shows a typical roughness curve.

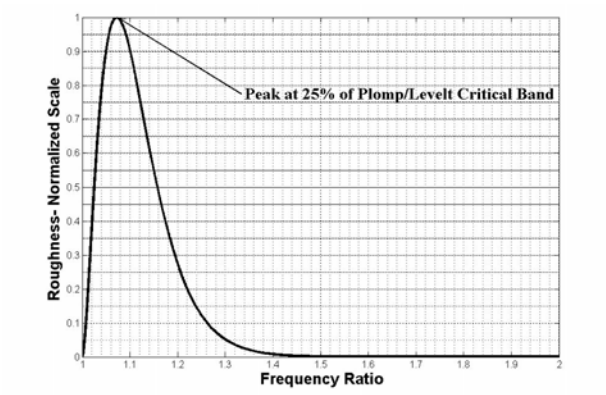

Figure 1- Typical Roughness Curve

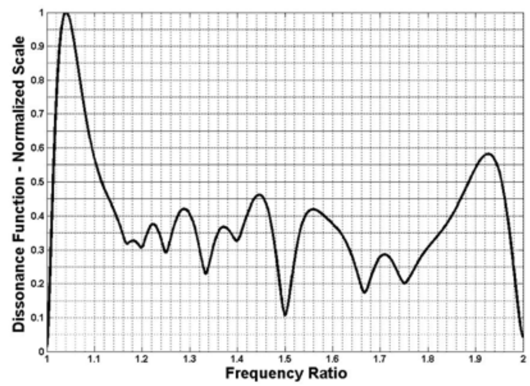

Figure 2 – Typical Dissonance Function

The peak of the curve occurs at 25% of the critical bandwidth, the frequency bandwidth at which two pure tones are perceived as distinct (see Zwicker et al. [10]). Below the peak beats are heard. Above the peak the tones are perceived more and more as two distinct tones. This dependence of perception based on beat frequency and the distinction between two pure tones, critical band theory, was first formulated by Helmholtz [11].

To generate a dissonance measure for intervals involving complex tones, those with harmonics, Sethares added the roughness curves for all pairs of pure tones weighted by the spectral amplitude of each pure tone. The result was the so-called Dissonance Function. Figure 2 shows a typical Dissonance Function. Peaks represent the most dissonant complex intervals and the sharp valleys represent the most consonant complex intervals.

Consider a Sethares-like Dissonance Function, $D_F(R)$, where R represents the frequency ratio of the fundamentals of the two complex tones of the interval and F represents the timbre of the complex tones. If,

$$M_F = \max\left\{D_F(x) \mid x \geq 1\right\}. \tag{3}$$

Then,

$$p_{F,b}\left(k,n\right) = \frac{M_F - D_F\left(b^{k/n}\right)}{n\left(M_F - C_{F,b}\left(n\right)\right)}, \tag{4}$$

represents the probability of the k^{th} partial out of n being "consonant" where $C_{F,b}(n) = \sum_{k=1}^{n} D_F(b^{k/n}) \big/ n$

3 The Integrated Scale Desirability Function

3.1 Discrete Case: We may now use the probabilities defined in equation (4) as the probabilities in equation (2) to define an "n^{th} partial desirability sum":

$$S_{F,b}(n,c) = 10 - 20\sum_{k=1}^{n} p_{F,b}(k,n)\left|\left\{\frac{ck}{n} + 0.5\right\} - 0.5\right| \tag{5}$$

where c is the chromatic cardinality of the equal tempered scale. Using equation (4) and simplifying yields:

$$S_{F,b}(n,c) = 10 - \frac{20}{n}\sum_{k=1}^{n}\frac{(1 - D'_F(b^{k/n}))}{(1 - C'_{F,b}(n))}\left|\left\{\frac{ck}{n} + 0.5\right\} - 0.5\right| \tag{6}$$

where $D'_F(b^{k/n}) = \dfrac{D_F(b^{k/n})}{M_F}$, the normalized Dissonance Function, and $C'_{F,b}(n) = \dfrac{C_{F,b}(n)}{M_F} = \dfrac{1}{n}\sum_{j=1}^{n} D'_F(b^{k/n})$.

3.2 The Continuous Case: We may define an Integrated Scale Desirability Percentage (ISDP) for a fixed base, b, timbre, F, and chromatic cardinality, c, as:

$$P_{F,b}(c) = 10\lim_{n\to\infty} S_{F,b}(n,c). \tag{7}$$

Substituting equation (6), with $k = n\,x_k$ *and* $\Delta x = 1/n$, in to equation (7) and taking the limit yields:

$$P_{F,b}(c) = 100 - \frac{200}{M_F - C''_{F,b}}\int_0^1 (M_F - D'_F(b^x))\left|\left\{cx + 0.5\right\} - 0.5\right| dx \tag{8}$$

where $C''_{F,b} = \lim_{n\to\infty} C'_{F,b}(n) = \int_0^1 D'_F(b^x)dx$. With $A_{F,b} = 50(M_F - 2C''_{F,b})$; $B_{F,b} = {}^{200}\!\big/\!_{(M_F - C''_{F,b})}$; and a simple change of variables the Desirability Percentage, equation (8), may be re-written as:

$$P_{F,b}(c) = A_{F,b} - \frac{B_{F,b}}{\ln(b)}\int_1^b \frac{D'_F(y)}{y}\left|\left\{c\,\log_b(y) + 0.5\right\} - 0.5\right| dy. \tag{9}$$

In this form it is apparent that the ISDP is a weighted average of the normalized Dissonance Function from the interval of the unison, $y_{\min} = 1$, to the interval of closure, $y_{\max} = b$. The weighting function: $\left|\{c\,\log_b(y) + 0.5\} - 0.5\right|\big/ y$ is a generalization of the original Desirability Function of Krantz and Douthett [1]. The Normalized Dissonance Function is based on the original, perception-based, Dissonance Function of Sethares [7].

3.3 An Important Property of the ISDP: In order to define a useful measure for the Integrated Scale Desirability we consider how, if at all, the timbre and base affect the "expected" desirability percentage of $P_{F,b}(c)$. We define the expected desirability percentage (EDP) as:

$$\left\langle P_{F,b}\right\rangle = \lim_{c\to\infty}\frac{1}{c}\sum_{j=1}^{c} P_{F,b}(j). \tag{10}$$

Remarkably, the expected desirability percentage is 50%; independent of the timbre, as reflected by the dissonance function used, or the closure interval, represented by the base b. This makes comparison to the EDP a universal measure for all scales.

3.4 The Integrated Scale Desirability Function: Given the universal property of the EDP; we, therefore, redefine the constants in equation (9) so that a value of zero represents the EDP measure, a value of 5 represents the most consonant scale possible, and a value of -5 represents the most dissonant scale possible. We define this function the Integrated Scale Desirability Function (ISDF), $K_{F,b}(c)$:

$$K_{F,b}(c) = A'_{F,b} - B'_{F,b} \int_1^b \frac{D'_F(y)}{y\ln(b)} \left| \left\{ c \log_b(y) + 0.5 \right\} - 0.5 \right| dy \,. \tag{11}$$

Shown in Figure 3 is a particular example of dissonance function, $D'_F(y)$ (solid), for intervals consisting of complex tones each having 7 harmonics of equal amplitude and the weighting function for a 12-tone equal-tempered chromatic scale (dashed) with octave closure (b=2). The product of these two functions is the integrand in equation (11).

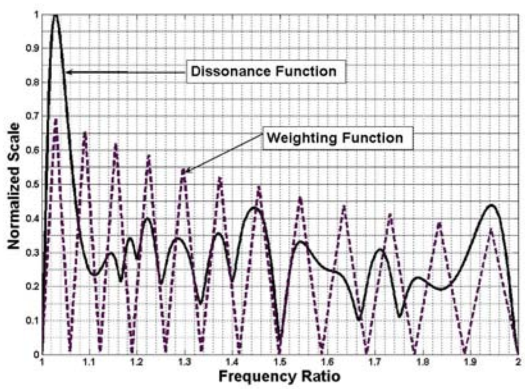

Figure 3 – 12-Tone Dissonance/Weighting Function

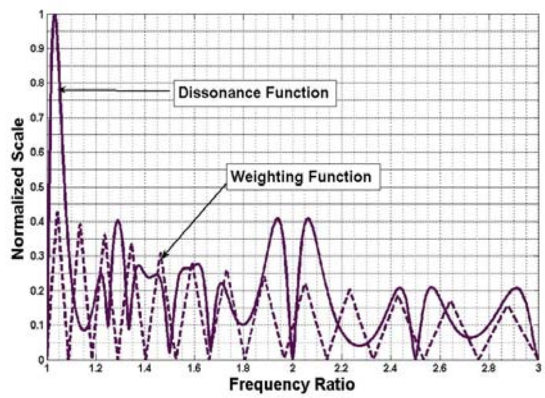

Figure 4 – 13-Tone Dissonance/Weighting Function

For comparison, Figure 4 shows the dissonance function (solid), for intervals consisting of complex tones each having 5 harmonics of equal amplitude and the weighting function for a 13-tone equal-tempered chromatic scale (dashed) with tritave, 3:1 (octave plus a fifth), closure. This represents the weighting function for the, well-known, Bohlen-Pierce scale [4, 5].

Comparison of the details of Figures 3 and 4 shows that the ISDF will depend, crucially, on the overlap of the dissonance function and the particular weighting function.

4 Preliminary Integrated Scale Desirability Function (ISDF) Analysis

In this section we use the ISDF to analyze the effects of: 1) the fundamental frequency, 2) the number of partials in the complex tones, and 3) the relative amplitudes of the partials making up the complex tones; on the overall scale consonance of equal-tempered scales of varying chromatic cardinality.

In each figure the fundamental of each complex tone is listed. For convenience, except in subsection 4.4, the fundamental of each tone is assumed to be the same. The closure interval for the equal-tempered scale is listed. For example, a closure interval of 2 represents equal-tempered scales with octave closure. A closure interval of 3 represents equal-tempered scales closed at an octave plus a fifth, the closure interval for the Bohlen-Pierce scale.

Also listed are the partials included in the complex fundamental tone and the partials included in the complex interval tone. For example, in Figure 5 below, the complex fundamental tone, as well as, the complex interval tone is each made up of 5 harmonics. These are listed after each respective spectrum. The relative amplitudes, of each harmonic in the complex spectrum, are also given. In Figure 5, each partial has the same amplitude.

4.1 The Effect of the Fundamental: In this subsection we show the effect of changing the fundamental frequency on scale consonance.

4.1.1 Octave Closure: Keeping spectral components and the amplitudes constant we changed the starting frequency of the complex tones making up the intervals in the dissonance function. We see, in Figures 5 and 6, that for octave closure the familiar 12 tone equal-tempered scale does not emerge as a consonant scale until higher frequency components are present. This indicates that our perception of consonance and dissonance depends on the presence of higher frequency components as observed by Sethares [7, 8].

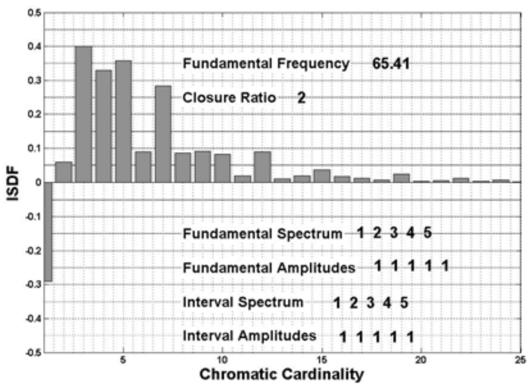

Figure 5 – ISDF Octave Closure (65.41 Hz)

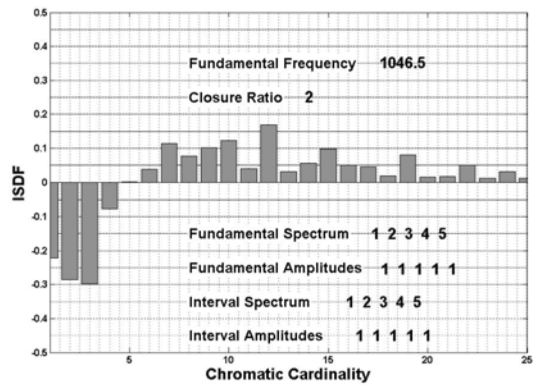

Figure 6 – ISDF Octave Closure (1046.5 Hz)

4.1.2 Tritave Closure: Shown in Figure 7 and 8 is a comparison of the effect of changing the starting tones on scales with 3:1 closure ratios. This closure interval is the basis for the Bohlen-Pierce scale. In keeping with the Bohlen-Pierce scale, we include only odd harmonics in the complex tones. Again, the amplitudes of the partials included are equal.

Although not as dramatic as the case for octave closure, we note again that the consonance of scales with larger chromatic cardinalities do not start to emerge until higher frequency components are included. Even in non-standard scales, it appears that the presence of higher frequency components is necessary for the perception of consonance and dissonance. Further analysis indicates that mixing in partials of all equal amplitudes, a clearly artificial case, obscures the effects of more complicated spectra. More on this in subsection 4.3.

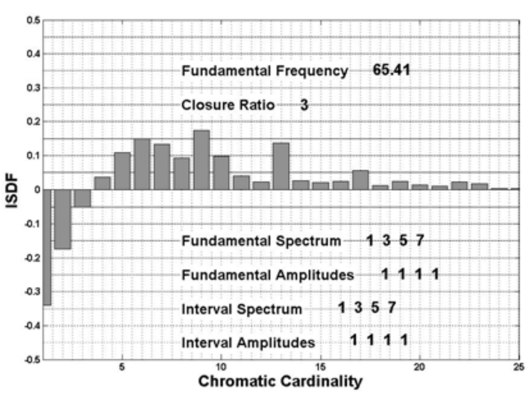

Figure 7 – ISDF Tritave (Odd Harmonics: 65.41 Hz)

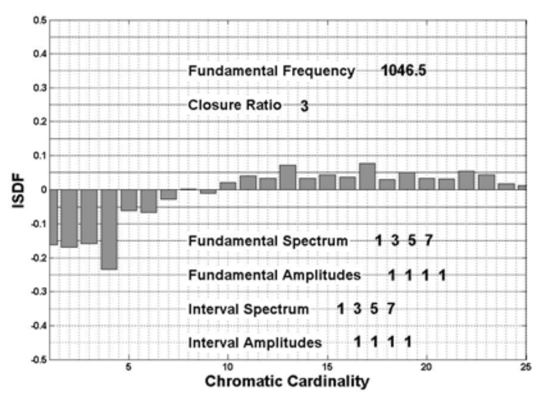

Figure 8 - ISDF Tritave (Odd Harm: 1046.5 Hz)

4.2 The Effect of the Number of Partials in the Complex Tones: In this subsection we show the effect of changing the number of partials on scale consonance.

4.2.1 Octave Closure: As shown in Figures 9 and 10, the effect of adding partials, of equal amplitudes, is similar to the effect of adding higher frequency components, as shown in Figures 5 and 6 above. Again, this indicates that adding higher frequency components adds to the perception of consonance of equal-tempered scales of higher chromatic cardinality, suppressing the consonance of scales with small cardinalities.

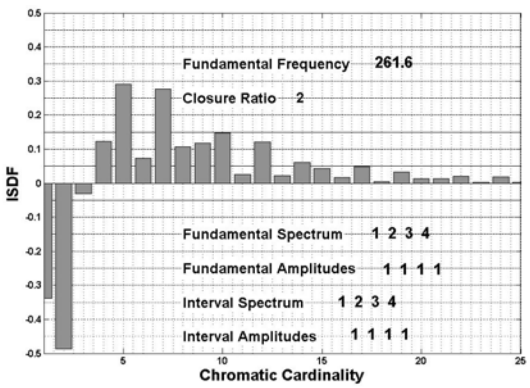

Figure 9 – ISDF Octave Closure (4 Harmonics)

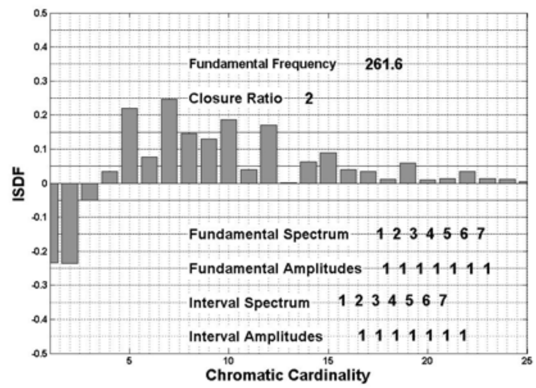

Figure 10 – ISDF Octave Closure (7 Harmonics)

4.2.2 Tritave Closure: The effect of the shifting of consonance to higher cardinalities with the inclusion of more partials of equal amplitude is also born out for the non-standard tritave scales as shown in Figures 11 and 12. As more partials are added, smaller chromatic cardinalities become more dissonant while sifting the more consonant cardinalities to larger numbers of equal-tempered notes per scale. Again, it appears that mixing in of higher partials with equal amplitudes obscures the effects due to including partials with varying amplitudes.

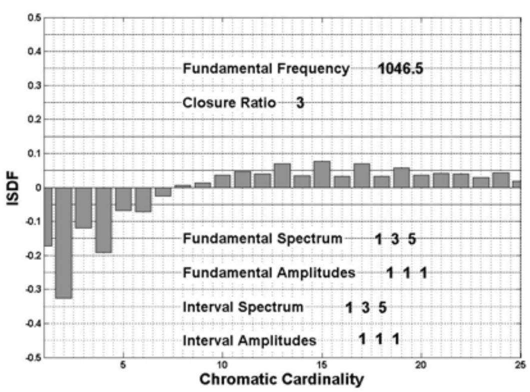

Figure 11 – ISDF Tritave (3 Odd Harmonics)

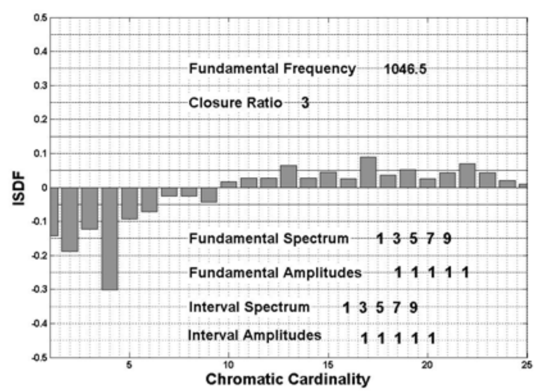

Figure 12 – ISDF Tritave (5 Odd Harmonics)

4.3 The Effect of the Amplitudes of the Partials in the Complex Tone: In this subsection we investigate the effect, on scale consonance, of including more partials with decreasing amplitudes.

4.3.1 Octave Closure: As shown in Figures 13 and 14 including more partials, even with smaller relative amplitudes, shifts the consonant scales to larger cardinalities. As more partials are included, the 12-tone equal-tempered scale emerges as the most consonant cardinality with octave closure. It appears that more complicated spectra are needed for our familiar 12-tone scale to be perceived as most consonant compared to other possibilities.

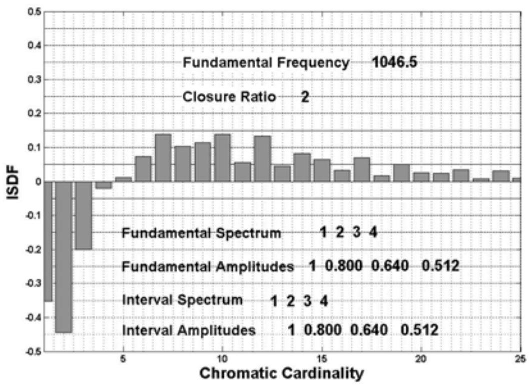

Figure 13 – ISDF Octave (4 Decreasing Harmonics)

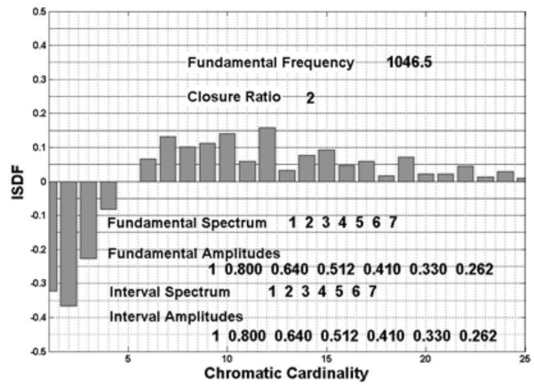

Figure 14 – ISDF Octave (7 Decreasing Harmonics)

4.3.2 Tritave Closure: A similar conclusion, that more complicated spectra move the consonant cardinalities to more notes per scale, is born out for the non-standard tritave scales shown is Figures 15 and 16 as well.

As we include more partials, 13 equal-tempered notes to the tritave emerges, even more strongly, as the most consonant of scales. It was 13 notes to the tritave that was used in the original Bohlen-Pierce scale [4, 5].

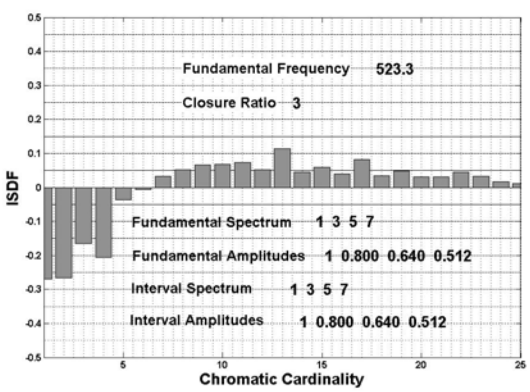

Figure 15 – ISDF Tritave (4 Decreasing Odd Harm.)

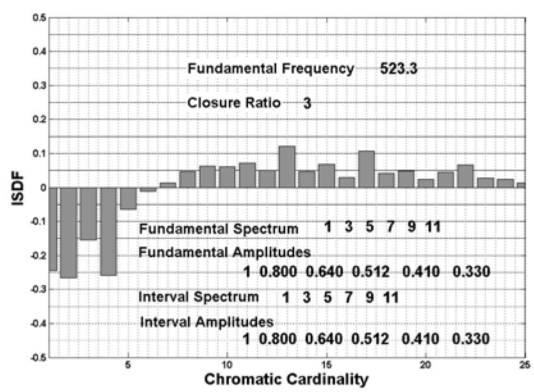

Figure 16 – ISDF Tritave (6 Decreasing Odd Harm.)

4.4 Sculpting Spectra: Shown in Figure 17 is the ISDF for a relatively complex fundamental and interval spectra. As long as we assume octave closure and spectra based on the harmonic series (each subsequent partial is a multiple of the starting tone) and the interval start tone is a pure fifth above the fundamental start tone, 12-tone equal-temperament emerges as the most consonant scale.

Even so, it appears that 19-equal-tempered notes to the octave could be important. This is born out in Figure 18. Even though each spectra is based on the harmonic series, 19-notes to the octave emerges as a consonant scale if the interval spectra starts on a tone a minor third above the fundamental start tone. Clearly, if spectra are "sculptured" one can get non-standard scales to be consonant. It should be pointed out that, historically, 19-tones to the octave has been used in compositions based upon the minor third.

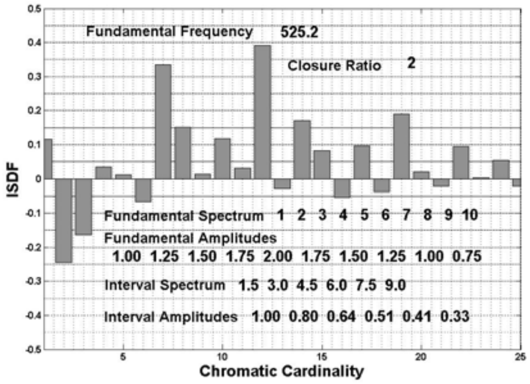

Figure 17 – Sculpted Harmonics Example 1

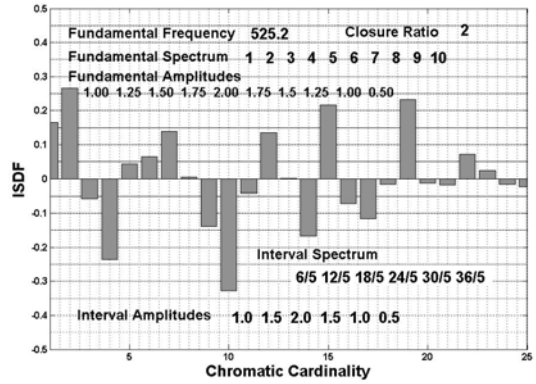

Figure 18 – Sculpted Harmonics Example 2

5 Summary

Based on previous work [1, 2, 3], we have developed a universal measure of the equal-tempered musical scale consonance, the so-called Integrated Scale Desirability Function which is based on the perception-based [9, 10] Dissonance Function of Sethares [7, 8]. Preliminary analysis shows that for scales made up of complex tones higher frequency components are necessary to explain the emergence of our usual 12-tone equal-tempered musical scale.

Preliminary application of the formalism to the historically important Bohlen-Pierce scale [4, 5] also shows that higher frequency components are necessary for the emergence of this well-known 13-note to the "tritave" scale.

We also show that complex interval spectra can be "sculpted" so that non-standard equal-tempered musical scales turn out to be perceived as consonant by our measure.

6 References

[1] R. J. Krantz and J. Douthett, "A measure of the reasonableness of equal-tempered musical scales," J. Acoust. Soc. Am. Vol. 95, No. 6, 3642-3650, June 1994.
[2] J. Douthett, "The Theory of Maximally and Minimally Even Sets, The One-Dimensional Antiferromagnetic Ising Model, and the Continued Fraction Compromise of Musical Scales," Ph.D. dissertation (University of New Mexico, Albuquerque, NM, May 1999).
[3] R. J. Krantz and J. Douthett, "Construction and interpretation of equal-tempered scales using frequency ratios, maximally even sets, and P-cycles," J. Acoust. Soc. Am. Vol. 107, No. 5, Pt. 1, 2725-2734, May 2000.
[4] M. V. Mathews, J. R. Pierce, A. Reeves, and L. A. Roberts, "Theoretical and Experimental Explorations of the Bohlen-Pierce Scale," J. Acoust. Soc. Am. Vol 84, 1214-1222, 1988.
[5] H. Bohlen, "13 Tonstufen in der Doudezeme," Acustica Vol. 39, 76-86, 1978.
[6] G. Balzano, "The Group Theoretic Description of 12-Fold Pitch Systems," Comput. Music J. Vol. 4, 66-84, 1980.
[7] W. A. Sethares, "Local consonance and the relationship between timbre and scale," J. Acoust. Soc. Am. Vol. 94, No. 3, 1218-1228, Sept. 1993.
[8] W. A. Sethares, Tuning, Timbre, Spectrum, Scale, Ch. 5, (Springer, 1998).
[9] R. Plomp and J. M. Levelt, "Tonal Consonance and Critical Bandwidth," J. Acoust. Soc. Am., Vol. 38, 548-560, 1965.
[10] E.Zwicker, G. Flottorp, and S. S. Stevens, "Critical bandwidth in loudness summation," J. Acout. Soc. Am. Vol. 29, 548, 1957.
[11] H. Helmholtz, On the Sensations of Tone, (1877). Trans A. J. Ellis, Dover, New York (1954).