A Braided Effort:
A Mathematical Analysis of Compositional Options

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Abstract:

Artist James Mai created a system of forms in the developmental stages of his work *Epicycles*. This system offered mathematician Daylene Zielinski opportunities to provide mathematical analysis and to contribute to the final compositional organization of *Epicycles*. A set of eight new permutational forms are developed from a revised interrogation of a previously developed system of eighteen forms. The new set of forms lends itself to a variety of compositional arrangements including, with contributions from Zielinski, a "braided" ordering that creates a coherent sequence of the forms in the final work. This paper not only explicates the system of forms used in the resulting work, but it also illustrates the benefits and insights gained from interdisciplinary interactions between an artist and a mathematician during the development of a mathematically based work of art.

1. From Antecedent To Subsequent Forms

For artist James Mai, a system of forms previously developed for one painting may lend itself to later elaborations for new and different paintings. In these cases, subsets or entirely new sets of forms may be discovered by refining the rules that gave rise to the original set. This paper examines how such a new set of forms is derived from the system used in an earlier painting, *Permutations: Earthly*, which includes eighteen distinct forms created from permutations of upward and downward arcing semi-circles [1]. The new set of forms opens fresh considerations of visual characteristics and organization. Both aesthetic and mathematical analyses of these new forms have culminated in the digital composition, *Epicycles*, shown in Figure 5.

The motivation for the new set of forms is to convert the earlier closed shapes, used in *Permutations: Earthly*, to open-ended curves.[1] These curves are comprised of three semi-circular line-segments that connect four nodes distributed along a line. The original eighteen closed forms were composed of semi-circular segments from the parent figure, shown at the top of Figure 1. Each closed form yields four 3-segment paths, giving a total of 72 new forms. After eliminating rotational and reflective symmetric duplicates, there remain 28 visually distinct 3-segment forms, shown in Figure 1. Eight of these, shown in the last row of Figure 1, consist of an upward arc followed by a downward arc, then completed by an upward arc. This subset of eight curves is the formal basis of *Epicycles* and the subject of the discussion that follows.

[1] These eighteen closed shapes were derived from a set of four permutational forms developed by Victor Flach. See [1] for details.

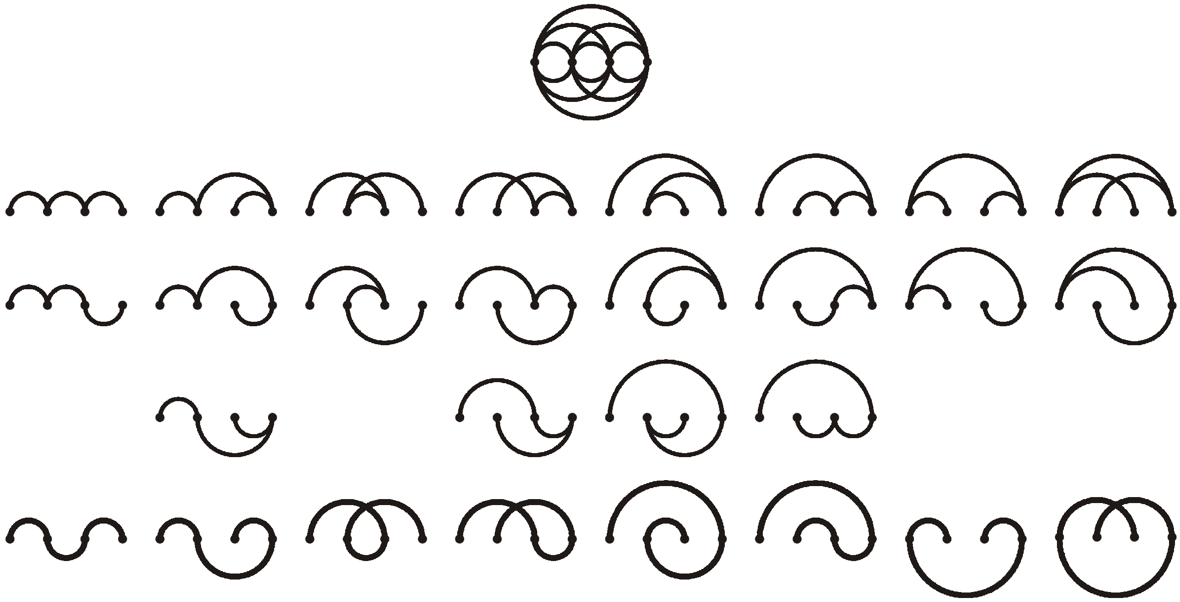

Figure 1: *Mai's set of 3-segment forms.*

2. Visually Understandable Characteristics

Mai employs a series of aesthetic strategies to make the permutational characteristics of his forms visually recognizable. The underlying logical relationships connecting the forms are converted into visual relationships of shape, size, location, and color so that they become available to the eye. For Mai, this translation of the conceptual to the perceptual is critical to the success of a mathematically based work of art. While a work of art may benefit from subsequent interpretation, Mai endeavors to give these logical relationships a fully visual existence in a "self-revealing" artwork that transcends dependence on verbal explanation.

The initial aesthetic strategies for that visualization included holding all forms to the same scale and stacking them vertically so that the eye may directly compare the different orderings and sizes of the component arcs. In so doing, the viewer can recognize that each form is both similar to the others in its construction from arcs and nodes and distinct from the others as a unique configuration of arcs. The vertical alignment also makes apparent the varying locations of endpoints among the eight forms, which suggest the next level of information to be visualized.

If we identify the beginning node of each form with the number 1, the next node along the curvilinear path with 2, and so on, then a left-to-right reading of the node numbers, along with the restriction that the semi-circles must follow the up-down-up pattern, uniquely determines each form. In this manner, each of the curves is an expression of a distinct permutation of the set {1, 2, 3, 4}. For instance, the first form in the last row of Figure 1 represents 1 2 3 4, while the second form represents 1 2 4 3, and so on. It should be noted that without the up-down-up pattern Mai has forced on each form, a permutation of {1, 2, 3, 4} does not translate into a unique visual form. All of the forms in the first column of Figure 1 can be seen as a visual expression of the permutation 1 2 3 4, and all forms in the second column are associated with 1 2 4 3. In fact, each column of forms displayed in Figure 1 is associated with a distinct permutation of {1, 2, 3, 4}, while each row is a distinct permutation of three directional choices from the set {up, down}.

Since there are 24 distinct permutations of the set {1, 2, 3, 4}, one might wonder why the final row of Figure 1 contains only eight forms representing the permutations 1 2 3 4, 1 2 4 3, 1 3 2 4, 1 3 4 2, 1 4 2 3, 1 4 3 2, 2 1 4 3, and 2 4 1 3. This is because all other permutations of {1, 2, 3, 4} correspond to forms that are merely rotations or reflections of one of the eight forms already appearing in this row of Figure 1. So, these particular configurations of the eight forms should be thought of as archetypes, where each of the eight forms represents all rotated or reflected versions of that configuration. Mathematically, these eight forms are a set of equivalence class representatives from the larger set of 24 forms where the equivalence is "is symmetric to." Interestingly, these equivalence classes are not all of the same size because each of the eight forms does not appear exactly three times in the full set of 24 forms. Each of the four symmetric forms appears exactly twice, while each of the four asymmetric forms appears four times.

When stacking the forms, Mai wanted to establish a vertical distribution of nodes that was equal to the horizontal distribution, creating a square grid of nodes upon which the forms would be arrayed. This resulted in severe overlapping of the semi-circular arcs of adjacent forms, causing visual confusion and obscuring the distinctness of each form. To accommodate this tight arrangement and eliminate the overlapping, Mai condensed each form vertically by changing the arcs from semi-circles to quarter-circles. This kind of proportional adjustment is not uncommon given the challenge facing the artist as he proceeds from considerations of the forms as independent objects to considerations of how the forms should be related collectively, while also retaining their individual characteristics, in the final composition.

One additional characteristic of the forms required visual clarification. The forms themselves offer no distinction between their endpoints, so an observer has no way to know which node corresponds to 1 and which to 4 in any permutation of {1, 2, 3, 4}. Mai solved this problem by using a fixed set of colors to encode the permutation in the nodes. Each form begins at an orange node, which is connected by an upward arc to a red node, which in turn is connected via a downward arc to a blue node, and is completed by an upward arc to a green node. While the different configuration of arcs makes each form visually distinct, the colored nodes provide the link to permutations of {1, 2, 3, 4}. For the purposes of reproduction in this paper, the color-coding has been replaced with a shape-coding for the nodes where the progression: orange, red, blue, green, has been replaced with: hollow circle, hollow diamond, solid diamond, solid circle.

3. Possibilities for Arrangements of Forms

After establishing the eight forms to be displayed, their sizes and vertical distribution, and the distribution and color-coding of the nodes, the set of forms initially offered no obvious order for their vertical arrangement. Compositionally, this ordering was the final formal question to be addressed. Mai's initial observation that there were four symmetrical and four asymmetrical forms presented an opportunity for arranging the eight forms into two groups. It seemed clear that the first form in the last row of Figure 1 should be at the top of the vertical composition, since it was the simplest permutation of {1, 2, 3, 4} and the most basic of the eight forms. Since it is one of the symmetrical forms, this suggested that the remaining three symmetrical forms should be grouped immediately beneath, followed by the four asymmetrical forms beneath those. This initial composition is shown in Figure 2a; recall that the color-coding of nodes has been replaced with a shape-coding in this paper for purposes of clearer reproduction.

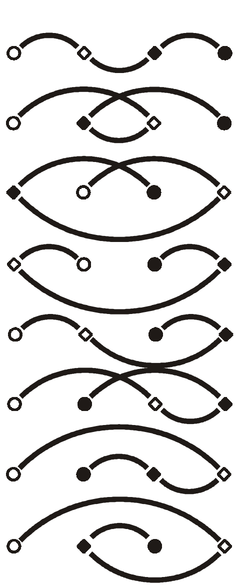

Figure 2a:
The forms.

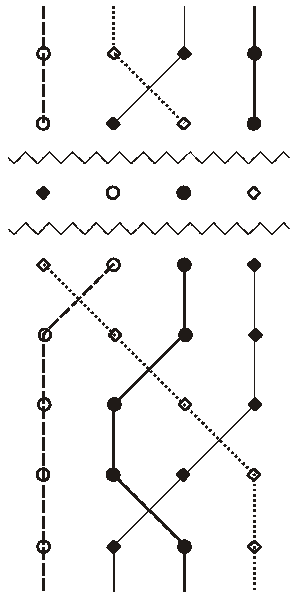

Figure 2b:
Almost a braid.

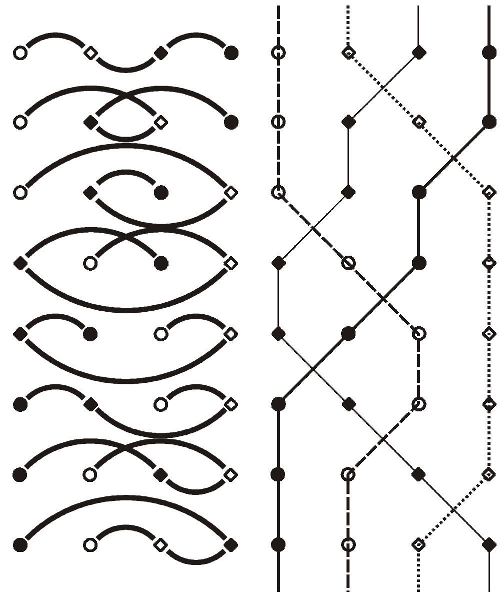

Figure 2c:
A braided sequence of the forms.

At this intermediate stage, when Zielinski saw the composition represented in Figure 2a, she immediately noticed that connecting the corresponding nodes from form to form almost produced a braid. The "almost" carries two distinct qualifications in this case. First, the third form cannot be created via a single braid move from the second form as indicated by the first break shown in Figure 2b, because a mathematical braid is allowed to cross only one pair of adjacent strands at a time. In addition to this obstruction, the node arrangement in the fourth form cannot be produced by any single braid move from the third form. Secondly, a formal mathematical braid must distinguish between over and under crossings, but Mai's forms have no visual characteristics that convey such crossing information. With a bit of trial and error, Zielinski found an ordering, seen in Figure 2c, that did create a braid void of crossing information. As we shall see later, this sequence is one of twelve braided orderings that begin with Mai's preferred form. For the remainder of this paper, we shall use the word braid in this looser context in which crossing information is disregarded.

Despite the lack of crossing information, the artist was immediately interested in this new ordering schema and posed a new question: Can the braided sequence of forms be made cyclical, whereby the bottom form can also be transformed via one braid move to the top form? This is primarily an aesthetic question for Mai, who frequently works with circuity and closed visual forms as expressions of wholeness and transcendence. Mai pursued the question by using a tree structure to diagram possible sequences of the eight forms. Mai's graph showed him that cyclical braiding was not possible when one begins with his preferred form. Furthermore, it suggested that there were only two braided sequences of the forms; as we shall see, this is correct. Further analysis by Zielinski established that cyclical braiding is indeed impossible and that, although there are twelve braided sequences that begin with Mai's preferred form, there are only two sequences of the forms that can be generated as braids on the nodes. To establish these and other facts, we need additional representations of the forms as graphs.

First, we will pull back to the full set of 24 configurations of the eight archetypal forms. The graph in Figure 3 shows exactly which configuration can be produced from any other via a braid move that swaps the left-most pair of nodes, the middle pair, or the right-most pair. Hence, the edges in this graph are labeled with an *L*, *R*, or *M* according to which move was used. As long as one is following a path of edges in the graph, the sequence of forms given by those vertices is a braided sequence. Thus, this graph displays for the artist all possible arrangements of these forms that can be produced through a sequence of braid moves. Consequently, any cycle in the graph gives a braided cyclic ordering of the forms. Hence, a fuller answer to Mai's question comes from investigating the eight-cycles of the graph in Figure 3. Though it is not difficult to find eight-cycles in this graph, it is tedious to confirm that not one passes through each of the eight archetypal forms.

Just as with Mai's tree, this analysis takes quite a bit of time and concentration. Before we redirect our analysis to a simplified version of the graph below, we need to take note of a particularly interesting, and mathematically relevant, fact. A handful of the forms produce the same type of new form regardless of whether the left-most or right-most nodes are swapped. An example of this can be seen by examining the relationship between what the authors refer to as the kink-form and the spiral-form. A kink-form can be found to the left of the top-most 'M' in the graph below, immediately to the left of Mai's preferred form, and one of the four spiral-forms can be found immediately to the left of that kink-form. It is not hard to confirm that each kink-form is connected to two distinct spiral-forms, even though each spiral-form is connected to only one kink-form.

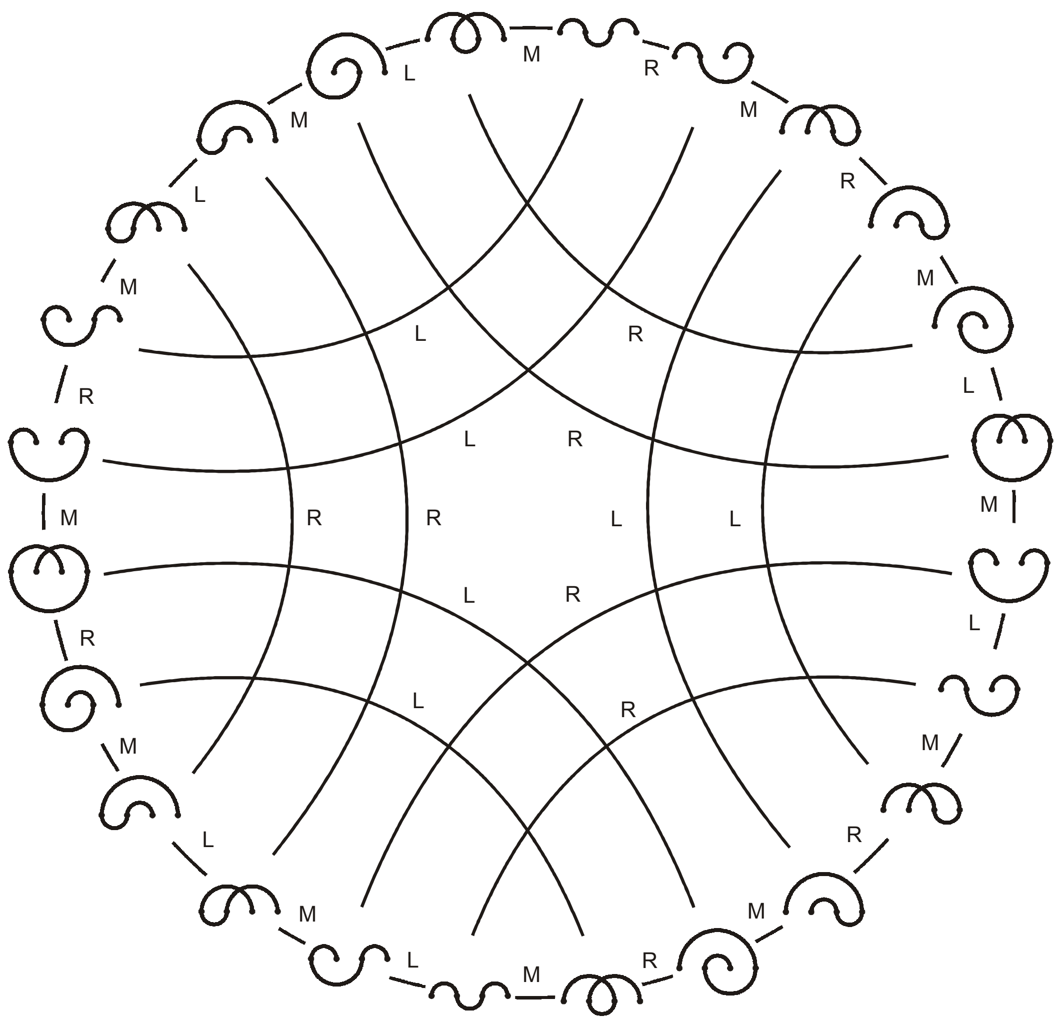

Figure 3: *All 24 configurations of the eight forms with related braid moves.*

A simpler graph arises when one disregards which braid move was used and which configuration of a particular form resulted. We do want to retain, however, the number of distinct paths that exists between each pair of archetypal forms. Thus, we will create a multi-digraph that summarizes the information in the more complex graph above. Our new graph has the prefix *multi-* since we will connect certain forms to others with multiple edges because of the phenomenon mentioned above. The prefix *di-* indicates that we have added directions to our edges. Each edge in Figure 3 is bi-directional, but we need some uni-directional edges in Figure 4a to indicate when we have choices and when we do not. The graph in Figure 4a shows each of the archetypes with edges connecting any two forms that have configurations that can be transformed to each other via a single braid move.

The question of looking for braided sequences of the eight forms boils down to looking for paths of edges in this graph that visit each vertex/form exactly once. Such a path is called Hamiltonian. A cyclic braided sequence of the eight forms would be represented by a Hamiltonian cycle in this graph, but not too much effort is required to see that no such cycle exists. One can see, however, that there are several Hamiltonian paths. Figure 4b shows all possible Hamiltonian paths that begin with Mai's preferred form. There are eight distinct paths shown in upper portion of Figure 4b and four more in the lower portion giving the total of twelve possible braided sequences of the forms mentioned earlier. Interestingly, some of these sequences differ only in their vertex colorings and not in either the order or the forms or even the configurations of the individual forms. Several other braided sequences of forms can be created beginning with forms different from Mai's preferred form. In fact, there are multiple Hamiltonian paths beginning from each of the forms that are connected to only two others in Figure 4a, but there are no such paths that begin at either of the two, left- and right-most, forms in Figure 4a, which are connected to three other forms. In total, there are 64 braided sequences of all eight forms.

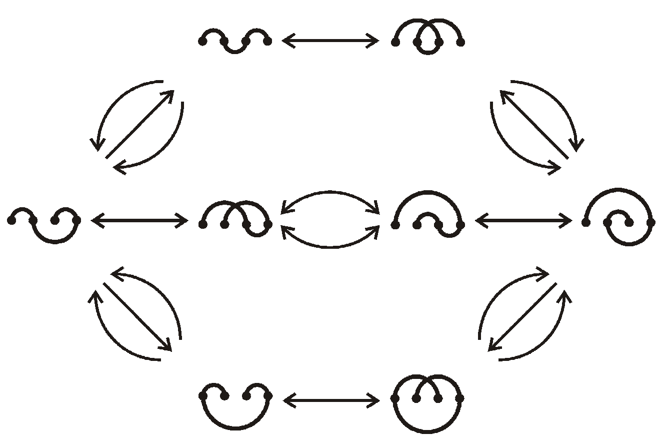

Figure 4a:
Multi-digraph of the eight archetypes
and the number of braid moves that relate them.

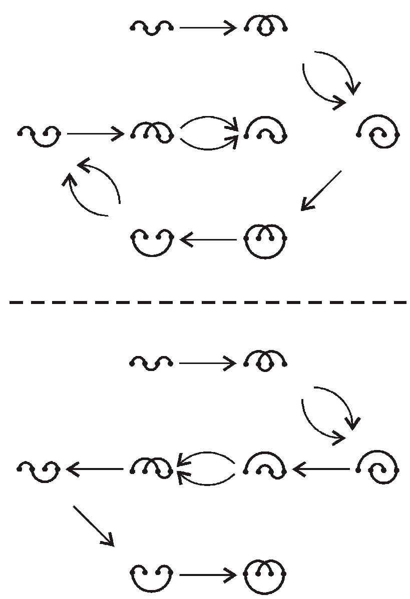

Figure 4b:
All Hamiltonian paths
beginning at Mai's preferred form

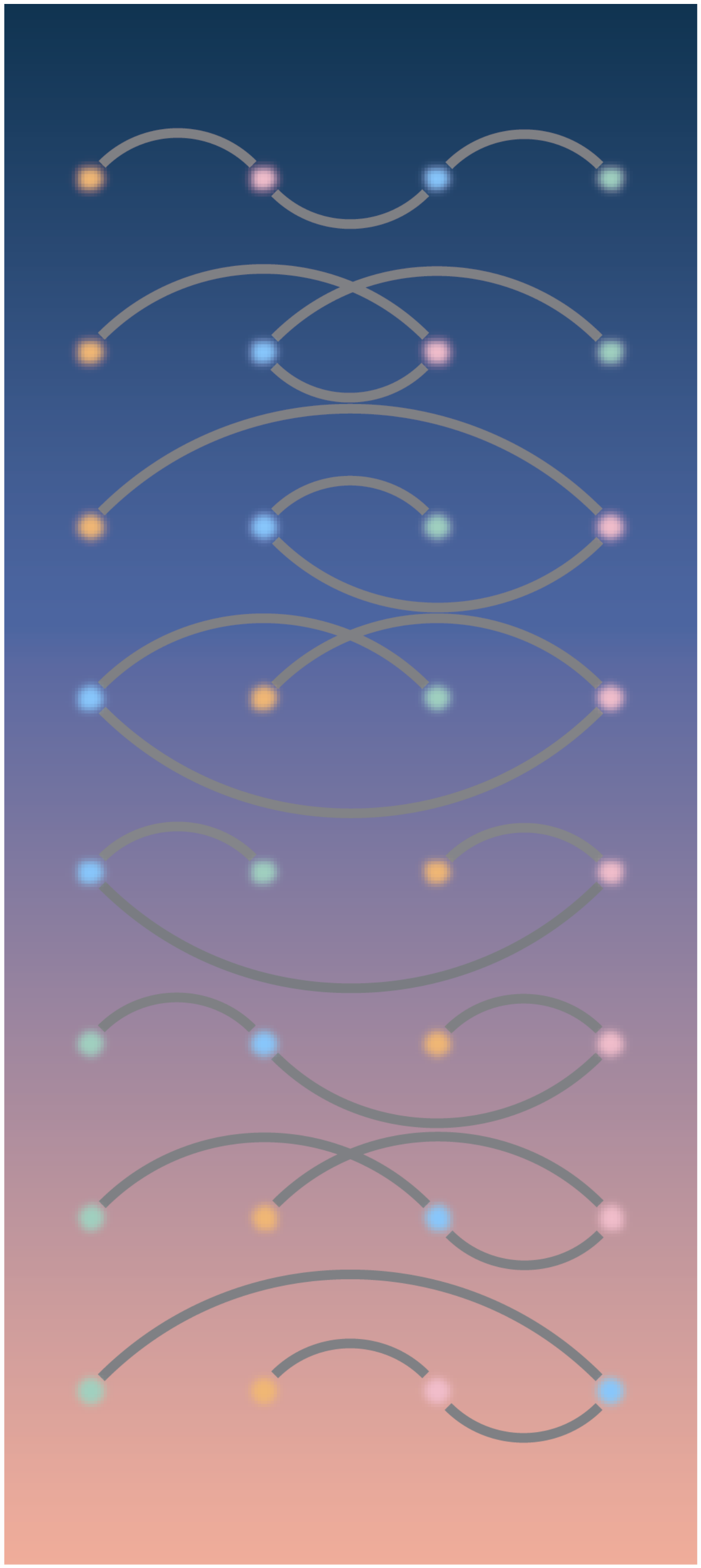

Figure 5: Epicycles*, digital print, 40 x 18"*

4. Particularizing the Abstract

Mai sees the task of art as not only to define new visual forms but also to suggest metaphoric associations between those forms and other events or experiences. Mai uses minimal figurative additions to his final compositions so as to stimulate imaginative allusion rather than to present straightforward illustration of a subject matter. The subject and title of this artwork refer to the efforts by early astronomers to understand the complex retrograde motions of planets, which at times trace out unusual looping and winding paths against the background of fixed stars. Working within the Ptolemaic model of the universe, early Greek astronomers posited epicycles, which are smaller circular orbits in combination with the larger orbit about the Earth, to explain a planet's back-and-forth motions across the sky [2]. Mai saw a similarity between his forms' nodes with their curvilinear connections on the one hand and the planets with their curvilinear retrograde motions on the other.

The permutational forms discussed in this paper, although clearly emerging from purely abstract, formal procedures, were lent this particular, figurative identity by means of just a few visual cues: a vertical color gradation in the background that matches the fading light of the sky at dusk, a slight blurring of the nodes to suggest the glow of planets as they appear to the naked eye, the combination of points and curves to suggest astronomical diagrams, and, of course, the title, *Epicycles*. The challenges faced by early astronomers to read accurately the order underlying the changing positions of the planets seems not altogether unrelated to the search by artists and mathematicians to discover new orders in the universe of form and number.

References

[1] J. Mai and D. Zielinski. Permuting Heaven and Earth: Painted Expressions of Burnside's Theorem. In *Conference Proceedings of Bridges: Mathematical Connections in Art, Music and Science, 2004*. Eds. Sarhangi, Reza and Séquin, Carlo.

[2] G. J. Toomer. Ptolemy and His Predecessors. In *Astronomy Before the Telescope*. Ed. C. Walker, pp. 74-76. New York: St. Martins Press, 1996.