Affine Regular Pentagon Sculptures

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Abstract

In this paper we shall describe how to apply symmetric linear constructions to a random non-planar pentagon to construct mathematically and artistically interesting sculptures, such as in Figure A. This process will always produce a nested set of affine regular stellar pentagons. This generalizes a procedure created by Jesse Douglas.

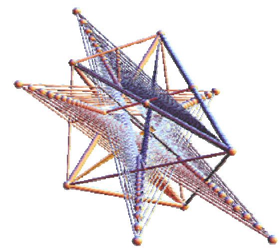

Figure A

Starting from a random non-planar pentagon, we shall describe a sequence of steps consisting of constructing line segments between pairs of known points followed by locating a new point on this line at a specified ratio of this length. At the completion of these steps, label the last point constructed v_1. We repeat the same process from the vantage point of each successive vertex of the original pentagon and label the last point of each sequence v_i respectively. By connecting the v_i in order we form a new pentagon. Although the final sculpture is not symmetric, since the process is symmetrical, this is called a *symmetric linear construction*. If this process is carefully selected, then the resulting pentagon will be planar and affine regular. By including variations on the theme, we can construct a collection of nested affine regular pentagons within the original pentagon. Affine regular pentagons, if oriented just right, will cast a shadow which is regular.

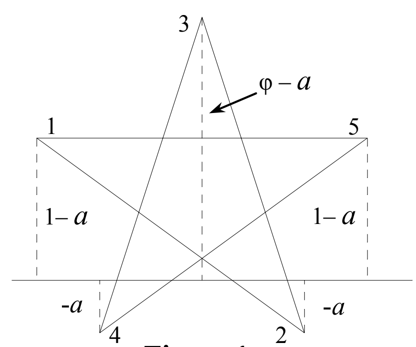

Figure 1: Calculating the weights

Given a random pentagon with vertices p_1, p_2, …, p_5, any finite sequence of steps of the type listed above can be written as a weighted average $w_1p_1+w_2p_2+\ldots+w_5p_5$ of these 5 points. By symmetry, all of the vertices of the final pentagon constructed can be written as $v_i=w_1p_i+w_2p_{i+1}+\ldots+w_5p_{i+4}$, where subscripts are modulo 5. If these weights w_i measure the distances from successive vertices of a regular stellar pentagon to an arbitrary line, with distances below the line measured as negative distances, then the pentagon constructed from these weighted averages will be affine regular stellar [1]. For our sculpture we shall place the line parallel to and a distance of a units above the base, as shown in Figure 1. By letting a vary, we shall be able to construct a nested set of affine regular stellar pentagons within our sculpture. If the distance between the base p_4p_2 and the horizontal line p_1p_5 is 1 unit then the height of the pentagon is the golden ratio $\varphi=(1+\sqrt{5})/2$. Thus, the desired weights, in order, are $(1-a)$, $-a$, $(\varphi-a)$, $-a$, $(1-a)$. While these weights will always generate affine regular pentagons, we must normalize these by dividing by $(2+\varphi-5a)$ before we can convert them into a sequence of linear steps. This gives us the following formula for calculating the location of the new vertices:

$$v_1=\frac{1-a}{2+\phi-5a}p_1+\frac{-a}{2+\phi-5a}p_2+\frac{\phi}{2+\phi-5a}p_3+\frac{-a}{2+\phi-5a}p_4+\frac{1-a}{2+\phi-5a}p_5 .$$

To convert these weights into a sequence of steps consisting of locating points on line segments we first rewrite the above in groupings of pairs of the form $(1 - c)x + cy$. The above is equivalent to:

$$v_1 = \left(\frac{2-4a}{2+\phi-5a}\right)\left[\left(\frac{1-a}{1-2a}\right)\frac{p_1+p_5}{2} + \left(\frac{-a}{1-2a}\right)\frac{p_2+p_4}{2}\right] + \left(\frac{\phi-a}{2+\phi-5a}\right)p_3 .$$

For any pair of points x and y, the point $(1- c)x + cy$ is located on the line xy at ratio c of the distance between x and y as measured from x towards y. For example, if c=½ then we find the midpoint; if c=1.5 then we go from x to y and then go an extra half the distance of xy past y; and if c= -½ then we travel half the distance of xy from x away from y. We start from any non-planar pentagon, such as the one shown in Figure 2. We can interpret the above formula, reading from the inside out, as: 1) Locate the midpoint $m_{15} = (p_1+p_5)/2$ as shown in Figure 3 below. 2) Construct the line segment p_2p_4 and locate the midpoint $m_{24} = (p_2+p_4)/2$. 3) Construct the line $m_{15}m_{24}$ between these midpoints and locate the point which is $-a/(1-2a)$ of the distance from the m_{15} towards m_{24}. Call this point q. 4) Construct the line from q to p_3 and locate the point which is $(\varphi-a)/(2+\varphi-5a)$ of the distance from q towards p_3. This last point is the desired point v_1. Repeating this in a symmetric manner for the other four points will construct points $v_2, \ldots v_5$. By connecting these in order, we obtain an affine regular stellar pentagon. Note that by letting a=0 we get the stellar construction first created by Jesse Douglas in [2]. To create the following sculptures, we repeat the above process for different values of a. Notice that steps 1), 2) and half of 3) do not need to be repeated for different values of a. Figures 4 and 5 consist of computer generated sculptures starting from the pentagon in Figure 2. Figure 6 consist of both the sculpture in Figure 4 and the sculpture in Figure 5.

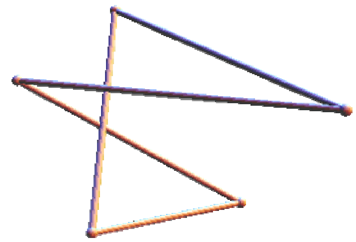

Figure 2: Non-planar pentagon

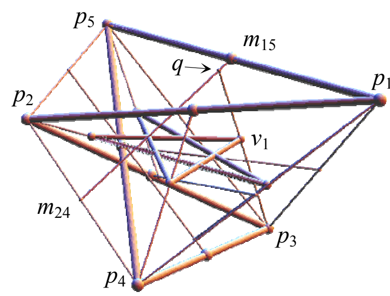

Figure 3: Construction $a = -0.1$

We finish with several observations. As the line in Figure 1 moves closer to the middle, $(2+\varphi-5a)$ gets smaller causing the resulting pentagons to get larger. This explains why values between 0.3 and 1.0 were ignored. As can be seen, the affine regular pentagons all lie upon the same plane, however, none of the lines constructed, other than the pentagons, lie on this plane. One wonders, what makes this plane special?

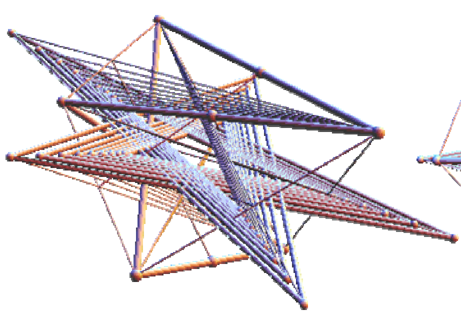

Figure 4: $1.0 \le a \le 1.4$

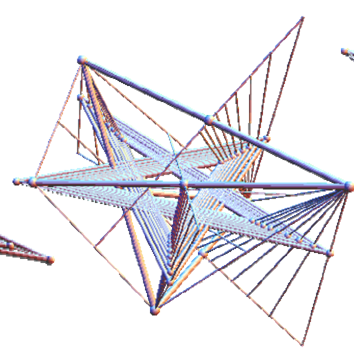

Figure 5: $-0.1 \le a \le 0.3$

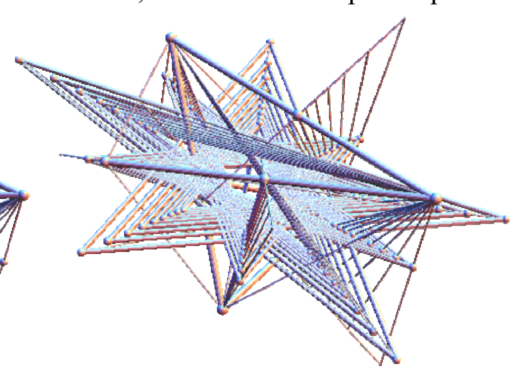

Figure 6: Combined Sculpture

References

[1] Douglas Burkholder, "Parallelogons and Weighted Averages," Revista de Física y Matemática (FISMAT) de la Facultad de Ciencias, Escuela Poltécnica Nacional (Quito-Ecuador), XII (2004).

[2] Jesse Douglas, *A Theorem on Skew Pentagons,* Scripta Mathematica, Vol. 25, 1960.