Playing Mathematics and Doing Music

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Abstract

Generally, the art and science of pattern making and pattern discernment are at the core of mathematics and music. The forms and shapes of music, the ways in which music is used to structure time "space" can be viewed as rich mathematics structures. For example, the musical features of pulse, subpulse, cycle, and harmonic rhythm can serve as aural representations of key mathematical ideas, potentially providing a context for a deepened understanding of and "feel" for these ideas. Conversely, the act of translating mathematics structures into music forms can lead to powerful music insights. Exploring these domains in concert leads to possibilities for exciting synergies that address both conceptual and affective considerations about what it means to do-make music and mathematics.

Provocations

Symmetrical Music. Generally, it is much easier to recognize visual symmetry than aural symmetry, particularly "reflective symmetry". "Translational symmetry" is somewhat taken for granted in music; repeating patterns can be thought of as being copied and translated across a temporal background, sort of like aural wallpaper. Reflective symmetry turns up in surprising places. One source comes from "harmonic rhythm", which can be created by layering music cycles of different lengths.

Harmonic rhythm refers to a phenomenon, sometimes called "polyrhythm", which exists in many African and African Diasporic music traditions. The term "harmonic rhythm", in contrast to "polyrhythm", is meant to capture a set of aural and functional qualities in a rhythmic context, analogous to features that distinguish polyphony from harmony in a tonal context.

For example, combining a 2-beat cycle, in which the first pulse beat is sounded, with a 3-beat cycle, in which the first pulse beat is sounded, results in:

X		X		X		X
X			X			X

This can also be represented as a number sequence, in which numbers represent the lengths of intervals that begin with a sounded pulse beat and end with the next sounded pulse beat. The above pattern, with the two cycles merged together, which happens when the two rhythmic patterns are heard "harmonically", is represented by the sequence: 2, 1, 1, 2. The number sequence for a 3-beat cycle merged with a 4-beat cycle is 312213.

Interesting rhythmic structures can be created by playing with layering cycles of different lengths. When all pairs of cycle lengths are *relatively prime*, congruence arithmetic can be used to compute the places where cycles align. The sound/feeling of cycles coming in and out of alignment can be powerful. The Chinese Remainder Theorem provides one mathematical means for determining places of alignment, particularly when you are interested in finding alignments that aren't limited to lining up "ones".

For example, one might choose to align the fifth beat of a 7-cycle, the third beat of 3-cycle, the first beat of a 5-cycle, and the second beat of a 4-cycle. This is equivalent to solving:

$$x \equiv 5(\bmod 7)...x \equiv 0(\bmod 3)...x \equiv 1(\bmod 5)...x \equiv 2(\bmod 4)$$

In general, the Chinese Remainder Theorem guarantees a unique solution to a system of congruences, $x \equiv a_i (\bmod m_i)$, for pairwise relatively prime m_i, $1 \le i \le n$. The unique solution, modulo M (M, the product of all of the m_is), is given by:

$$x \equiv [a_1e_1 + a_2e_2 + ... + a_ne_n] (\bmod M), \text{ where } e_i = M/m_i \text{ x } (M/m_i)^{-1} (\bmod m_i).$$

The solution to the above system is $x \equiv 306 (\bmod 420)$. Musically, this means if the 7, 3, 5, and 4 beat cycles are begun simultaneously, the desired alignment occurs on beat 306, with subsequent alignments occurring every 420 beats.

Investigating symmetry and asking mathematics questions to be answered in aural-temporal contexts leads to numerous and open-ended creative possibilities.

Aural Mapping and Transformational Music. Thinking of aural and/or rhythmic vocabulary as set elements and defining functions that map elements from one set to another leads to additional insights for playing with music structures. Thinking analogically is a critical aspect of exploring these possibilities, recognizing that there is rarely, if ever, one right way to translate musical ideas into mathematical ideas and vice-versa.

One possibility is to use set theory to map from one genre/tradition to another. For example, in the Ewe drum tradition of the Upper Volta region of West Africa, the master drum is called Atsimewu. The sonic vocabulary of Atsimewu consists of:

Dza (hand playing the center/bass tone, stick playing the side)
Dzi (hand playing the edge in a muted fashion, stick playing the side)
Ge (hand or stick playing the high tone)
T (hand pressing the edge, while stick strikes the center)
Re ("roll")

One might define a function that maps from 7 notes in a Western octave to the "notes" of Atsimewu in the following way:

C → Dza, D → Ge, E → Ge, F → Dzi, G → T , A → Dza, B → Re

Such a mapping provides a means for translating a Western melody into a West African "drum song."

The use of linear transformations is an additional tool for "extending the mileage" of music structural ideas. For example, a function, such as:

f: $R^2 \rightarrow R^3$ defined by $f\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x+2y \\ 2x-3y \end{bmatrix}$

can be used to inform, motivate and stretch some music choices and possibilities. It can be interesting to consider what it means to map from 2 dimensions to 3 dimensions in the context of a music composition. One way of making this function musically meaningful is having the x-axis represent time, the y-axis represent pitch, and the z-axis represent volume (loudness). Clearly, there are other possibilities. Exploring these ideas opens up avenues for creating aural models, enabling other means for multi-dimensional representations beyond the means of visual graphs.

Conclusion

Perhaps and hopefully, the above explorations and provocations, which admittedly have skimmed the surface of possibilities, have, nonetheless, provided some insights into ways of traversing back and forth between mathematics and music.