The Borromean Rings - A Tripartite Topological Relationship

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1. Introduction

The following dialogue is between Bertie and George. Bertie and George are following an evolutionary path upward from the Void. In this episode they encounter a frighteningly beautiful ternary relation. The following is an account of their conversation and the epistemological issues that emanate from this experience.

The reader will note that one might interpret Bertie as a precursor to the twentieth century philosopher Bertrand Russell and George as a precursor to the twentieth century maverick mathematician George Spencer-Brown. Such interpretations should be taken lightly.

2. An Encounter with a Ternary Relation

Bubbling up from the void, you never know what sorts of structures are likely to come into view. Old Bertrand here thinks that it should all happen in completely orderly fashion, with the emergence of binary relations as the basis of everything.

Bertie, I said, why do you think that binary relations are the primary generator of all form?

He says to me, well look here George, distinctions are made. Distinctions are made within and without the spaces of the distinctions that have already been made. Everything comes out of that process of distinguishing, and distinguishing distinguishing. A distinction connotes a binary relation between its parts. One part is dominant (marked) to the other. There is an ordered binary relationship between the two sides. This leads to higher forms and all of Art and Mathematics.

Amazing fellow this Bertie, that he could make a speech like that when he was still an amoeba but he did, waving a primordial flagellum from time to time. We were all together at one point when the argument started, and that argument went on, hot and heavy, while we underwent uncountable transformations of form in our journey out of the void. But this time I figured I had him. There were new forms underfoot. I said:

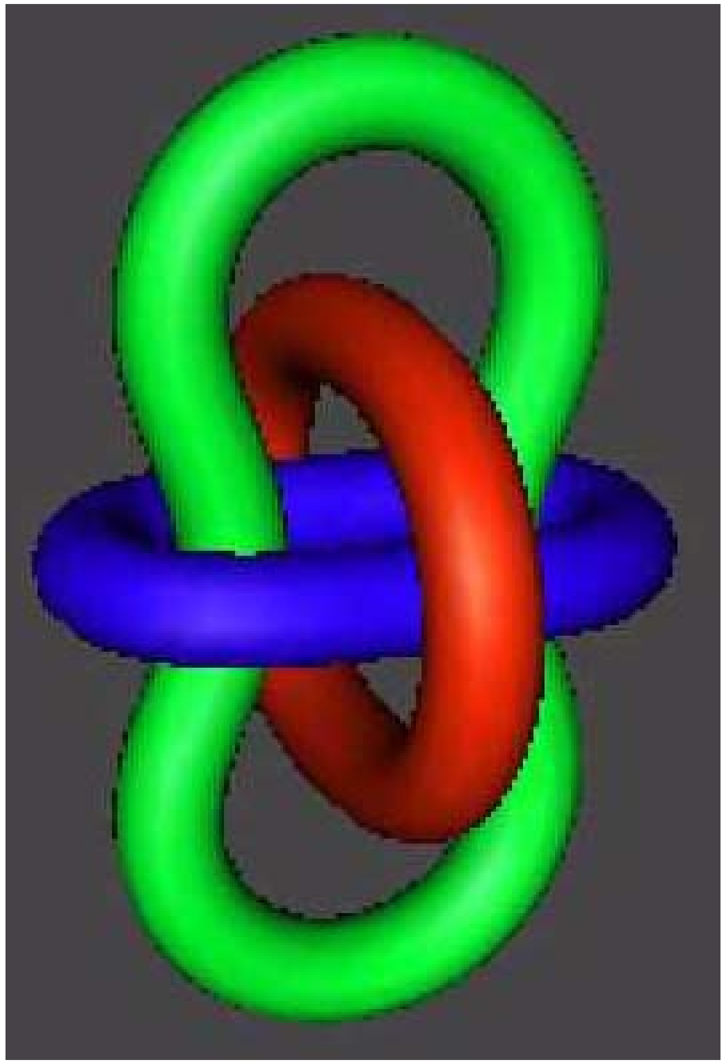
Bertie!! Look sharp! There! In red, blue and green!!!!

And there she was indeed, in all her curvaceous beauty, *a topological tripartite relation in full bloom*.

Bertie, I says, look at her. She's autopoetically constructed from three colored toroids, red, blue and green. If you remove any one toroid, she'll just float apart and drift away. There's binary relations in there, but she's a true integral tripartite relation. Look at her structure:

Red surrounds blue.
Blue surrounds green.
Green surrounds red.

Figure 1. *Borommean Rings*



Bertie, if I were to ask you to make a circularity like that in your binary logic, why you'd probably blow a gasket and send off a packet of paradice (paradoxes, paradoxen? (does anybody know the plural of paradox?)). She's held together by a circularity. Floating out there in the void of topological space girdled in glorious triparticity. One and yet Three. Made from Three, All One. A Trinity in the pregeometry prior to space and time.

I paused and Bertie started up. He says: Good Lord George she is beautiful and triparitous indeed. A vision of the Trinity, an angel of circularity. Almost enough to make me a believer. You would have me believe that she's an *elemental form*. You would have me take her and my beloved empty set , and put them in the same category. But I shall have none of that! She is a *composite*. Mark my words and I shall prove it to you.

Wait Bertie! (I says.) Before you go into an interminable analysis of this situation, think! You agree that the Borommean Rings (That's what we called them then in our amoebic and paramecium states respectively. I t was not until millennia later that the Borommeo Family in Italy (who would have guessed "Italy" way back there at the beginning?) used the Rings as a coat of arms. But somehow, we did call them by that appellation. Time is an illusion.) are a single entity and yet composed of three entities, the individual rings. So of course they are composite. Your binary relations are composite as well, being composed of the two sides of the distinction that gives rise to them. See Figures 1 and 2 [PC].

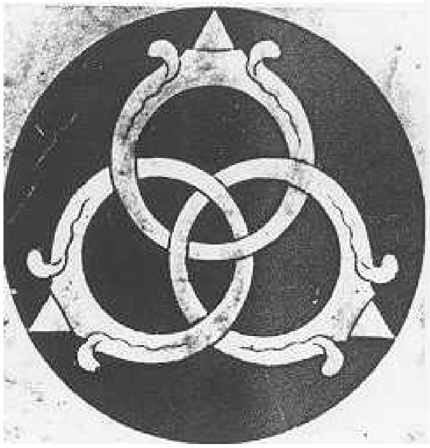
 

Figure 2. (Left) *From the Capella Rucellai in the Church of San Pancrazio in Florence* (1467). *Designed by Leon Batista Alberti, the rings are said to be a symbol of the Medici Family.*

Figure 3. (Right) *Odin's Triangle, or the Walknot, used by the Norse people of Scandinavia.*

So George, you regard these Borommean Rings as equally fundamental with the first distinction? Says Bertie. I do, says I. You can analyze them every which way, and you'll come back and realize that there is fundamental topology occurring at the bottom of the world. Those rings were co-created along with your empty set and your notion of binary distinction and binary relation.

Bertie goes on: A little set theoretic magic, a few equations and the Borommean Rings come into existence with no more than a continuum of binary distinctions! They are a beautiful example of a composite form unfolding in the (not yet written) 137th volume of Principia Mathematica.
There goes Bertie again, only an amoeba but going on about the 137th volume of Principia Mathematica, his great future work on the emanation of the World and Mathematics from pure Logic! I say to him. Bertie, I do not doubt it. But look, how about the empty set, isn't that also a composite?

Bertie: The empty set stands for a distinction in the void. The empty set itself is distinct from the void. Its contents are void. The empty set is a composite of nothing and the first something. It is a true composite. Without the empty set there would be nothing at all.

George: That reminds me of a riddle: What is better, eternal happiness or a ham sandwich?

Bertie: Well, nothing is better than eternal happiness, and a ham sandwich is better than nothing. Therefore, a ham sandwich is better than eternal happiness.

George: That's Logic for you.

Bertie: But look here, you do agree that the Rings are complex!

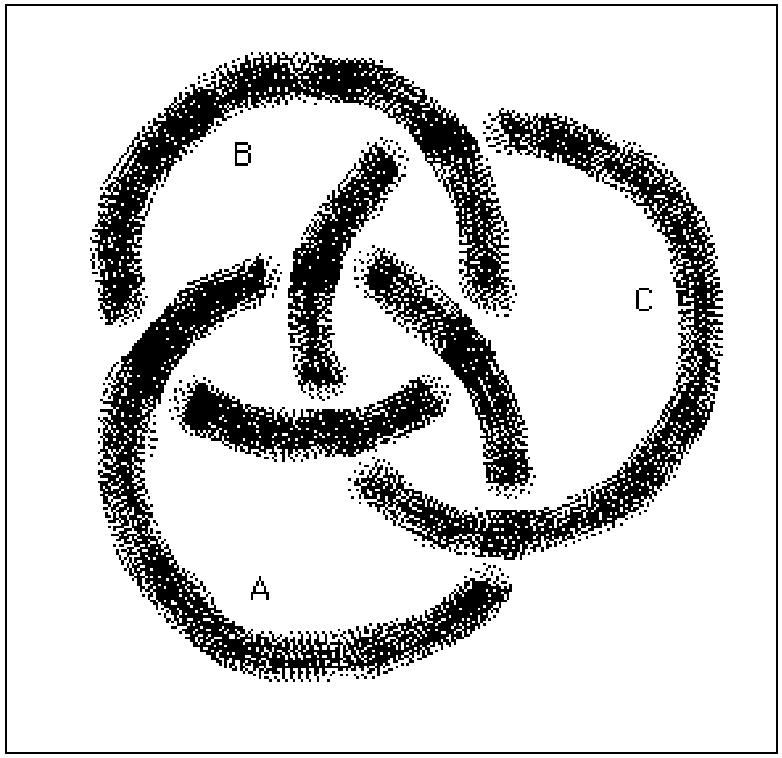
George: I agree, but they are not so complex as you might think.
I am going to tell you about knot set theory [KL], and how a generalization of it, captures the Rings.

Bertie: You'd best do that in the next section.

George: Well yes, but first let me just draw a diagram.

Bertie: You mean the diagram below?

A over B
B over C
C over A



George: Well, now its above us. They keep shifting our sentences down the page, and we are only located in those sentences after all.

Bertie: We used to be an amoeba and a paramecium, and now we are just a pair of alternating disembodied bits of text.

George: Well I am still a sign of myself!

Bertie: Yes, yes. But what about the diagram above?

George: Well it is an example of a link diagram, and it represents the Borommean Rings.

Bertie: I see that. I suppose you are now going to give the Rings purely syntactic existence inside a language of diagrams.

George: Of course. Wouldn't you like to exist in a language of diagrams?

Bertie: And be tied in reference to some particular form of diagram? That's not for me. I will take my chances in these sentences. Sentences are based ultimately on the binary relation of marked/unmarked. Nothing unremarkable there, and I feel quite safe. Your diagrams make me nervous.

George: Oh Bertie, take a leap. Read the next section.

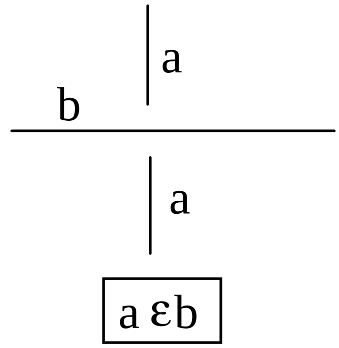
Bertie: That would indeed be a leap. How is a text supposed to read another text? Am I to become interpretant as well as sign and signifier?

George: You interpret all the time. Let's go to the next section.

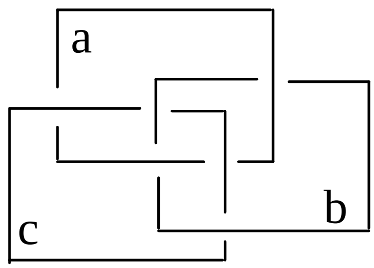
3. Knot Sets, Ordered Knot Sets and the Borommean Rings

We shall use knot and link diagrams to represent sets. More about this point of view can be found in the author's paper "Knot Logic" [KL]. Diagrams were first used in this way by Flatlanders before the invention of the third dimension. After that, it turned out that the diagrams represented knotted and linked curves in space, a concept far beyond the ken of those original flatlanders.

Set theory is about an asymmetric relation called *membership.* We write **a** $\in$ **S** to say that **a** is a member of the set **S.** In this section we shall diagram the membership relation as follows:



This is *knot-set notation.*



a = {b,b}
b = {c,c}
c = {a,a}

Here again are the *Borommean Rings.* The Rings have the property that if you remove any one of them, then the other two are topologically unlinked. They form a topological tripartite relation. Their knot-set is described by the three equations in the diagram. Thus we see that this representative knot-set is a "scissors-paper-stone" pattern. Each component of the Rings lies over one other component, in a cyclic pattern.

4. Commentary

Bertie. Interesting manuscript. Where did you get it?

George. I told you. It is the third section of this paper.

Bertie. No no! I mean where did you find it before it was included in this paper. Surely that document has a different source than our conversation.

George. I don't know about that, but you are right. The document was found written on parchment and attached to an old bronze copy of the Borommean Rings. It could be centuries old.

Bertie. How could that be? It refers to a paper written by Kauffman in 1994!

George. You are naive as always. This is a result of time travel.

Bertie. Time travel you say?

George. Yes. The old breed of mathematicians, before the twentieth century were prone to travel forward in time, sometimes stealing the work of future mathematicians, sometimes just referring obscurely to their papers. In this case the author of this paper on knot-sets has borrowed from Kauffman's 1994 paper on Knot Logic, but he or she has been kind enough to give a reference. Other time-travelers could then find the paper and read it.

Bertie. What happened? Why don't we time travel anymore?

George. Actually, Bertie, it is your fault. Some mathematicians from the 19th century went forward and read your now-famous Russell Paradox. They were so shocked, that they quit time travel. Of course others had gone farther forward in time, but there was some singularity associated with the Russell paradox. It induced a one-way blindness that kept mathematicians then on from traveling forward into the future and taking future results from our contemporaries.

Bertie. You're pulling my leg again George. In my opinion, this manuscript proves my point. It constructs the Borommean knot set. All this occurs in a formal system where all the language is built from binary relations. Why I warrant that if Whitehead were born yet, the two of us could build a mathematical system based on logic and binary relations so that these Rings would be one of the simpler constructions of the system. Why I wager I will call this system Principia Mathematica!

George. Well you will have to evolve a bit for that Bertie. It is deucedly difficult for an amoeba to write a book. I am sure that any such endeavor will be incomplete, and you will always have to ask yourself whether the Rings in Principia are really the same as the beautiful vision of colored rings surrounding one another that we have seen today. Perhaps I shall be able to reduce the noise level of your work after you have done it. That will be a good future for a paramecium.

References

[KL] L. H. Kauffman. Knot Logic. In *Knots and Applications* ed. by L. Kauffman, World Scientific Pub. (1994) pp. 1-110.

[PC] Peter Cromwell, University of Wales, <http://www.liv.ac.uk/~spmr02/>.

[GSB] G. Spencer-Brown, "Laws of Form", Julian Press, New York (1969).