Mathematical Measures of Syncopation

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Abstract

Music is composed of tension and resolution, and one of the most interesting resources to create rhythmic tension is syncopation. Several attempts have been made to mathematically define a measure of syncopation that captures its essence. A first approach could be to consider the *rhythmic oddity property* used by Simha Arom to analyze rhythms from the Aka pygmies. Although useful for other purposes, this property has its limitations as a measure of syncopation. More elaborate ideas come from the works by Michael Keith (based on combinatorial methods) and Godfried Toussaint (based on group theory). In this paper we propose a new measure, called the *weighted note-to-beat distance* measure, which overcomes certain drawbacks of the previous measures. We also carry out a comparison among the three measures. In order to properly compare these measures of syncopation, we have tested them on a number of rhythms taken from several musical traditions.

1 Introduction

Music is emotional and has the power to create complex universes of psychological feelings. An important question brought up by psychologists, critics, musicologists, composers, interpreters and listeners in general is how music actually stirs the emotion, that is, what the specific processes that transform sonorous material into emotions are like. In the last few decades researchers from various disciplines have shown a growing interest in this question and others not less fascinating, e.g., the problem of meaning in music (designative meaning or embodied meaning), the role of learning in the musical experience, the description of the psychological changes prompted by music, to name but a few (see [10, 6, 8]).

Psychologists of music have discovered that emotion caused by music may have its origins in a continuous process of creation and relaxation of tension [10, 7]. This process comprises the stimuli themselves, the expectations music arouses in the listeners (undoubtedly determined by their familiarity with the given musical style and their past experience among other factors), and finally the tension created between those expectations and its actual resolution in the musical piece.

The presence of tension/resolution occurs at all levels in the musical phenomenon. It can be encountered in the melodic, harmonic and rhythmic elements as well as in timbre and musical form. Normally, tension is well balanced among all the distinct musical elements.

This work is concerned with the rhythmic devices that create tension in a musical piece. In particular, we turn our attention to syncopation, one of the most striking and transgressive mechanisms to produce rhythmic tension. Syncopation is easy to perceive but difficult to define properly as its manifestations can be numerous and of distinct nature. In the following section, we formally define syncopation within an abstract framework. Next in section 3, we address the problem of mathematically formalizing syncopation; we review previous works and introduce our measure of syncopation, the so-called *weighted note-to-beat distance* measure. In section 4, our measure is tested on several rhythms (basically timelines or *claves*) taken from widespread musical traditions.

2 Definition of Syncopation

The authoritative *Harvard Dictionary of Music* [13] contains the following definition of syncopation, that we believe captures its essence: "Syncopation: a momentary contradiction of the prevailing meter or pulse." This very dictionary details further its definition and adds that "syncopation may be created by the type of note-values themselves or by accentuation, articulation, melodic contour, or harmonic change in the context of an otherwise unsyncopated succession of note-values." That clarifies two subtle points, namely: firstly, for a contradiction to exist there must be a template of regularity to contrast with; secondly, that contradiction may be revealed through various musical elements, not only purely rhythmic aspects. Moreover, syncopation can be materialized either as a momentary change of the primary character of the meter or as a contradiction between strong and weak beats against other parts of the musical texture whose metrical context is fixed.

The former kind of syncopation, the change of meter, may be produced through a transformation from duple to triple (hemiola), or others of similar type. This rhythmic device was much used in cadential progressions by composers up to and including the Baroque period; it is also frequently found in the music of Beethoven. In Figure 1 we have an example from Handel's Concerto Grosso no. 4, bars 97 to 99. In this example we notice a binary grouping on the upper voices against a ternary grouping on the lower voices. That creates a tension through two conflicting meters, whose resolution is reached on the final A minor.

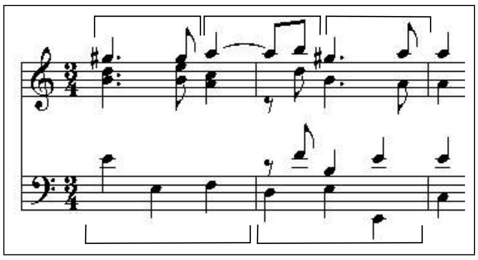


Figure 1: Hemiola as a form of syncopation.

The other type of syncopation involves the attacks between beats instead of having them on the beats as the main form of contradiction. As stated above, there must be a fixed metrical context, in this case a fixed pattern of strong and weak beats on top of which syncopation becomes prominent.

The above complex rhythmical devices used by composers in the Baroque and Classical periods naturally have their origins in music from earlier periods of Western music history, stretching back to the first forms of precisely notated rhythm of the Mediaeval period. As sacred choral music developed from the single line of Gregorian Chant to several independent parts singing simultaneously, composers needed to synchronize these parts using a fixed metrical pulse and precise temporal relationships between the parts. This resulted in concepts of *on the beat* and *off the beat* which were exploited by composers in a device called the Hoquetus (also known as Hocket). Here a single melody is shared between two parts, one singing on the beat and one on the off-beat:

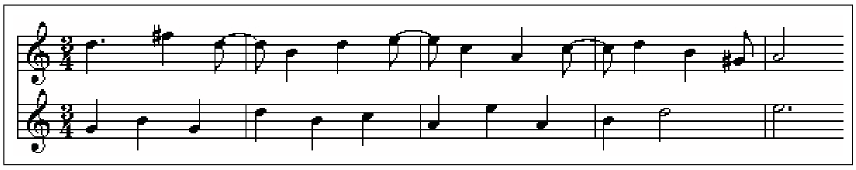


Figure 2: Mediaeval music example.

Although the overall resultant rhythm may be considered not to be syncopated, the individual strand of the upper voice, as it is off the beat apart from the last note, lends a particularly syncopated flavor and added rhythmic vitality characteristic of syncopation.

This device can also be found in the music of the Baroque period and the music of J. S. Bach (for example, see his invention no. 1). In that work, it can been observed that although syncopation is present, the overall effect is one of balance between *on the beat* and *off the beat*, reflecting Bach's compositional concern with creating a balanced and ordered universe.

One feature of the concept of *off-beat* is that it does not necessarily need to be half-way through the beat, as in the above examples. Beethoven placed his offbeat three quarters of the way along, in other words closer to the next beat than the beat to which it belongs. This device is known as anticipation and in this example produces a lopsided and slightly jazzy effect which may even be considered to be humorous. An example of that can be found in his 'Tempest' Sonata Op.31, no.2, 3rd movement. This lopsided placing of the offbeat produces an effect of imbalance which lends a sense of drama and tension characteristic of Beethoven. Here one can see a composer exploiting and stretching notions of on- and off-beat to produce a heightened sense of syncopation and rhythmic unpredictability.

However, such rhythmic experimentation can be considered to be mild compared to the revolutionary techniques used by the Russian composer Igor Stravinsky in his 1912 ballet *The Rite of Spring*, in the *Auguries of Spring (Dances of the Young Girls)*. Although here we can see an unchanging 2/4 meter with an unchanging quaver pattern, the patterns of accents (stressed notes) are ever-changing and unpredictable. In the classical music of previous generations, melodic line was paramount. In this example, the melodic line is reduced to a single note and syncopation takes center stage. The accentuation pattern produces quaver groups in the following sequence: 9,2,6,3,4,5,3. Stravinsky claimed that *The Rite of Spring* was a product of his intuition and that the piece came to him in a dream. Indeed, although the work has been extensively analyzed, there has been no foolproof evidence of any rational systems behind the music, although with the above pattern certain observations can be made. Firstly, even though the numbers are ever-changing, the total number of beats in the cycle is one often found in classical music: 32. In a waltz by Johann Strauss, for example, a phrase of music may last 32 beats or a section may total 32 bars. However, in the waltz one would expect the section of 32 to divide into two equal halves of 16, while the Stravinsky example has no such halfway division. Rather, the result is two unequal halves of 17 and 15, and the natural propensity of composers to subdivide into 16, 8, 4 and 2 beats (or bars) is replaced by an irregular sequence of 7 numbers only two of which divide into 32. For this reason, some theorists have found it more constructive to analyze this and other works by Stravinsky by taking the smallest beat-unit to which the other notes in the group are added, as opposed to taking larger group of beats (the bar or phrase) and sub-dividing, the latter being a set feature of previous classical composition. The concept of working from the smallest unit is known as *additive* rhythm, whereas that of regular divisible meters is known as *divisive* rhythm. The propensity for creating irregular rhythmic structures was taken up widely by composers of the early 20th century including the Hungarian composer Bela Bartok, whose piano piece *Syncopation* is a maze of rhythmic twisting and turning, unexpected silences and interrupted patterns which, as in the Stravinsky example take precedence over melodic and rhythmic concerns.

By no means was Stravinsky the first musician to use additive rhythms in music. In folk traditions from around the world this way of making rhythm has been in existence, as in the folksongs of Russia, Bulgaria and the music of the Aka Pygmies of Central Africa, whose music has been extensively researched by Simha Arom (see [1]). The percussion music of the Aka pygmies, like that of the Brazilian Samba, for example, consists of a web of cyclic rhythmic patterns layered in several individual strands. However, as in the following example, it can be seen that one such strand has a complex internal rhythmic structure (see Figure 3):

Figure 3: Music from the Aka pygmies.

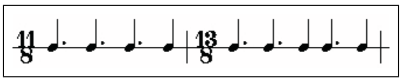


Figure 4: Music from the Aka pygmies notated with a different meter.

The Aka pygmies have no tradition of notated music and this example has been translated to Western music notation. The music fits neatly into 4 bars of 6/8 time, giving a total of 24 quaver beats. However, on inspecting the internal groupings of beats, we see that the 24 beats are not divided evenly - rather we get an irregular sequence as follows: 3,3,3,2,3,3,3,3,2. Notationally, the example comes across as highly syncopated, although, in fact, it may be misleading to represent this music in Western notation with its given features on strong and weak beats, features which may not be an integrated part of the Aka musical tradition. It is also possible to represent the same rhythm but using irregular time-signatures, such as those Stravinsky may have used for his additive rhythmic structures (see Figure 4). Once again, from these groupings we can see that the cycle is not divided equally through the middle (12+12) but that we get two unequal halves of 11 and 13. This principle of a rhythmic cycle consisting of two unequal halves is found in much Aka music, resulting in a unique form of complex rhythmic tension. Once again, it seems that rhythms are created from the smallest unit in multiples of 2s and 3s, rather than from equal subdivisions of the cycle length of 24 beats which raises parallels with Stravinsky's use of additive rhythm. The question as to whether the Stravinsky and Aka examples are syncopated is made more complex due to the nature of additive rhythm. As divisive rhythm creates predictable notions of on-beat and off-beat, so the music can be perceived as going with the beat (non-syncopated) or against the beat (syncopated). As the Stravinsky and Aka examples do not set up any such binary relationship of beat and offbeat, such a template does not exist for the listener and so rather than hearing on- and off-beats he may perceive the rhythm in the additive sense of smaller beats grouping together to form larger beats. In this sense it could be maintained that syncopation is in its nature more a feature of divisive rhythm than additive rhythm.

Another type of 20th century music which makes explicit use of syncopation is jazz. Here the music more often than not relies on a steady underlying pulse given by the drummer and bass player against which the musicians can react against the beat by playing on the off-beat in a syncopated manner. This is a characteristic of the *swing* style of Duke Ellington whose famous composition *It Don't Mean a Thing* is defined by this feature. *Swing* jazz consists of the off-beat being moved slightly beyond half-way towards the following beat (the precise distance is hard to measure, giving rise to the jazz musician's maxim that swing needs to be *felt* rather than measured). In music notation this offbeat is represented either on the half-beat or three-quarters of the way through the beat, although this is strictly speaking inaccurate. As western notation cannot accurately represent *swing*, the word is normally inserted at the top of the score for the musician to translate the notational approximations into the jazz idiom. As the jazz styles emerged, progressive jazz composers pushed the boundaries of rhythmic irregularity as Stravinsky had done within the classical tradition. However, usually even these composers kept within the generally divisible frameworks of 12 or 16 bars. This framework provides the template for the listener to measure syncopation, no matter how complex the internal rhythms are. A good example of this can be found in *Evidence* by Thelonious Monk.

As we pointed out at the outset of this work, syncopation may have complex and numerous manifestations, as we have seen in this brief review of musical examples. Hence, an attempt to measure syncopation always involves certain risks. Here we restrict ourselves to the second type of syncopation, that is, the contradiction between strong and weak beats against a fixed metrical context.

3 Measures of Syncopation

Much formalization and study of music from a mathematical standpoint has been undertaken, but it seems that scales, chords, melodies and other pitch-related phenomena have received much more attention and effort [9, 2, 3] than rhythmic questions. Several authors have remedied this situation by studying some open and fascinating questions on rhythm (see, for example, [12, 1, 9, 5, 14, 15, 16, 4]). Very few authors have addressed some of the many problems arising around syncopation. For example, given two rhythms with the same meter, which one is more syncopated? Or, is there a measure of syncopation that agrees with human perception? In the final chapter of [9], Keith considered the problem and defined a combinatorial measure of syncopation. Also, in [16] a measure of preference for Sub-Saharan African music was put forward, the so-called *off-beatness* measure, based on group theory. It turns out that not only is the off-beatness measure useful as a preference measure, but it may also serve as a measure of syncopation. The *Rhythmic oddity property* [1] may also be regarded as an approximation to a measure of syncopation. It is based on partitioning rhythms with certain properties. In this work we define a new measure of syncopation which is based neither on combinatorics nor on group theory, but rather on the concept of duration distances between notes. In the following we review the previous measures of syncopation and formally define the *weighted note-to-beat distance* measure.

3.1 The Rhythmic Oddity Property

Simha Arom [1] discovered that the *Aka* Pygmies use rhythms that have what he calls the *rhythmic-oddity* property [5]. A rhythm with a time span consisting of an *even* number of time units, has the rhythmic-oddity property if no two onsets partition the cycle (entire time span) into two sub-intervals of equal length. Such a partition will be called an *equal bi-partition*. Note that the rhythmic oddity property is defined only for time spans of *even* length. For *odd* length spans all rhythms have the property, thus rendering it useless. Nevertheless, this property is a first step towards a mathematical definition of some sort of limited syncopation. Consider, for example, the ten fundamental West (and South) African bell timelines composed of seven onsets in a time span of twelve units, with five intervals of length two and two intervals of length one (see [15] for more details). The ten rhythms and their interval vectors are as follows: *Soli* = (2 2 2 2 1 2 1), *Tambú* = (2 2 2 1 2 2 1), *Bembé* = (2 2 1 2 2 2 1), *Bembé-2* = (1 2 2 1 2 2 2), *Yoruba* = (2 2 1 2 2 1 2), *Tonada* = (2 1 2 1 2 2 2), *Asaadua* = (2 2 2 1 2 1 2), *Sorsonet* = (1 1 2 2 2 2 2), *Bemba* = (2 1 2 2 2 1 2), *Ashanti* = (2 1 2 2 1 2 2). See Figure 11 for the conventional notation of these rhythms. All ten rhythms are obtained by suitable rotations of one of three cannonical patterns. None of these ten rhythms, nor any of the other eleven that are not used but belong to the same three generating cannonical patterns has the rhythmic-oddity property. Furthermore, among the group of ten that are used, some are more syncopated than others, but the rhythmic oddity property does not make this differentiation.

3.2 The *Off-Beatness* Measure

Consider first the rhythms defined in a 12-unit time span. A twelve-unit interval may be evenly divided (with no remainders) by *four* numbers greater than one and less than twelve. These are the numbers six, four, three and two. Dividing the twelve unit circle by these numbers yields a bi-angle, triangle, square, and hexagon, respectively. African music usually incorporates a drum or other percussion instrument on which at least one or portions of these patterns are played. Sometimes the music is accompanied by hand-clapping rhythms that use some of these patterns. For example, the *Neporo* funeral piece of Northwestern Ghana uses the triangle, square, and hexagon clapping patterns [17]. In any case the rhythm has a pulse which we may associate with position "zero" in the cycle. In polyrhythmic music these four sub-patterns form the possible underlying beat-patterns. Two of the patterns (bi-angle and square) are binary beat-patterns and two (triangle and hexagon) ternary beat-patterns. Therefore, notes played in other positions are off-beat in a strong polyrhythmic sense. There are four positions not used by these four pulse patterns. They are positions 1, 5, 7, and 11. Onsets at these positions will be called *off-beat* onsets. A rhythm that contains at least one off-beat onset will be said to contain the *off-beatness* property. A measure of the *off-beatness* of a rhythm is therefore the number of off-beat onsets that it contains. These off-beat positions (1, 5, 7, and 11) also have a group-theoretic interpretation. The twelve positions of the possible notes in the cycle form the cyclic group of order 12 denoted by C_{12}. The four off-beat position values correspond to the sizes of the intervals that have the property that if one traverses the cycle starting at, say "zero" in a clockwise direction in jumps equal to the size of one of these intervals, then one eventually returns to the starting point after having visited all twelve positions. Conversely, if the lengths of the jumps are taken from the complementary set (2, 3, 4, 6, 8, 9, 10) then the start point will be reached without having visited all twelve positions in the cycle. For this reason the elements (1, 5, 7, and 11) are called the *generators* of the group C_{12}.

The numbers indicating the position of the off-beatness onsets in the cycle also have a number-theoretic interpretation. Consider a rhythm with a time span of n elements. The indices of the off-beatness onsets are known in number theory as the *totatives* of n. The totatives of n are the positive integers less than n that are *relatively prime* to n. Two numbers are relatively prime if the only positive factors (divisors) they have in common is 1. The Euler *totient function* denoted by $\phi(n)$ is the number of totatives of n, and is therefore the maximum value that the off-beatness measure can take for a rhythm with a time span of n units.

Returning to the ten West-African bell patterns in 12/8 time discussed in the preceding, the off-beatness measure not only discriminates better than the rhythmic-oddity property in terms of syncopation, but it shows that the more a rhythm is syncopated, the more it appears to be appreciated. The *Bembé* rhythm is the most frequently used of these patterns. Among these ten rhythms, the highest value of off-beatness is three and only the *Bembé* realizes this value.

Since every cyclic group C_n has a set of generators, the off-beatness measure described in the preceding generalizes to rhythms defined on n-unit time spans for other values of n. Although the measure works best for even values of n, it also has some applicability for odd n.

3.3 Keith's Measure

In [9] Keith examines several rhythmic phenomena and carries out a mathematical analysis of them. In particular, he tackles the problem of measuring the degree of syncopation of a given rhythm. His definition of syncopation leans upon the distinction of three events: *hesitation*, when a note starts on a beat but ends off a beat; *anticipation*, when a note starts off a beat but ends on a beat; and *syncopation*, which is conceived as a combination of the two previous events (see Figure 5; from left to right: hesitation, anticipation and syncopation).

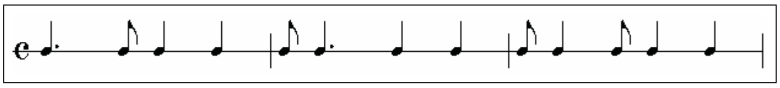


Figure 5: Hesitation, Anticipation and Syncopation.

Keith's measure has a certain weaknesses due to its assignments of weights to the above events. Hesitation receives a value of 1, anticipation is given 2, and syncopation obtains 3. This assignment seems, at least, subjective and Keith himself says that "deciding on the relative rhythmical "strength" of a hesitation, anticipation, and syncopation is an interesting philosophical problem."

Nevertheless, his definition suffers from a more serious drawback. It is limited to meters composed of n beats, where n is a power of 2. Let us see how his definition goes in order to understand this limitation. His main idea is, given a musical event (note), to find out whether it is off the beat by comparing it with a template of strong and weak beats depending on the size (duration) of the event. For example, if $d = 3$, there will be $2^3 = 8$ eighth-notes and the templates are the following (S stands for strong and W for weak): [S W W W W W W W], [S W W W S W W W], [S W S W S W S W] and [S S S S S S S S].

This measure requires that the duration δ of the event be measured in terms of the units of the 2^dth-notes. The start time S is also measured in this way. Then, D comes into play, which is defined as δ rounded down to the nearest power of 2. It is at this precise point, through the computation of D, when the level of metrical subdivision is chosen. Here D goes from $1, 2, \cdots, 2^d$ and gives the number of strong beats of the metrical subdivision Then, the event starts off the beat if $D \not| S$, where $|$ means divisible by; similarly, the event ends off the beat when $D \not| (S + \delta)$. The value s_i of syncopation of the i-th note of the rhythm is given by:

$$s_i = \begin{cases} 0 & \text{if } S \text{ and } S + \delta \text{ are on a beat,} \\ 1 & \text{if } S \text{ is off the beat and } S + \delta \text{ is on a beat,} \\ 2 & \text{if } S \text{ is on the beat and } S + \delta \text{ is off a beat,} \\ 3 & \text{if } S \text{ and } S + \delta \text{ are off a beat.} \end{cases}$$

Finally, the measure of syncopation of a rhythm is obtained by adding up the values s_i over all i.

3.4 The *Weighted Note-to-Beat Distance* Measure

In defining of our measure, we have paid attention to attacks between strong beats rather than to the structure of metrical levels (as Keith does), or to the number of generators of C_n, where n is the number of beats of the rhythm (as occurs with off-beatness). Our definition is based on distance and is more flexible inasmuch as it allows us to quantify the level of syncopation of a broader range of rhythms.

The *weighted note-to-beat distance* measure (from now on *WNBD* measure) is defined as follows. Firstly, notes are supposed to end where the next note starts. Let e_i, e_{i+1} be two consecutive strong beats in the meter. By strong beats we just mean pulses. Also, let x denote a note that starts after or on the strong beat e_i but before the strong beat e_{i+1}; we first define $T(x) = \min\{d(x, e_i), d(x, e_{i+1})\}$, where d denotes the distance between notes in terms of duration. Here the distance between two adjacent strong beats is taken as the unit and, therefore, the distance d is always a fraction. For example, quarter-notes in 4/4 time are the strong beats, and, if the notes in Figure 6 are referred to the nearest strong beat, then the distances $T(x)$ are, respectively, $\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}$.

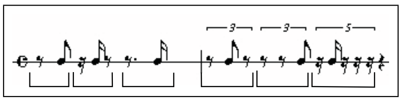


Figure 6: Measuring syncopation with the *WNBD* measure.

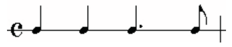
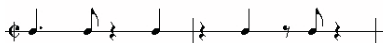
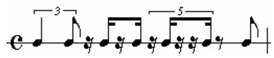
Rhythm	Musical Scores	WNBD
Hesitation		1/2
Anticipation		1/2
Syncopation		1.2
Triplet		0.857
Bembé		3
Bossa-Nova		4
Irregular		5

Figure 7: Examples of our measure.

The *WNBD* measure $D(x)$ of a note x is then defined as follows: 0, if $x = e_i$; $\frac{1}{T(x)}$, if note $x \neq e_i$ ends before or at e_{i+1}; $\frac{2}{T(x)}$, if note $x \neq e_i$ ends after e_{i+1} but before or at e_{i+2}; and $\frac{1}{T(x)}$, if note $x \neq e_i$ ends after e_{i+2}. Let n denote the number of notes of a rhythm. Then, the *WNBD* measure of a rhythm is the sum of all $D(x)$, for all notes x in the rhythm, divided by n. The table in Figure 7 lists the WNBD values for various rhythms.

The weights that appear in the definition of the measure deserve some explanation. It is reasonable to give some weight to a note that occurs off the strong beats. Our measure gives less weight when the note occurs exactly in the middle of two strong beats, and as the note approaches a strong beat, it gains more weight. However, it makes a big difference if the note crosses over a strong beat or just ends before or on the next strong beat. In the former case, there is a stronger feeling of syncopation than in the latter case. Therefore, it receives more weight in order to reflect this musical situation.

4 Measuring Syncopation of Rhythms

In this section, we examine and compare the aforementioned measures of syncopation on several rhythms. Certainly, it seems difficult to choose a family of rhythms that one could qualify as representative for such a task. A first idea would be to select rhythms that comprise an essential part of a musical genre. In so-called World Music abundant examples may be found. In [14, 15], Toussaint compiles the main timelines or *claves* from African, Cuban and Brazilian traditions and performs a rhythmic study on them. In these traditions there is a timeline, often played with an iron bell, a pair of claves or a wood-block, which is maintained throughout the piece and whose main function includes rhythmic stabilization and organization of phrasing [11]. In the classical music tradition, a concept closer to *clave* would be that of *rhythmic motif* and *ostinato*, as we can appreciate, for example, in Purcell (*Dido and Aeneas*), Marin Marais (*Sonnerie de Sainte Geneviève du Mont de Paris*), Beethoven (*Tempest* piano sonata opus 31, no.2), Ravel (*Bolero*), Holst (*Mars* from *The Planets*), and many other composers. In order to obtain a more comprehensive study, we incorporate some common rhythmic motifs taken from the classical music tradition.

The *claves* have been divided into two groups, the binary claves and the ternary claves, that will be studied separately. Since these timelines are played on bells or hard wooden *claves*, that yield attacks rather than sustained notes, the measure of Keith is calculated by taking into account only whether D divides S, to find out whether a given note is off the beat.

4.1 Binary Rhythms

The six fundamental binary *claves* are: *Shiko*, *Son*, *Rumba*, *Soukous*, *Bossa-Nova* and *Gahu*. We refer the reader to [14] and the references therein for thorough information on these timelines. In Figure 8 we can see the musical scores and their measures of syncopation computed with Keith's measure, off-beatness and the WNBD measure.

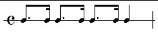
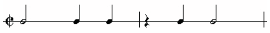
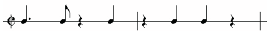
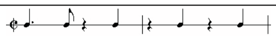
Rhythms	Musical Scores	Keith's M.	Off-Beatness	WNBD M.
Classical-1		0	3	1.8
Classical-2		3	3	1.71
Shiko		1	0	1.2
Son		2	1	2.8
Rumba		2	2	3.6
Soukous		3	2	3.6
Gahu		3	1	3.6
Bossa-Nova		3	2	4

Figure 8: Binary rhythms.

Clearly, *Shiko* is the least syncopated rhythm, *Son* is more syncopated than *Shiko*, and *Rumba* is more syncopated than *Son*. However, it is not clear which one among *Rumba*, *Soukous* and *Gahu* is more syncopated. *Bossa-Nova* is certainly the most syncopated. In general, the three measures bear out these conclusions, although there are some divergences that allow us to draw interesting conclusions on their usefulness.

Keith's measure in most cases returns reasonable values of syncopation, although it does not fully go into detail. For example, *Bossa-Nova* feels more syncopated than any other rhythm, but Keith's measure gives the same value to it as to *Soukous* and *Gahu*. Surprisingly it attributes the same amount of syncopation to both *Rumba* and *Son*. It is also somewhat odd that this measure assigns a value of 3 to rhythmic motif, *Classical-2* , putting it on the same level with *Bossa-Nova* or *Gahu*.

The off-beatness appears to be closer to human perception of syncopation than Keith's measure, although it yields some debatable conclusions too. For example, *Gahu* receives a value of 1, but feels more syncopated than *Son*, which also receives 1. It is disconcerting that the two classical rhythmic motifs obtain a higher score than any of the remaining claves, including *Bossa-Nova*.

The WNBD measure suggests reasonable conclusions and in general shows a greater agreement with human perception of syncopation. *Bossa-Nova* achieves the highest score in the group. The measure puts *Rumba*, *Soukous* and *Gahu* within the same category. At a lower level, we find *Son*, and below this, we find *Shiko*, the least syncopated rhythm.

We come back to the delicate issue of choosing the weights for the WNBD measure. In the definition of $D(x)$, when a note x crosses a strong beat and ends within the next strong beat, the distance is $\frac{2}{T(x)}$. Why $\frac{2}{T(x)}$? It could be $\frac{3/2}{T(x)}$ or $\frac{5/2}{T(x)}$, for example. Suppose that the distance $D(x)$ in that case is given by $\frac{a}{T(x)}$, where a is an arbitrary weight with $a > 1$. Then, the following table in Figure 9 contains the classification of the binary claves.

Rhythms	Weights
Shiko	$2a + 2$
Son	$6a + 2$
Soukous	$6a + 6$
Gahu	$8a + 2$
Rumba	$8a + 2$
Boss-Nova	$8a + 4$

Figure 9: Weights for the binary clave patterns.

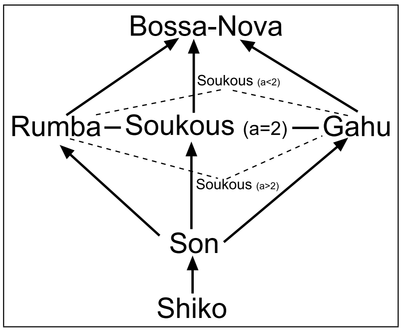


Figure 10: Sorting the binary clave patterns in terms of a.

Consider the graph depicted in Figure 10. Here each level denotes a measure of syncopation. No matter what values of $a, a > 1$, are chosen, the sorting is preserved, except for *Soukous*. A few algebraic manipulations with inequalities shows that this statement is true. With respect to *Soukous*, if $1 < a < 2$, then *Soukous* would be more syncopated than *Gahu* and *Rumba* (this situation is shown in Figure 10 with dashed lines); if $a = 2$, then the three of them would be equally syncopated; finally, if $a > 2$, then *Soukous* is deemed less syncopated (see the other dashed lines in Figure 10).

4.2 Ternary Rhythms

As mentioned in the preceding, the ten fundamental ternary timelines are: *Soli*, *Tambú*, *Bembé*, *Bembé-2*, *Yoruba*, *Tonada*, *Asaadua*, *Sorsonet*, *Bemba* and *Ashanti*. See [15] and its references for a complete description of these rhythms. The musical scores and the corresponding off-beatness and WNBD values are shown in Figure 11.

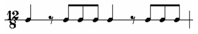
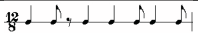
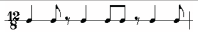
Rhythms	Musical Scores	Off-Beatness	WNBD M.
Classical - 1		2	1.5
Classical - 2		2	1.5
Soli		1	2.142
Tambú		2	2.142
Bembé		3	3
Bembé-2		2	2.142
Yoruba		2	3
Tonada		1	2.142
Asaadua		1	2.142
Sorsonet		1	3
Bemba		2	2.142
Ashanti		2	3

Figure 11: Ternary rhythms.

Again, both measures seem to make global sense on these rhythms. In the case of off-beatness, only *Bembé* obtains the highest score, whereas WNBD measure has *Bembé*, *Yoruba*, *Sorsonet* and *Ashanti* as the most syncopated rhythms. After *Bembé*, off-beatness leaves *Tambú*, *Bembé-2*, *Yoruba* and *Bemba* in the second position (value 2). Here off-beatness is rather coarse. *Yoruba* is like the *Bembé*, except for the last note, which is played an eighth-note before. Therefore, its level of syncopation is high. However, *Bemba* feels less syncopated than *Yoruba*. Notice that *Bemba* has notes on three strong beats and only one note crosses over a strong beat (the fourth one).

All ten timelines belong to three distinct cannonical necklaces [9, 15]. Cannonical pattern number I corresponds to *Sorsonet* only; cannonical pattern number II generates *Soli*, *Tonada* and *Asaadua*; and cannonical pattern number III includes *Bembé*, *Bembé-2*, *Tambú*, *Yoruba*, *Bemba* and *Ashanti*. Off-beatness classifies the rhythms originated by cannonical patterns number I and II as the least syncopated ones. All the timelines generated by the cannonical pattern number III, except for the *Bembé*, whose measure of off-beatness is the highest, would be in second place. The WNBD measure groups the rhythms in a different manner; it only distinguishes two groups. The group with the highest value of syncopation is formed by *Bembé*, *Yoruba*, *Ashanti* and *Sorsonet*; in the other group we find *Tambú*, *Bembé-2*, *Bemba*, *Soli*, *Tonada*, and *Asaadua*. This last group comprises all the rhythms from the cannonical pattern number II plus *Tambú*, *Bembé-2*, and *Bemba*. Then, the question that naturally arises is how the WNBD measure actually classifies the claves into these groups. Let us remark that all rhythms in the least syncopated group have three notes on strong beats and one note crossing over the fourth strong beat. In contrast, all the rhythms in the most syncopated group have two notes on strong beats and two notes crossing over the other strong beats. However, off-beatness seems to miss those facts and takes *Tambú*, *Bembé-2*, and *Bemba* from the least syncopated group to the most syncopated one. On the contrary, *Sorsonet*, which is considered as syncopated by WNBD measure, is placed by off-beatness in the least syncopated group.

Comparison between the claves and the classical rhythmic motifs leads to more consistent results than in the case of the binary claves. Off-beatness considers the two motifs as syncopated as *Yoruba* or *Ashanti*, but this appears to be erroneous. The WNBD measure correctly detects that they are less syncopated than the claves.

Weighting the WNBD measure has no bearing on the relative ordering of the ternary rhythms, because all of these measures are of the form $3a + c_i$, where c_i is an additive constant for rhythm i.

5 Concluding Remarks

We derive some conclusions from the empirical results obtained in the previous section, starting with the drawbacks of each measure.

The main drawbacks of Keith's measure are: (1) It cannot measure rhythms framed in meters whose number of notes is not a power of 2; (2) It cannot be used for measuring irregular rhythms; (3) The choice of weights in his definition is subjective; (4) It shows limited agreement with human perception of syncopation.

With respect to off-beatness, we find the following drawbacks: (1) It is limited in its applicability, since for meters with a prime number of notes all notes are off-beat; (2) It cannot be used for measuring irregular rhythms; (3) It does not measure syncopation in all its generality. For example, in a 12/8 meter, the off-beats are at positions $1, 5, 7, 11$. For a rhythm to be as syncopated as possible it must have a note on the eleventh position. However, a rhythm can be very syncopated without having a note there; (4) It is independent of the number of notes. For example *Bembé* and [x x . . . x . x . . . x] have off-beatness 3, but a different number of notes; (5) Although it shows more agreement with human perception of syncopation than Keith's measure, it still proves to be limited.

The only drawback that the WNBD measure appears to have is the ambiguity in the choice of weights. However, the choice of weights does not appear to have as dramatic an outcome as it has for Keith's measure. A better algorithm for choosing these weights would be welcome.

Finally, we conclude that the WNBD measure has greater agreement with human perception of syncopation than the other measures, as well as a greater degree of applicability.

We close with some open problems. It would be interesting to generalize Keith's measure so that it can handle general meters, and off-beatness so that it can handle prime numbers. With regards to the WNBD measure, we would like to obtain additional empirical results so that a more precise algorithm for choosing the weights can be found.

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