A Physical Proof for Five and Only Five Regular Solids

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Abstract

Physical models are invaluable aids for conveying concepts in geometry, particularly concepts about polyhedra. In this paper physical polygons, polygon corners, polygon flat regions, polygon cylinders, polygon nets and polyhedra are all used to show a physical proof that there are five, and only five regular solids. Students from grade school, high school, undergraduate and graduate levels have been presented with this physical proof, and all of them have taken away something worthwhile related to their existing level of knowledge.

1. Introduction

The Regular or Platonic solids can serve as a substantive foundation for the study of three-dimensional space. The better a student understands Regular solids, the better that student is able to understand more sophisticated concepts about space. Additional understanding of these solids is achieved by adding the sense of touch to the sense of vision and students who are able to handle Platonic solids reinforce the knowledge gained through their vision. The size of the models is also important. A model that fills the hands of a student gains her or his attention more than a considerably smaller or a considerably larger model.

Regular polygons are simple shapes with straight and equal length edges, as well as, equal angles between each adjacent pair of those edges. An equilateral triangle, a square, a regular pentagon, and a regular hexagon as seen in **Figure 1**, are four regular polygons that we see in our everyday lives. They frequently appear in designs used in flat visual art.

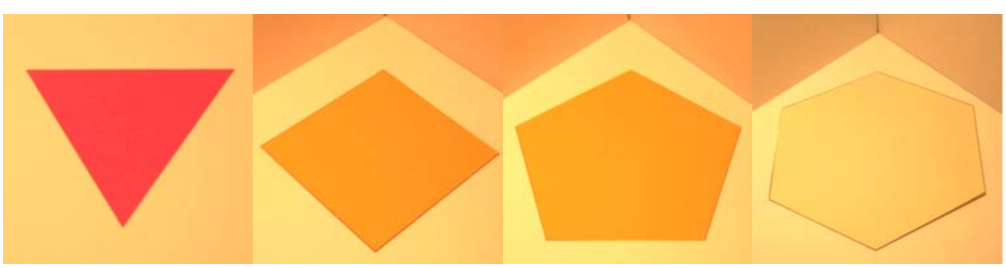

Figure 1: An equilateral triangle, a square, a regular pentagon, & a regular hexagon.

Regular solids also have a simple definition. A regular solid consists entirely of a single-sized regular polygon. All vertices of a regular solid have the same number of regular polygons meeting at that vertex. Each vertex looks the same as every other vertex of a regular solid. This condition implies that such vertices have equal solid angles. In Coxeter's Regular Polytopes[1], page 15, it states that solids with regular faces and regular solid angles are regular solids.

The remainder of this paper provides a physical proof that there are five and only five regular solids. These five regular solids are known as the Platonic solids. Using a number of equilateral triangles, squares, regular pentagons, and regular hexagons and some adhesive tape, I will begin this physical proof.

2. Regular Polygons Used to Form Corners for Regular Solids

The simplest regular polygon is an equilateral triangle, with three equal angles and three equal sides. Three equilateral triangles taped together form a corner as seen in **Figure 2**. Four equilateral triangles taped together form a second corner as seen in **Figure 2**. Five equilateral triangles taped together form a third corner as seen in **Figure 2**. Six equilateral triangles taped together form a hexagon. These six triangles are flat, so they do not form a corner. Forming potential corners for regular solids with equilateral triangles has now been exhausted. There are three and only three corners from equilateral triangles.

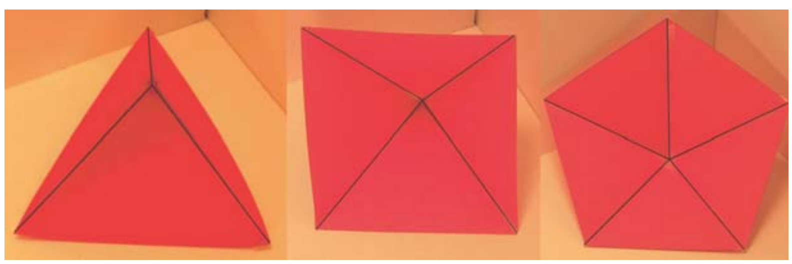

Figure 2: 3, 4, 5 equilateral triangles as corners.

A square is another regular polygon. Three squares form a corner as in **Figure 3**. However, when four squares are taped they are again flat, thus squares provide only one corner. Another regular polygon is a pentagon and three regular pentagons form another corner as seen

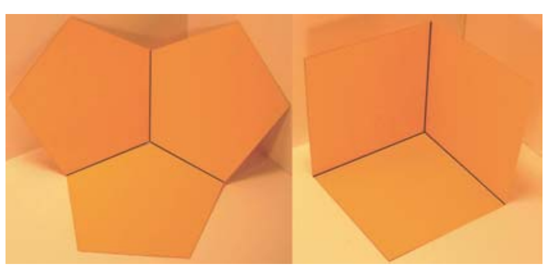

Figure 3: 3 squares as a corner, and 3 regular pentagons.

in **Figure 3**. A regular pentagon can be used to produce only one corner. Finally, to continue this physical proof, a regular hexagon is considered. Three regular hexagons taped together are flat. In all, five corners have been produced for regular solids from three regular polygons. Three corners from equilateral triangles, one corner from squares and one corner from regular pentagons have been produced.

3. Corners with Regular Polygons Used To Form Regular Solids

The first corner of three equilateral triangles has an opening that is an equilateral triangle of the same size. An equilateral triangle can be taped to the three equilateral triangles of the corner to cover this opening. These four equilateral triangles have now been taped together to form a regular solid. This regular solid is known as a regular tetrahedron. Tetra is from the Greek meaning four. If we hold this regular tetrahedron in our hands we can see each of its four vertices, six edges, four faces and all of its symmetries. Each vertex is formed when three equilateral triangles are joined together. This same corner configuration produces an identical solid angle for all four vertices of the regular tetrahedron.

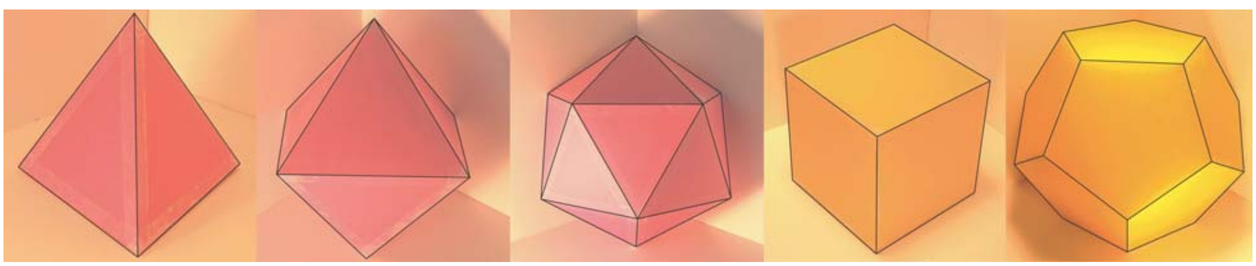

Figure 4: Tetrahedron, Octahedron, Icosahedron, Cube, Dodecahedron.

The second corner using four equilateral triangles has an opening which can be covered with a square having the same edge length as the edge lengths of the equilateral triangles. This combination of polygons forms a pyramid with a square base. Even though this does not form a regular solid, it is an interesting shape. However, if we were to duplicate the corner of four equilateral triangles, we would have two corners with a total of eight equilateral triangles. A second regular solid can now be formed, by aligning the openings of the two four-triangle corners in an edge to edge manner, and taping them together. This regular solid is known as a regular octahedron, with the Greek root “octa” for eight. If we hold this octahedron in our hands and look at every vertex, edge, and face, we can see that all of the six vertices, twelve edges, and eight faces are identical.

The third corner, consisting of five equilateral triangles, will be used to form a regular solid. The opening of this corner is a regular pentagon and can be closed by taping a regular pentagon in the opening to form a pyramid with a pentagonal base. This is also an interesting shape, but it is not a regular solid. As in the previous corner example, we can duplicate this corner of five equilateral triangles and once again tape these two corners or “pent caps” together by aligning their edges. Now we have formed a solid, known as a regular pentagonal di-pyramid.

It has four equilateral triangles meeting at the vertices where the two “pent caps” are joined together. Again, this is not a regular solid.

Ten equilateral triangles of the same size as those of the “pent caps” can be taped together side by side to form a strip. If we alternate the orientation of adjacent triangles we can form a strip of triangles where the top and bottom edges of the strip are parallel to each other. If an edge at one end of this strip is taped to the edge at the other end of the strip, a cylinder of ten triangles is produced. When we look at the top and bottom of this shallow cylinder, the openings have the shape of a regular pentagon. At this point we could form a pentagonal anti-prism by taping regular pentagons into the opening at the top and bottom of this cylinder. Instead, we take two “pent caps” and tape one to the top of the cylinder and one to the bottom.

To review this construction, we have joined two “pent caps” of five equilateral triangles each, and a cylinder of ten equilateral triangles, for twenty triangles. This regular solid of twenty triangles is known as a regular icosahedron, with the Greek root “icosa” meaning twenty. By handling this icosahedron we see that all of the twelve vertices, thirty edges, and twenty faces are identical.

A corner of squares can be duplicated to help form a regular solid. Placing three “open” vertices of this corner on a flat surface one vertex of the corner will point up. Looking further we can see that in between each of the three “open” vertices there are three vertices formed where two squares meet. These two corners can be oriented so that three “open” vertices are aligned with the three vertices formed from the two squares. The six edges near each other, one from each of the corners, can be taped together to form a regular solid. This regular solid is known as a cube and also as a regular square hexahedron, with the “hex” root being six from the Greek. Taking this shape in our hands we can see its eight vertices, twelve edges and six faces. The general population thinks of this regular polyhedron as a “box”.

The fifth and final corner is formed with three regular pentagons. The corner can have a vertex pointing up, and three edges resting on a flat surface. This corner can be duplicated to form two corners with a total of six faces. By taking a strip of six regular pentagons, taping the ends together to form a cylinder, much like the central section of the icosahedron, and then attaching both three-pentagon corners to the top and bottom of the cylinder, another regular solid is produced. This regular solid is known as a dodecahedron, where the Greek “dodeca” means twelve. We can handle the dodecahedron to see each of the twenty vertices, thirty edges, and twelve faces. Note that twelve regular pentagons make up the regular dodecahedron, and perhaps this is why regular dodecahedrons are used for displaying the months in a calendar year.

4. Handling a Regular Polyhedron

Platonic solids have multiple axes of symmetry that encourage handling. I can use an index finger from each hand to hold a solid. Other fingers can help to rotate the solid. A regular tetrahedron has two axes of symmetry. I can put one index finger on a vertex and another index finger in the middle of the face is opposite that vertex. When I rotate this tetrahedron I get a view of the relationship between a vertex and its opposite face. A second view of the regular tetrahedron can be achieved by placing one index finger on the middle of an edge with the other index finger in the middle of the edge on the opposite side of the tetrahedron. Note that these edges are parallel as well as perpendicular. These two edges are diagonals of squares that are on opposite sides of a cube.

A second Platonic solid to handle is the octahedron. If I place one index finger on a vertex and the second index finger on an opposite vertex and spin the octahedron around, a set of four edges, describes an *equator* for the octahedron. Next, if I place an edge of the octahedron on a flat surface, there will be an edge that is opposite to this one on the flat surface. If I place one index finger in the middle of the edge that is up and one index finger that is in the middle of the edge on the flat surface I will observe that these edges are parallel to each other and oriented in the same direction. A third rotation of an octahedron is attained by placing one of the faces on a flat surface where a second face is parallel and rotated by sixty degrees with respect to the first face. This orientation can help one more easily see that the octahedron is a triangular anti-prism. When rotating the octahedron about the centers of opposite faces, we can see six triangles forming a midsection with triangles alternately pointing up and pointing down.

The third regular solid to be handled is the regular icosahedron. If I start with an index finger on a vertex there will be a vertex on the opposite side of the icosahedron for my other index finger. Here I can see that I have each finger on a pent cap and that they are on opposite sides of the icosahedron. These pent caps are also oriented with a rotation of thirty-six degrees or one tenth of a circle of three hundred and sixty degree. A strip of ten triangles can be seen to connect the two pent caps with five triangles pointing up and five triangles pointing down. The icosahedron can also be rotated about the midpoints of opposite edge and faces for interesting viewing.

A cube can also be rotated about three axes of symmetry through opposite vertices and midpoints of opposite edges and faces. When rotating about opposite vertices, the cube has the other six vertices in an up and down pattern. Holding opposite edges shows squares in a diamond shape at either end of the cube. Rotating about fingers in the middle of opposite faces displays the cube at its simplest, with all sides appearing as squares with edges at the top and bottom.

The dodecahedron can be oriented so that it has one vertex pointing up and one vertex pointing down. If I hold an index finger from one hand on the vertex pointing up, and an index finger for the other hand on the vertex pointing down, I can turn the regular dodecahedron to get some interesting viewing. In addition, I can place one of the pentagons on a flat surface and see that there is another face that is parallel to the face lying on the flat surface. I can once again place an index finger of one hand in the middle of the face that is on top, and another index finger of the other hand in the middle of the face to do some turning.

5. A Regular Polygon Net Folds into a Regular Polyhedron

A polygon net of four triangles, having three triangles sharing an edge with a central triangle, can be easily imagined folding into a tetrahedron. A second net of four equilateral triangles in a rectangular strip can also be folded into a tetrahedron. This proves that nets are not unique, not even for the most simple of regular solids.

Since nets are not unique, they can have their own order and symmetry imposed on them to enhance the appearance of the net. A regular polygon net for a regular polyhedron has an even number of edges on the perimeter of the net. When there is an even number of edges, placing tape on alternating edges of the perimeter of the net provides one piece of tape for each edge when folding it into a regular polyhedron.

6. Conclusion

I have presented this spatial proof using physical models to students ranging from elementary school to graduate school for a period of more than twenty years. The presentation has always been well received by both students and teachers. The teachers have shown their appreciation by wanting to keep the models to hang in their classrooms and serve as constant examples for future study. This month I presented the models and the proof to seventh and eight grade students at the McGillis Elementary School in Salt Lake City, Utah. These classes had recently been divided along gender lines. The girl’s class was very attentive and well focused throughout, easily asking questions of their interest. The boy’s class was spending energy in attempts to be humorous and I needed to provide discipline for them to stay on tasks. They were, however, interested in taking the proof home to their parents in order to prove a point. The classes were identifiably different but the highlight of the presentation for both classes was the physical handling of the models.

Each group of students, depending on their educational background, takes away something different, even though the materials are exactly the same and the wording is very much the same. Emphasizing the tactile experience of handling each of the regular solids provides the students with a recognizably different experience than merely viewing three-dimensional images of the regular solids on a piece of paper or on a computer screen

Many years after I started to give this presentation I saw fundamentals of this physical proof in Holden’s Shapes, Space and Symmetry[2], on page 2. It strengthened my conviction that presenting this work to as many students as possible continues to have value.

Acknowledgements

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References

[1] Coxeter, H.S.M., Regular Polytopes, Third Edition, Dover Publications, Inc., NY, 1973.
[2] Holden, Alan, Shapes, Space and Symmetry, Columbia University Press, New York, 1971.