Paper Polylinks

George W. Hart
Computer Science Department
Stony Brook University
Stony Brook, NY 11794 USA
E-mail: george@georgehart.com

This workshop presents three activities to make attractive symmetric constructions. Each activity involves cutting out hollow paper polygons and assembling them into a geometric weave. The equipment required is paper, a copier or computer printer to print the templates, scissors to cut them out, and a small amount of clear tape to assemble the parts. I recommend proceeding in sequence from simplest to hardest:

Activity 1: Four Triangles (related to a tetrahedron)
Activity 2: Six Squares (related to a cube)
Activity 3: Six Pentagons (related to a dodecahedron)

I have used these as individual and group activities in a classroom. They develop geometric ideas, visualization, and mathematical communication skills. But be warned: *these constructions are trickier puzzles than they might first appear*. As with many 3D constructions, they are difficult to explain with text and 2D illustrations. So don't expect your students to solve them if you haven't solved them first, because you will probably need to give pointers. It is helpful if you can bring in a completed model as an example that they can copy.

Activity 1. Four Triangles

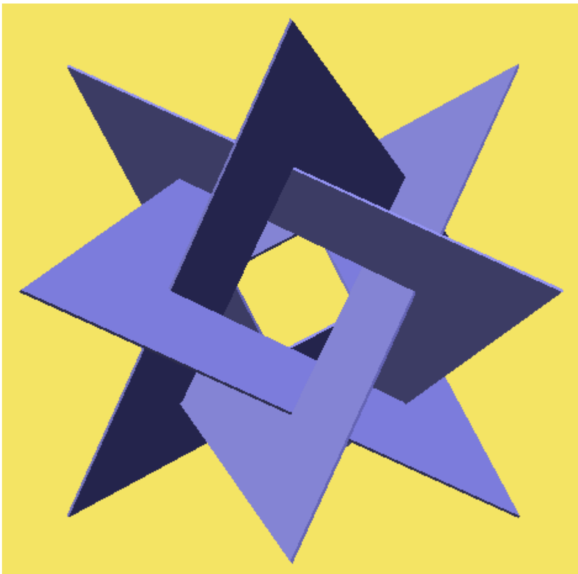


Figure 1: *Four hollow equilateral triangles.*

Figure 1 shows the construction of four triangles. Notice that each triangle is flat (planar) and each links with the other three. To make the triangles, enlarge and print Template 1 onto card stock. (Card stock is a heavy paper that makes a more rigid model than ordinary paper, yet it is thin enough to travel through the rollers of a printer or copy machine.) Cut out the four triangles and cut out their holes. Neatness counts! It is allowable to slit three or four triangles to access their interior holes, because you will need a slit to link them. One triangle at a time, create the linkage by following the structure shown in Figure 1. You will have to bend the paper sometimes to manipulate the parts around each other, but when properly assembled, the triangles are again flat. After linking, tape the slits back together so the triangles can not untangle. If students work in small groups, they will have enough hands to hold the earlier parts in the proper relative positions as new parts are added.

For guidance, observe how each of the twelve edge midpoints nestle inside the "V" of another triangle's interior. If your construction has an edge midpoint not inside another triangle's "V", or if it has an interior "V" that doesn't touch the midpoint of some edge, then the weaving is incorrect. The finished form is very elegant for its simplicity and symmetry, yet can be frustrating to achieve. When partially complete, with only two or three triangles in place, it is difficult to visualize how to orient the parts and where to place the next piece. Developing the 3D intuition for this is, of course, the goal. In a classroom, students who finish first can be encouraged to assist their neighbors by explaining the steps, which helps develop mathematical communication skills.

Activity 2. Six Squares

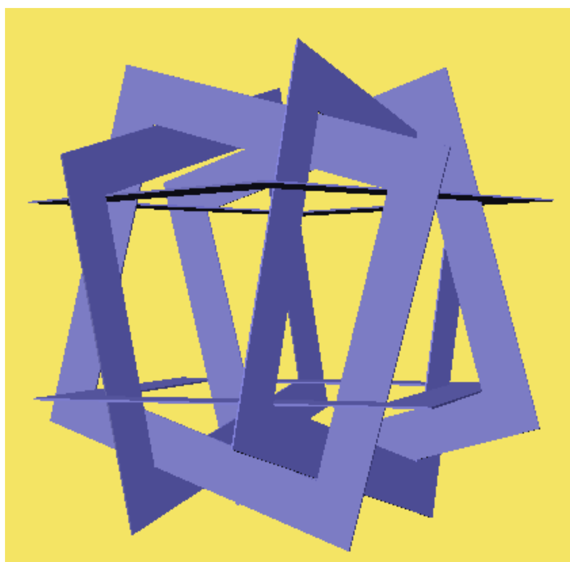


Figure 2: *Six hollow squares.*

Figure 2 shows the construction of six squares. Notice that each square is flat (planar) and each links with four of the other five. To make the parts, enlarge and print Template 2 onto two sheets of card stock. Cut out six squares and cut out their holes. Again, it is OK to slit through the bodies of the polygons to access their interior holes, because you will need a slit to link them. One piece at a time, create the linkage by following the structure shown in the image. Again, some temporary bending is required to achieve the weave, but the polygons are planar in the final configuration. After linking, tape the slits back together so the squares can not untangle. Again, if working in groups, having many hands may make it easier to hold the first few parts in the correct relative position.

It is important to see how the form is closely related to a cube. The six squares correspond to the six faces of a cube, except that they are each rotated clockwise somewhat. And each square links with four squares that are cube-neighbors, but does not link with the opposite, parallel square. It may be useful to make the model in three colors, with the top and bottom squares of one color, the left and right squares of a second color, and the front and back squares of the third color; then squares of the same color do not link. (After understanding how this form relates to the cube, it may be insightful observe that the four triangles of Activity 1 relate analogously to a tetrahedron.)

As with the triangle construction, each edge of each square nestles into an interior "V" of another square. However, in this construction the contact point is not the midpoint of the edge, but is a point about a third of the way along the edge.

Activity 3. Six Pentagons

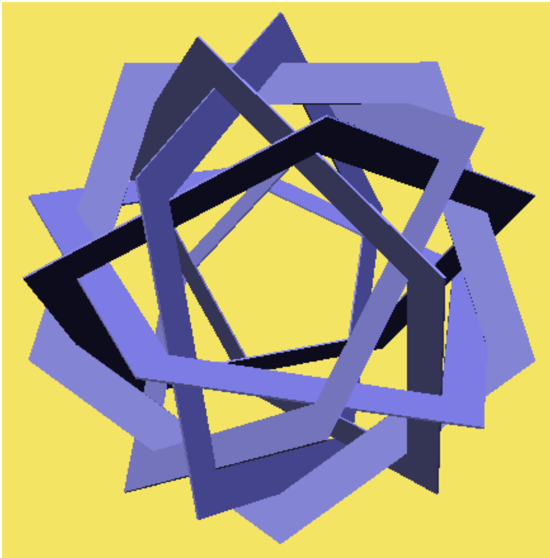


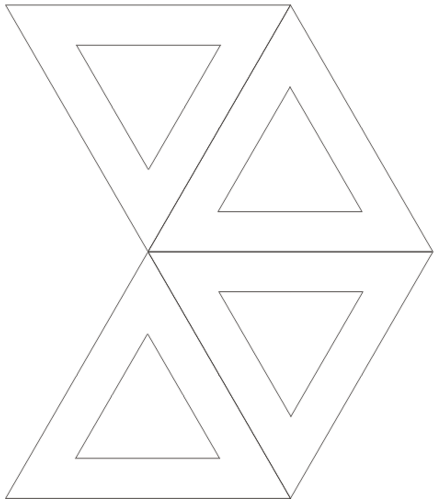
Figure 3: *Six hollow regular pentagons.*

Figure 3 shows the construction of six pentagons. The assembly steps are analogous to the first two forms, but more difficult because it has the less familiar symmetry of a dodecahedron. (The six planes of these pentagons correspond to the planes halfway between pairs of opposite faces of a dodecahedron.) As with the triangle construction, the midpoint of each edge of each pentagon nestles into an interior "V" of another pentagon.

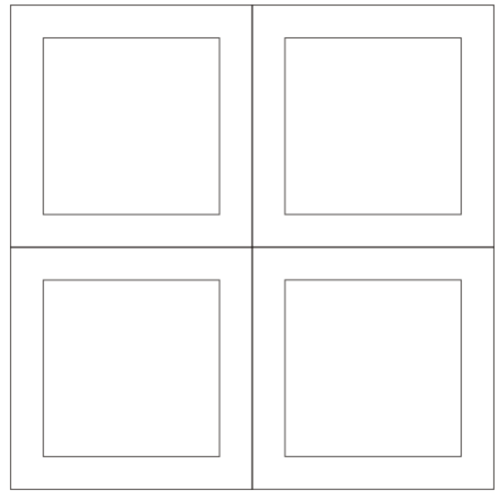
Observe that there are twelve five-sided openings, and each is surrounded by three-sided openings. There are no four-sided openings. The most common mistake when first attempting this form is to mis-weave in a way that creates one or more four-sided openings. So look carefully for four-sided openings in the completed model, which indicate an error. But four-sided openings are expected in the partial configurations before the final pentagon is added.

Variations for All Three Activities

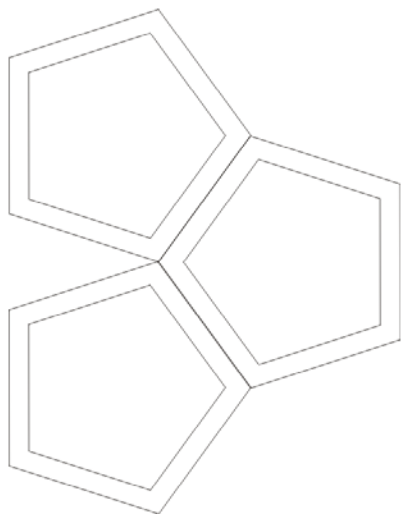
- Students may enjoy making these with a different color paper for each polygon.
- The polygons can be decorated.
- After mastering a paper construction, some students may enjoy scaling up the template to a larger size, in a more rigid material such as cardboard.
- Each construction comes in a left-hand and right-hand form. After making one version, look at it in a mirror and try making its mirror image version.
- Reference [3] describes free software for generating additional "orderly tangles".



Template 1. *Four triangles.*



Template 2. *Six squares required.*



Template 3. *Six pentagons required.*

References

These forms are examples of “regular polylinks” or “orderly tangles”, which were first presented briefly in a 1971 book by Alan Holden [1]. He then developed the idea into a book-length exposition in 1983 [2]. I have explored his idea further and have written software that produced the images and templates in this paper, and can be used to generate many related forms. For additional background, examples, and references, see [3].

[1] Alan Holden, *Shapes, Spaces and Symmetry*, Columbia Univ. Pr, 1971, (Dover reprint, 1991).

[2] Alan Holden, *Orderly Tangles: Cloverleafs, Gordian Knots, and Regular Polylinks*, Columbia U., 1983.

[3] George W Hart, “Orderly Tangles Revisited,” in Proceedings of *Renaissance Banff*, July, 2005 (to appear).