Symmetrical Hamiltonian Manifolds
on Regular 3D and 4D Polytopes

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Abstract

Hamiltonian cycles on the edge graphs of the regular polytopes in three and four dimensions are investigated with the primary goal of finding complete multi-colored coverages of all the edges in the graph. The concept of a Hamiltonian path is then extended to the notion of Hamiltonian two-manifolds that visit all the given edges exactly once. For instance, the 4D simplex can be covered by a strip of 5 triangular facets that form a Moebius band! The use of Hamiltonian cycles to create physical dissection puzzles as well as geometrical sculptures is also investigated. The concepts are illustrated with computer graphics imagery and with small maquettes made with rapid prototyping techniques.

Figure 1: *Path-based sculptures by K. Verhoeff and R. Roelofs*

1. Introduction

The edges of the regular polytopes in three and higher dimensional spaces form highly symmetrical graphs. The edge graphs of the 3D Platonic and Archimedean solids have stimulated some artists, such as K. Verhoeff and R. Roelofs, to build impressive constructivist sculptures by sweeping a regular polygonal cross section along some or all of the edges (or cords) of a regular or semi-regular polyhedron and making sure that the corners are nicely mitered (Fig.1). Often the subset of edges is selected in such a way that a single, branchless, closed-loop path results. A sequence of edges that visits all the vertices of a graph exactly once is called a *Hamiltonian path*, or a *Hamiltonian cycle* if the path closes into a loop. These sculptures have inspired me to ask what Hamiltonian cycles might exist on all the regular polytopes in three and four dimensions, and in what ways such paths might lead to new and attractive sculptures. The potential for interesting new configurations is enormous: While the most complex graph of a 3D Platonic solid has only 30 edges, on a 4D regular polytope the number of edges can be as high as 1200.

This study starts with an overview of the Hamiltonian cycles on the 3D Platonic solids and investigates what kind of geometrical sculptures or puzzles might be derived from them. This includes not only prismatic sweeps along such Hamiltonian cycles, but also minimal surfaces suspended in such loops, as well as polyhedral dissections resulting from a cut that passes through a Hamiltonian subset of the edges.

The discussion then focusses on the six regular polytopes in 4-dimensional space and the existence of symmetrical Hamiltonian cycles on suitable projections of these graphs into 3D space – after all, at some point one would like to enjoy actual, physical sculptures, not just abstract mathematical constructs. One other requirement is added to the desired Hamiltonian cycles on these projected graphs: We would like to find cycles of a special shape, so that a collection of congruent copies of them will cover all the edges of the graph. After a multi-year effort [3,5], such cycles have now been found for all the 4D regular polytopes, including the 120-Cell and the 600-Cell.

Subsequently I extend the concept of the piecewise linear (1-manifold) Hamiltonian path to the notion of Hamiltonian 2-manifolds; these are partial surfaces or closed shells of facets that cover all the edges of a given polytope. For instance, the 4D simplex can be covered by a strip of five triangular facets that form a Moebius band. And two copies of this Moebius band can then cover the whole surface of that simplex.

Throughout this presentation I will rely heavily on computer graphics and on maquettes made by layered manufacturing on a rapid prototyping machine to assist visualization of these fascinating geometrical objects. Hopefully, some of these models will inspire courageous artists to create one or two large-scale sculptures of the most attractive constructions.

2. Hamiltonian Cycles and Dissections of the 3D Platonic Solids

To set the stage for the study of the graphs of the 4D regular polytopes, I first review on the more amenable 3D Platonic solids. When looking for Hamiltonian cycles on these objects of genus zero, it helps to realize that such a path must split the surface into two regions that are topologically equivalent to disks. We can thus look for a suitable connected subset of half the faces of a polyhedron, so that the edge of this region touches all vertices exactly once. We can then go one step further and ask whether this partitioning can be continued through the volume of the solid, so that two pieces result that can be physically separated. This then provides another attractive way to visualize the Hamiltonian cycles on these polyhedra (Fig.2).
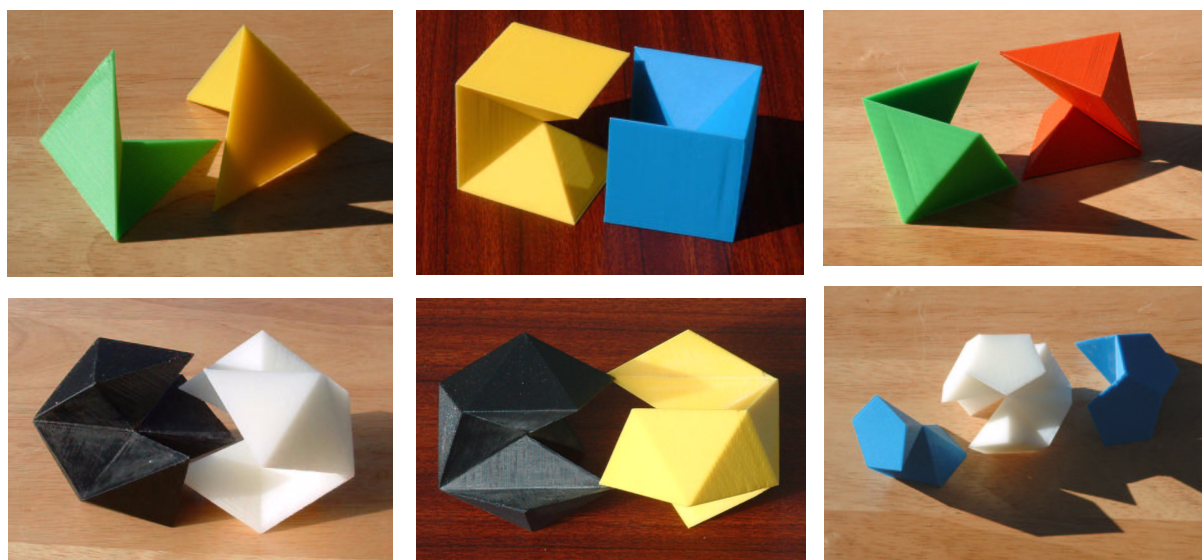
Figure 2: *Hamiltonian dissections of the Platonic solids.*
(Some net templates for these objects can be found in an appendix to the electronic paper version)

On the simplest regular solid, the **tetrahedron**, there is only one possible cycle, composed of four edges that surround two adjacent triangles, and this polyhedron can readily be dissected by connecting the four edges to the centroid (Fig.2a).

The **cube** also admits only one possible cycle: eight edges surrounding three in-line squares. The extension of this cut to the centroid creates two congruent parts that slide together with a snug fit (Fig.2b).

There exist two possible Hamiltonian cycles on the **octahedron**: a path with D_2-symmetry in the shape of a double-Z (Fig.2c), and a 3-fold zigzag around the sides of the octahedron when the latter is viewed as a 3-sided antiprism, exhibiting D_{3d} symmetry. The corresponding half-shells correspond to two of the three possible ways in which four triangles can be assembled in the plane. Both cycles readily yield dissections when extended to the centroid.

To form two congruent half-shells surrounded by a Hamiltonian cycle for the **dodecahedron**, we have to assemble six pentagons. Only one arrangement, forming an S-shaped path of tiles, will work. However, the two half-shells are so intertwined that there is no movement that allows the faces to be separated without some of them passing through some other ones. On the other hand, this object can be partitioned into three separable pieces by conically extruding 6, 3, and 3 pentagons to the centroid, respectively (Fig.2f).

The **Icosahedron** presents a different challenge. We need to look for all possible ways in which ten equilateral triangles can be connected so that their joint perimeter visits every vertex exactly once. There is one fairly obvious Hamiltonian cycle with S_6-symmetry, and it readily forms a dissection when stretched to the centroid (Fig.2d). Furthermore, there are also seven Hamiltonian cycles splitting the icosahedron into two congruent halves (Fig.3) while exhibiting C_2-symmetry around the center of one of the edges in the path (marked by a dot). In addition, there are several cycles that lead to two non-congruent half-shells. A few cycles, e.g., the one corresponding to the rightmost net, marked * in Figure 3, also lead to a dissection of the solid polyhedron, if we modify the internal partitioning surface to have more edges parallel to the direction in which we plan to slide the two pieces apart (Fig.2e).
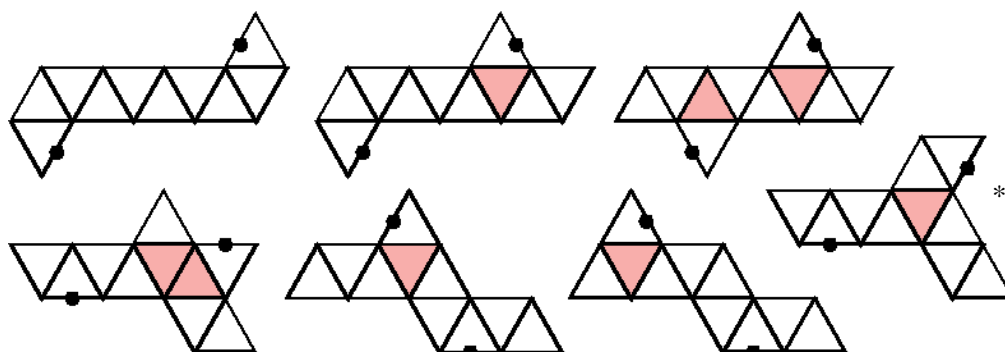

Figure 3: *The nets of the seven C_2-symmetrical paths that cut an icosahedron into congruent half-shells.*

3. Uniform Edge Coverage with Congruent Hamiltonian Cycles

In preparation for our ultimate goal in 4D, we will try to fully cover the above edges graphs with congruent copies of a Hamiltonian cycle. However, the octahedron is the only Platonic solid that possesses vertices with an even valence. Indeed, two copies of the double-Z path (Fig.2c) can jointly cover all the edges of this graph [5, Fig.2b]. Clearly, for the other solids with odd valences there is no hope to achieve such coverage. However, if we are willing to consider a uniform **double** coverage of all edges, then we have a reasonable challenge. In the tetrahedron, cube, and dodecahedron the number of vertices (and thus the length of the Hamiltonian cycle) is equal to two thirds of the total number of edges. Thus we should try to use 3 copies of a suitable Hamiltonian cycle to cover all polyhedron edges exactly twice. For the icosahedron this ratio is 2/5, and thus we should look for a set of five copies to obtain uniform double coverage.

It turns out that three of these four puzzles have satisfactory solutions. As an illustration we show the dodecahedron and the icosahedron with double edges (Fig.4a,b). The individual Hamiltonian cycles alternate between outer and inner edges. The cube is a special case; its unique Hamiltonian cycle cannot be replicated in a manner to cover all edges twice. But by being a little more open-minded we can also get around this problem: If we allow an open-ended Hamiltonian path, terminated at each end by two half edges, the necessary uniform coverage can be achieved (Fig. 4c); the three paths together form one 3-fold symmetric cycle that covers all edges twice.
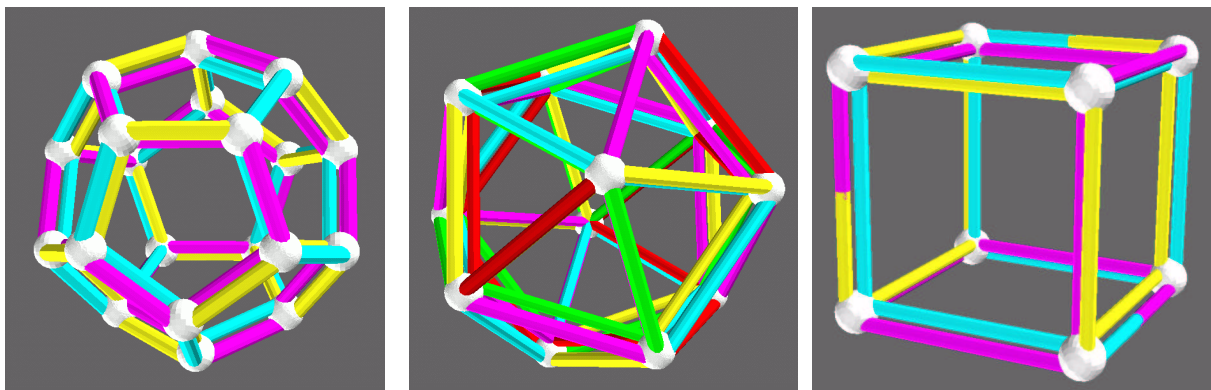

Figure 4: *Double edge coverage of (a) the dodecahedron and (b) the icosahedron with 3 and 5 cycles, respectively, and (c) of the cube with 3 congruent Hamiltonian paths.*

4. Sculptures Inspired by Hamiltonian Paths

This section samples some artistic constructions resulting from these mathematically determined models.

4.1 Sweeps along 3D Hamiltonian Paths

Figure 5a shows a simple prismatic sweep along the unique Hamiltonian cycle of the dodecahedron using a 6-sided cross section suitably modified to match the dihedral angle of 116.5° of the Platonic solid.
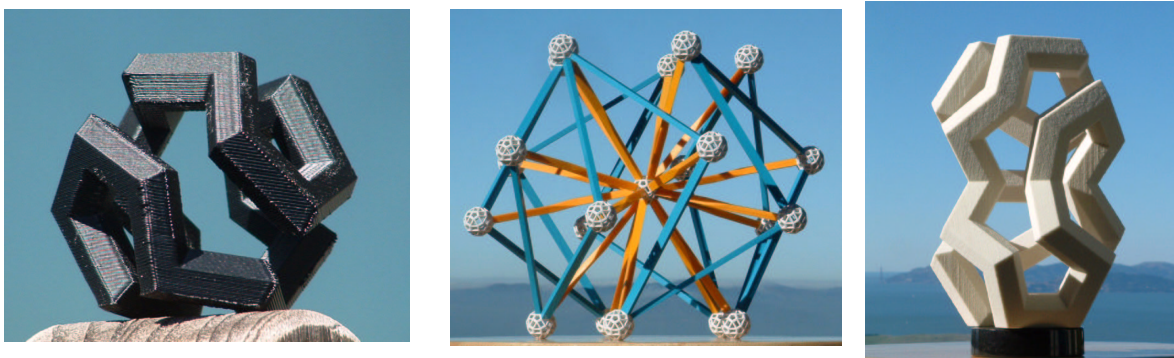
Figure 5: *Sweep sculptures derived from Hamiltonian cycles on the dodecahedron.*

In the spirit of Roelofs' sculpture (Fig.1b), instead of traversing the outer edges of a Platonic polyhedron, one can use a set of **space diagonals** to create a path that visits all 20 vertices exactly once. As a variation of this theme, Figure 5b shows a highly symmetrical Hamiltonian cycle that is composed of 20 **face** diagonals on a dodecahedron. Sculptures often gain in drama and impact if some of their inherent symmetries are judiciously broken. Figure 5c shows an example of a sculpture derived from a stack of two dodecahedra, where the Hamiltonian path has been made as irregular as possible.

4.2 Embedded Soap Films

As an extension of the *Volution* surfaces presented in Granada [4], we can create similar objects based on Hamiltonian cycles on the edges of Platonic solids. Using such a cycle as a wire frame, a surface topologically equivalent to a disk can be suspended in this loop, and a program such as the Surface Evolver by Ken Brakke [1] can be used to smooth its geometry into the shape of a minimal surface. To make the resulting sculptures more interesting, additional tunnels connecting different flanges can be introduced, and the surface optimization process can be repeated. This leads to *Volution* surfaces of higher genus. Figure 6 shows results based on the dodecahedron, and on the icosahedron with 2 and 4 tunnels, respectively.

Figure 6: *Volution shapes based on Hamiltonian cycles on (a) the dodecahedron and (b,c) the icosahedron.*

5. Hamiltonian Cycles on the Regular 4D Polytopes

In four dimensions there exist six regular polytopes [2]. Table 1 summarizes some of their salient geometric features and lists the valence v of the vertices, the number w of faces (or cells) sharing each edge, the number n of sides on each face, and the number and type of cells that makes up the shell of each polytope. Many different symmetric edge projections from 4D to a 3D subspace have been discussed and illustrated in [3]. We restrict ourselves to close-up perspective cell-first projections, where the whole 3D image is completely contained within a single outer cell. These projections maintain a high degree of symmetry and have no coinciding vertices or edges.

**Table 1: Characteristics of the Regular Polytopes in 4D**

	Simplex	Tesseract	16-Cell	24-Cell	120-Cell	600-Cell
#Vertices	5 (v=4)	16 (v=4)	8 (v=6)	24 (v=8)	600 (v=4)	120 (v=12)
# Edges	10 (w=3)	32 (w=3)	24 (w=4)	96 (w=3)	1200 (w=3)	720 (w=5)
# Faces	10 (n=3)	24 (n=4)	32 (n=3)	96 (n=3)	720 (n=5)	1200 (n=3)
# Cells	5 (tetra)	8 (cube)	16 (tetra)	24 (octa)	120 (dodeca)	600 (tetra)

An ongoing quest for the last couple of years has been to find congruent Hamiltonian cycles that will fully color the edge graphs of these six polytopes. The results for the four simpler ones has been presented last year [5], but the solutions for the two large graphs of the 120-Cell and the 600-Cell have eluded us until recently. Here is a quick review of the simple cases: The **4D simplex** can be covered by two identical Hamiltonian cycles that nicely complement each other (Fig.7a). The **4D cross polytope** (16-Cell) requires three colors, since all its vertices are of valence 6. A very attractive symmetrical arrangement of three congruent paths has been found (Fig.7b). The **hypercube** requires only two Hamiltonian cycles, since all its vertices are of valence 4. Several different valid coloring schemes have been found. The one shown in Figure 7c transforms one path into the other with a simple 90°-rotation around the horizontal axis shown.
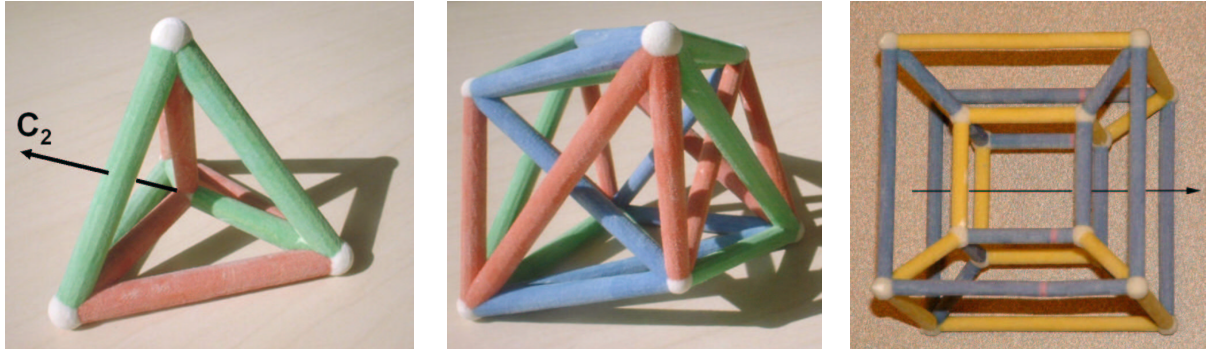
Figure 7: *Congruent Hamiltonian cycles on (a) the 4D simplex, (b) the cross polytope, (c) the tesseract.*

Four Hamiltonian cycles are needed to cover the **24-Cell** since all vertices are of valence 8. The most symmetric solution has the octahedral edge symmetry shown in Figure 8a. A rotation around any of the four C_3-axes leaves one color in place, while the other three colors are cyclically permuted for each rotation step of 120°. The solution was found by considering separately the individual shells in this object and finding colorings for them that adhere to the chosen overall symmetry scheme [5]. For each shell there were only a rather finite number of possible solutions, and it was then just a matter of trying out combinations of these shells that would connect all the edges of the same color into a single cycle. This dramatically reduced the dimension of the search space. Figure 8c shows one isolated cycle in which all the edges are colored differently according to the particular shell that they belong to.
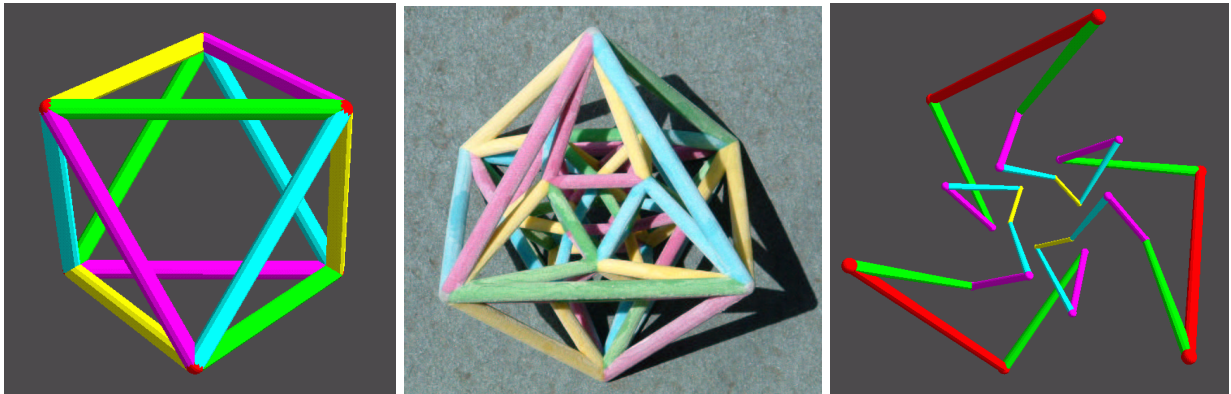
Figure 8: *(a) Basic color scheme chosen; (b) four congruent Hamiltonian cycles on the 24-Cell; (c) a single cycle colored by the respective shells that each edge belongs to.*

6. The 600-Cell

The shell-based approach also helped us to find eventually a desired coloring of the 600-Cell. We analyzed the perspective projections of this polytope by sorting all vertices into shells according to their distances from the origin, and sorting all edges into either *intra-shell* edges (connecting vertices of the same shell), or *inter-shell* edges (connecting two different shells). For the 600-Cell, we find that there are 15 discrete shells. Eight of them have intra-shell edges. But they are relatively sparse; no shell has more than 12 edges. In addition, there are 41 "connector shells" that contain edges going from one vertex shell to a different one; these shells have a maximum of 24 member edges. A complete table can be found in [5].

To maintain congruence among the six Hamiltonian cycles, we set up five transformations that copy one prototype path into its five differently colored siblings so that we obtain different colors on the six edges of the outermost tetrahedron. All the above edge-shells were analyzed separately as to all possible

coloring patterns that are compatible with this chosen overall color symmetry. It turns out that the 24-edge shells can be treated like two shells with 12 edges, since their respective edges cannot map into one another under the chosen global coloring scheme. For each such shell of 12 edges we then have to assign two prototype edges of one color, so that five differently colored copies of this pair can be placed without causing any interference; we always find exactly 12 different solutions.

The shell structure also shows an inside-outside symmetry, representing a point inversion symmetry at the centroid of the original 4D object. We gambled (successfully) that the desired Hamiltonian cycle would maintain this symmetry, which allowed us to cut in half the depth of our search tree for a viable path.

The search algorithm we used was a simple depth-first search on all option combinations of the different shells. We began with the smallest, most constrained shells and progressively tried various option combinations on other shells to incrementally build up a viable Hamiltonian cycle. Two checks need to be made in each such step. First, no node in the graph must ever admit more than two edges of the same color ("color conflict"). Second, we must determine whether adding the latest edge would prematurely create a cycle ("loop conflict"), since the Hamiltonian cycle must only be closed with the very last edge added. For the latter check, Daniel Chen implemented a union-find algorithm on a disjoint-set forest data structure. All vertices are given a link pointer, which initially points to that same vertex. For each chain of connected edges, there is a single representative vertex that points to itself. When we add an edge, we change the representative of the net to which the first vertex belongs to the representative of the net of the second vertex, and we record this change in a separate list for easy backtracking. Now loop conflicts are easily detected when both vertices of a new edge reference the same representative vertex, and that move can then readily be un-done.

7. The 120-Cell

On the perspective projection of the 120-Cell with all valence-4 vertices, we needed to find two congruent Hamiltonian cycles of length 600. This was an even harder task, because the search space was so much deeper and there were fewer constraints from the differently colored sibling paths. The only way the two complementary paths can transform into one another is by a point-mirror operation at the geometrical center of the edge graph. Possible plane-mirror operations or C_2 rotations are excluded, because they all map some edges of the dodecahedron back onto themselves. Trying to reduce the depth of the search tree, we looked for inherent symmetries that we might find in the prototype path itself. However, neither 3-fold nor 5-fold symmetry will be possible: Consider, for instance, the edges circling a pentagonal face around a potential C_5 symmetry axis. The only way to maintain 5-fold symmetry with a 2-color scheme would be to make them all the same color. However, this would form a sub-cycle of length 5 – which is not allowed.

We can also rule out inside/outside (I/O) symmetry. To see that, we look at the central intra-layer in vs7. These edges form 12 pentagons. These rings of odd count can neither be of just one color, nor of strictly alternating colors. Thus there must be vertices where the edges going off to vertex shells vs5 and vs9 respectively (see Table 2) will be of the same color, and others where they will be of different color. The overall complementarity of opposite edges, together with I/O symmetry would force all these edge pairs to be of different color; on the other hand, together with I/O anti-symmetry it would force them all to be of equal color. But as we just saw, they cannot possibly be all of the same type; thus neither type of I/O symmetry can exist.

Table 2 shows the complete set of all edges in the various shells. However for the 120-Cell, the shell approach is not so well suited. Some shells have up to 120 edges which need to be assigned to one of two colors. There are simply too many possible combinations to enumerate them all! After trying many different approaches, we finally achieved success with a mixed strategy. We tried to make the color assignments on a shell by shell basis, but we constructed the shells on the fly, using a backtracking search on a per edge-pair basis. This approach has the problem that on the last few shells to be colored, many of the assignments

were forced by the avoidance of color conflicts; however, these assignments often produced multiple loops. The backtracking depth needed to undo these loop conflicts appeared to be very deep, and the search never finished in a reasonable time (a few days). A modification to our strategy fixed this problem: Whenever an edge reaches a vertex that already has an edge of the same color or two edges of the complement color – which thus fixes the coloring of all the edges at that vertex – we make these assignment immediately, and recursively continue these forced paths, regardless of what shells these edges belong to. This strategy discovers any "forced" loop formation at a much earlier stage, where a smaller backtracking effort is sufficient to correct the situation. This turned out to be the crucial component of our approach! Now it only takes a few seconds to find a solution. With that strategy we can find solutions even by processing the edges in random order without concern what shells they are part of. However, the random approach takes somewhat longer than the structured approach, and it does not work well on graphs of higher valence; thus it does not find a solution for the 600-Cell in a reasonable amount of time.

Table 2: Shell and Connector Schedule for the 120-Cell

	vs0	vs1	vs2	vs3	vs4	vs5	vs6	vs7	vs8	vs9	vs10	vs11	vs12	vs13	vs14
vs0	30														
vs1	20	0													
vs2		60	0												
vs3			60	60											
vs4				60	30										
vs5					**120**	0									
vs6						60	0								
vs7						60		60							
vs8							20		0						
vs9								60	60	0					
vs10										**120**	30				
vs11											60	60			
vs12												60	0		
vs13													60	0	
vs14														20	30

Thus there is a suspicion that there may be millions of viable Hamiltonian cycles. We did one experiment to obtain an idea what fraction of all colorings with no color conflicts might yield the desired pair of two congruent cycles. The two largest shells with 120 edges, connecting vertex shells vs4 to vs5 and vs9 to vs10, respectively, form twelve decagon-rings each. These rings can be colored in an alternating ABAB... pattern. When we flip this pattern to the complementary BABA.... state, the resulting coloring is still legal, but the connectivity of chains of same-colored edges is changed dramatically – most likely breaking some loops and forming others. There are 4096 different combinations of the state of these 24 rings; it turned out that when we started from a complete solution, 46 of these combinations also produced viable Hamiltonian cycles.
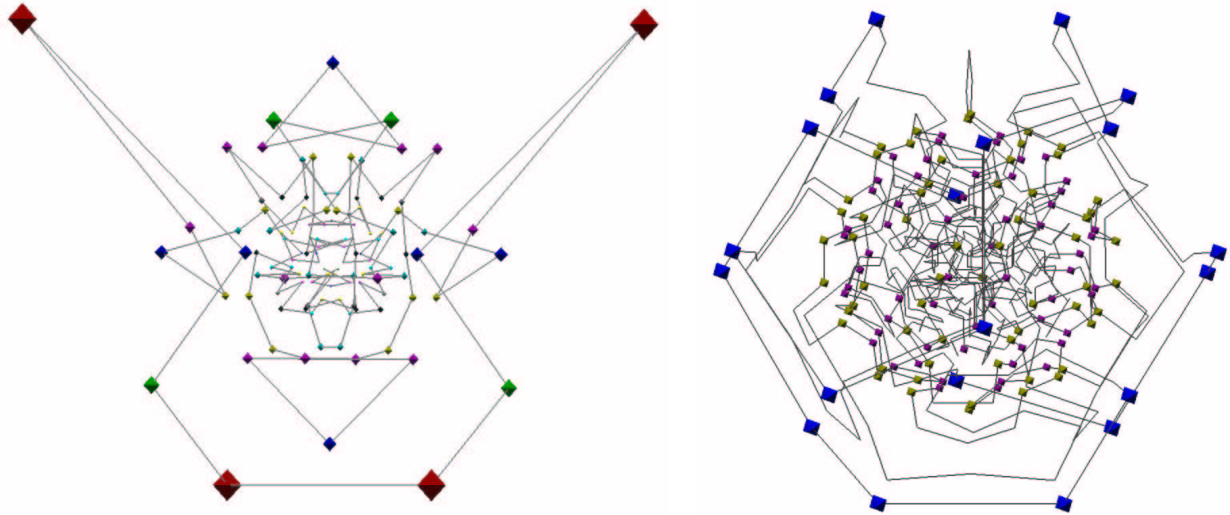
Figure 9: *Hamiltonian prototype paths (a) on the 600-Cell and (b) on the 120-Cell.*

8. Hamiltonian 2-Manifolds

A Hamiltonian cycle can be understood as a closed contiguous 1-dimensional manifold that visits all the zero-dimensional elements (vertices) of a given graph (or polyhedron). Stepping up from 3D to 4D, we now consider what 2D-manifolds one can find that visit all the 1D-elements (edges) of a polytope. These 2D-manifolds may be closed "Hamiltonian shells" that **pass through** all the given edges, or they may form "Hamiltonian surfaces" that have some borders, i.e., edges in which the surface terminates. Just as we tried to find congruent sets of Hamiltonian cycles or paths that cover all edges uniformly (once or multiple times), we now will try to find ways to cover all the faces of the 4D-polytopes uniformly with multiple copies of such a Hamiltonian shell or surface. If the edges are of odd valence, i.e., if an odd number of 3D cells is clustered around it, then the surface needs to have some borders in order to obtain single coverage.

On the **4D Simplex or 5-Cell** all edges are of valence 3, we thus need to look for a surface with a border. Using five triangles, i.e., exactly half of the available facets, we can form a Moebius band with five internal edges between the triangles and with a single border of length 5 forming a Hamiltonian cycle on this polytope. To obtain full coverage, two such Hamiltonian surfaces can now be joined so that every edge is an internal edge in one Moebius band and also part of the border of the other one (Fig.10a). If built with translucent facets of two different colors, this makes a very attractive object.

On the **Hypercube or Tesseract,** the most natural and most symmetrical shell that visits all the edges is a torus with four 4-sided segments. This torus dissects the 3D-"surface" of the tesseract into two intertwined rings of four cubes each. It uses 16 out of the 24 square faces; thus we need three copies of the 3D torus (at right angles to one another) to obtain a uniform double-coverage of all faces. Alternatively, we can remove 4 square windows from this torus to obtain a Hamiltonian surface of genus 5 (Fig.10b). In the 4D polytope, two copies of this surface cover all faces exactly once.

The **4D Cross Polytope** has edges of valence 4. We can form a borderless surface using 16 out of the 32 triangles. Euler's formula: $24\{\#E\} - 8\{\#V\} - 16\{\#F\} + 2 = 2*$genus, tells us that this 2-manifold should have the topology of a torus. It dissects the 3D-"surface" of the 16-Cell into two rings of eight tetrahedra. Two copies of this Hamiltonian shell cover all 2D faces. Figure 10c shows a 3D model with S_2 symmetry.

The **24-Cell** has edges of valence 3. A closed shell that passes through all edges would have to comprise 64, i.e., 2/3 of all triangular faces. Euler's formula: $96\{\#E\} - 24\{\#V\} - 64\{\#F\} + 2 = 10$, yields a genus of 5. Ideally this shell should partition the 24-Cell into two sets of twelve octahedra. To yield con-

nectivity of genus 5, eight octahedra might serve as 3-way connectors, while the other four would just contact two neighbors. Alternatively, 16 octahedra with the connectivity of a hypercube provide the most symmetrical solution, but their combined surface is no longer truly manifold, since all vertices get touched by that surface more than once. An alternative coverage might be obtained from a triangle strip closed into a loop of 48 triangles, with two borders of length of 24 each. But again, all vertices will be touched more than once by these borders. Work continues to find the most satisfactory solution.

Future work will also perform a similar analysis for the **120-Cell** and for the **600-Cell** with the goal of finding the most symmetrical configurations of such Hamiltonian 2-manifolds.
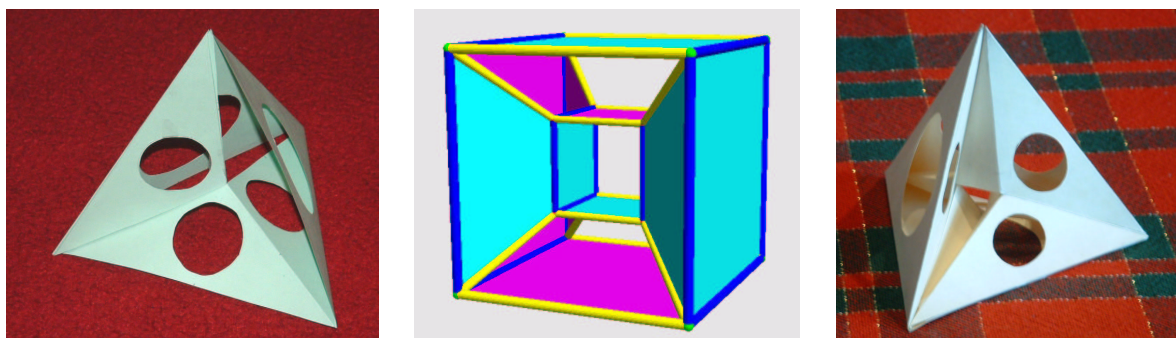
Figure 10: *Some Hamiltonian 2-manifolds on (a) simplex, (b) hypercube, (c) cross polytope. (Net templates for these objects can be found in an appendix to the electronic paper version)*

9. Conclusions

Our original goal was to find Hamiltonian cycles for all six regular 4D polytopes so that congruent copies of them would yield complete edge coverage. Finding such cycles for the two most complex graphs relied heavily on computer graphics visualization and even on physical models of the projections to 3D generated on a rapid prototyping machine. Insight gained from these modeling efforts were crucial to finding a structured approach for the search of such cycles. We then extended the notion of a Hamiltonian cycle to include 2-manifolds that contain all the edges of these graphs. Interesting new geometrical structures are emerging from this quest. Trying to find the most symmetrical configuration is our general goal, but we have little knowledge as to how close we have come to that goal in an absolute sense with our first emerging solutions for the three more complicated regular polytopes. When it comes to symmetry, many intriguing questions are still open.

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