Factor Group Transformations on Escher Patterns

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Abstract

The artist M.C. Escher intuitively expressed many mathematical concepts in his graphic designs. One of these concepts was that of transformations between different factor groups over the complex plane. We describe a method whereby any tiling that can be expressed as a rectangle system can be mapped to a multiplicative factor group over the complex plane.

1 Introduction

Ever since its creation in 1956, the blank spot in M.C. Escher's *Prententententoonstelling,* or *Print Gallery*, has fascinated enthusiasts of his work. Why is the sense of limit, so present in his other works, absent here? In 2003 Hendrick Lenstra and B. de Smit described how to express a version of the *Print Gallery* as a factor group over the complex plane [1]. This version of the picture repeats itself as its twists and shrinks down endlessly toward the center. The repeated picture is rotated clockwise by 157.62 degrees and scaled down by a factor of 22.58.

Though Escher's process was guided by his intuition and artistic skill, his result was much the same as Lenstra's: Images that utilize an isomorphism from one factor group of the complex plane to another. Escher started with a series of sketches that expressed a Droste effect scaled down by a factor of 256. A Droste effect occurs when an image contains a copy of itself. The term is named after the Dutch chocolate maker Droste for the visual effect on its boxes of cocoa. Any good Dutch dictionary will contain a reference to this effect [2]. Escher's sketches manifest the multiplicative factor group over the complex plane represented by $\mathbb{C}^*/\langle 256\rangle$. He then used a spiraled grid to transform the picture into one that is very closely modeled by the multiplicative factor group $\mathbb{C}^*/\langle\omega\rangle$, where $\omega = (2\pi\mathbf{i} + \ln 256)/(2\pi\mathbf{i})$.

Though *Print Gallery* was the first print of its kind, it was neither the first nor the last print that Escher did that can be replicated as factor group transformations (see *Development II* and *Smaller and Smaller*). Utilizing an exponential function on Escher prints over the complex plane similar to Lenstra's, complex factor group transformations can be realized in Escher's other works. The most notable of these is the *Path of Life* series of prints.

2 Path of Life Isomorphisms

Let us begin by looking at Escher's Regular Division Drawing 102 (Figure 1), done in 1958. This drawing was used as a basis for both Escher's *Path of Life I* and *Path of Life II*. To create a transformation of this drawing over the complex plane, we first need to explain how it will be placed on the complex plane. We proceed by laying this regular division upon the complex plane in such a way that eight tiles fit perfectly in a distance of 2π along the imaginary axis. This will give us the additive factor group $\mathbb{R}/\langle\mathbf{a}\rangle\oplus\mathbb{R}\mathbf{i}/\langle\pi/4\rangle$, with $\mathbf{a}$ being the period in the real direction (in this case $\mathbf{a}$=0.523598...).

By applying the exponential map $f(z)=e^z$ to Figure 1, we carry out an isomorphism from the additive factor group $\mathbb{R}/\langle\mathbf{a}\rangle\oplus\mathbb{R}\mathbf{i}/\langle\pi/4\rangle$ to the multiplicative factor group $\mathbb{C}^*/\langle e^{\mathbf{a}}\rangle$. If we were working with purely rectangular tiles, $e^{\mathbf{a}}$ would be the scaling factor between concentric circles in the resulting picture. In this case we will have each circle being $e^{\mathbf{a}}$=1.68809 times larger than the circle inside of it.

Figure 1: Escher's Regular Division 102, 1958.

In looking at the resulting picture (Figure 2), we can see a striking resemblance to Escher's print *Path of Life I* (Figure 3). The noticable difference is where Escher chose to have his outermost fish curving back into the center. Otherwise, the differences are negligible.

Figure 2: Exponential mapping of Figure 1 with 8 tiles fitting in 2π distance along the imaginary axis.

Figure 3: *Path of Life I,* 1958.

The print *Path of Life II* can also be seen to possess a similar transformation. The difference is, when we place the tiling upon the complex plane we use four tiles to fit into the distance of 2π along the imaginary axis instead of eight. In comparing the result (Figure 4) to Escher's original (Figure 5) we can again see the similarity. This transformation would be an exponential mapping from $\mathbb{R}/\langle \mathbf{a} \rangle \oplus \mathbb{R}\mathbf{i}/\langle \pi/2 \rangle$ to $\mathbb{C}^*/\langle e^{\mathbf{a}} \rangle$, where **a** is the period in the real direction (in this case **a**=1.04719...).

Figure 4: Exponential mapping of Figure 1 with 4 tiles fitting in a 2π distance along the imaginary axis.

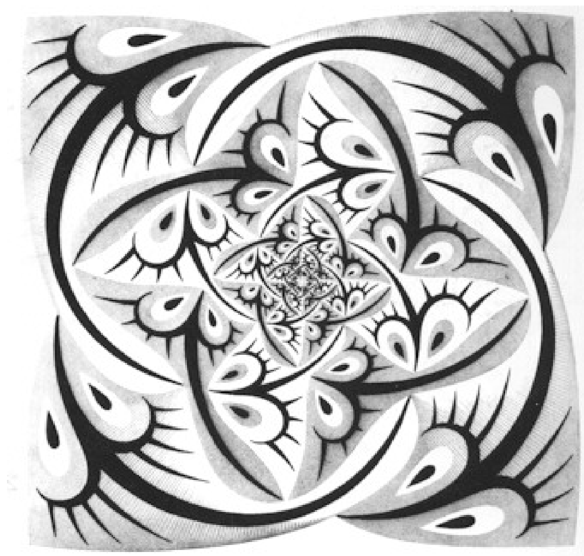

Figure 5: *Path of Life II,* 1958.

Path of Life III, the last drawing in the series, can also be seen as a factor group transformation over the complex plane. The basis for this print is Escher's Regular Division Drawing 125 (Figure 6), done in 1966. We lay this division on $\mathbb{C}$ in such a way that we have six tiles fitting in the distance of 2π along the imaginary axis. We then use the exponential mapping to transform the image to $\mathbb{C}^*$. The result (Figure 7) can be compared to Escher's original (Figure 8). This transformation would be an exponential mapping from $\mathbb{R}/\langle \mathbf{a} \rangle \oplus \mathbb{R}\mathbf{i}/\langle \pi/3 \rangle$ to $\mathbb{C}^*/\langle e^{\mathbf{a}} \rangle$ (in this case **a**=1.03199...).

Figure 6: Escher's Regular Division 125, 1966.

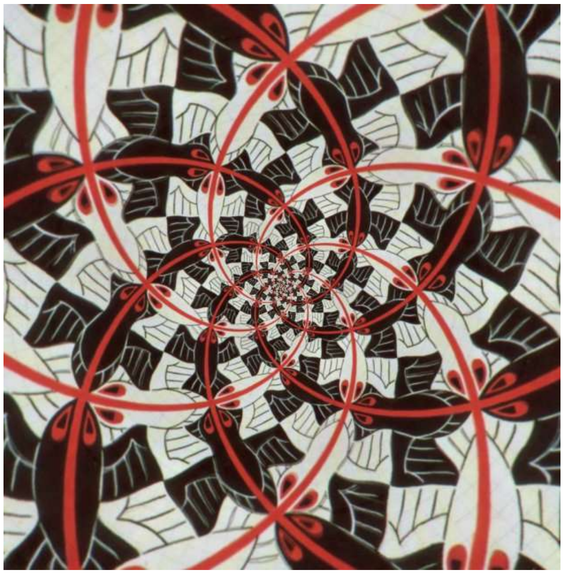

Figure 7: Exponential mapping of Figure 1 with 6 tiles fitting in a 2π distance along the imaginary axis.

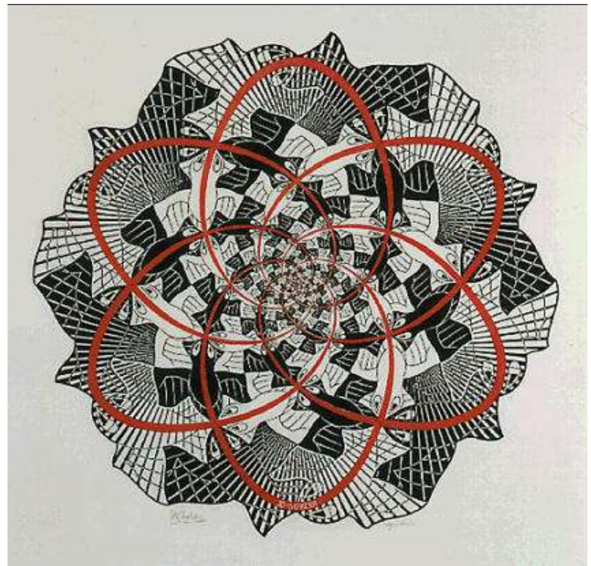

Figure 8: *Path of Life III,* 1966

3 General Method

If one wishes, one can take any tiling that can be expressed as a rectangle system and create a factor group transformation from $\mathbb{R}/\langle \mathbf{a} \rangle \oplus \mathbb{R}\mathbf{i}/\langle 2\pi/\mathbf{n} \rangle$ to $\mathbb{C}^*/\langle e^{\mathbf{a}} \rangle$. Where **a** is the period in the real direction and **n** is the number of tiles fitting in the distance of 2π along the imaginary axis. For example, take Escher's Regular Division number 102 (Figure 9), done in 1941. We can perform an exponential transformation to take it from $\mathbb{R}/\langle \mathbf{a} \rangle \oplus \mathbb{R}\mathbf{i}/\langle 2\pi/4 \rangle$ to $\mathbb{C}^*/\langle e^{\mathbf{a}} \rangle$ (Figure 10). We can also take it to $\mathbb{C}^*/\langle \omega \rangle$, where $\omega=(2\pi\mathbf{i}+\mathbf{a})/(2\pi\mathbf{i})$, with $f(z)=e^{\omega z}$ (Figure 11).

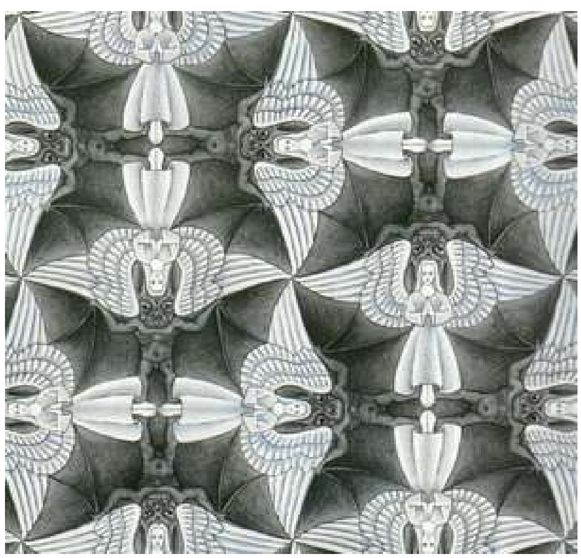

Figure 9: Escher's Regular Division 45, 1941.

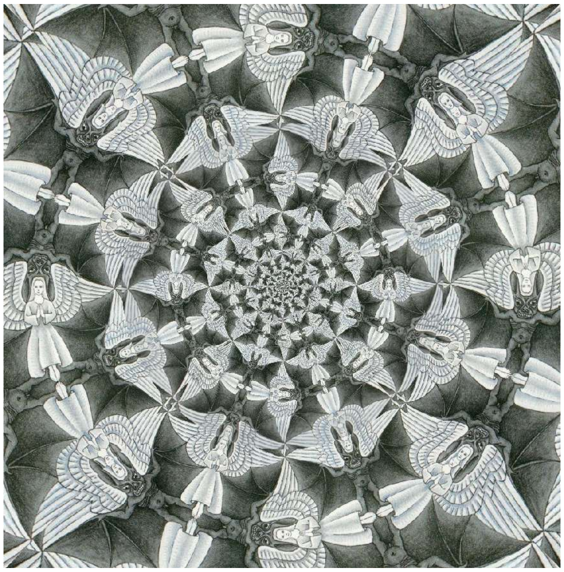

Figure 10: Transformation from $\mathbb{R}/\langle a\rangle \oplus \mathbb{R}i/\langle 2\pi/4\rangle$ to $\mathbb{C}^*/\langle e^a\rangle$.

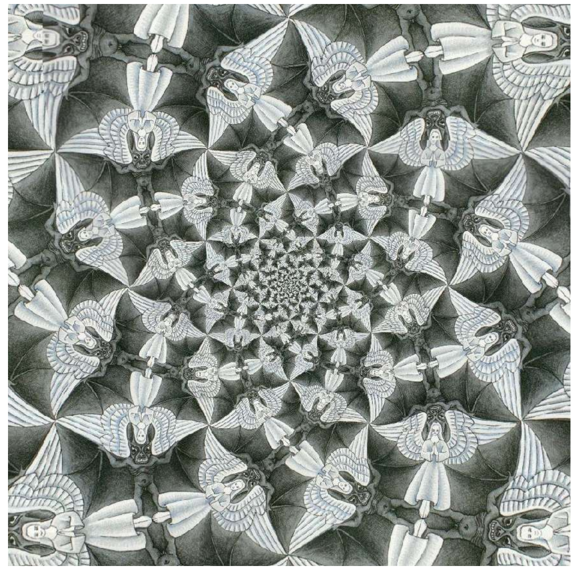

Figure 11: Transformation from $\mathbb{R}/\langle a\rangle \oplus \mathbb{R}i/\langle 2\pi/4\rangle$ to $\mathbb{C}^*/\langle \omega\rangle$

There is a simple and more intuitive way of understanding these transformations. Imagine curling up the plane along the real axis into an infinitely long, evenly lit cylinder with **n** tiles wrapping around the circumference of the cylinder [3]. Looking down the center of the cylinder will give us a view much like Figure 10. Now, if we chose to wrap up the plane at a diagonal to the real axis such that the tiles still wrapped seemlessly around the circumference of the cylinder, we would get something more like Figure 11.

4 Conclusions

We have presented a method whereby any rectangular tiling can be mapped to a multiplicative factor group over the complex plane. By utilizing exponential mappings on complex tori we we can gain an understanding of Escher's intuitive process.

Since Escher intuitively performed many different factor group transformations by hand, one could very well believe that the missing piece in the *Print Gallery* was made by design rather than accident. But this would conflict with how meticulously he crafted his other works. Would he, with the stated goal on this print being "to express a cylclic expansion or bulge without beginning or end" [4], have intentionaly left out the center of the *Print Gallery*? From the evidence presented in his works, we could infer that if Escher knew of the possibility of continuing to the limit, he would have done so.

Acknowledgements
Special thanks to Dr. Joseph Gallian and Dr. Douglas Dunham for their support on this project, and thanks to Mark Probst for his technical assistance on the software that performed the transformations.

Appendix I
The transformations in this paper were done with the MathMap GIMP Plug-In written by Mark Probst of the Institute of Computer Languages at Vienna University of Technology. The code that was used for the transformations is listed below. Note that "//" represents a comment that should not be included in the code. Also note that the MathMap language determines the color of a given pixel based on transformations on the coordinates of that pixel. Therefore all functions used are the inverses of the functions in the above text.

Code:

```
//n is the number of tiles in a 2π distance along the imaginary axis
//ytile is the number of pixels in a tile in the y direction
//xtile is the number of pixels in a tile in the x direction
unit=n*(ytile)/(2*pi);
xmod=xtile;
ymod=2*pi*unit;
twist=0;
xoff=t*xmod;
p=ri:[x/unit,y/unit];
X1=(2*pi*I + twist*xmod/unit)/(2*pi*I);
q=X1*log(p);
ij=xy:[(unit*sum(q+conj(q))/2 + xoff)%xmod,(unit*sum(q-conj(q))/2)%ymod];
origVal(xy:ij)
```

References

[1] B. de Smit and H.W. Lenstra, The Mathematical Structure of Escher's Print Gallery, *Notices of the AMS*, April 2003, pp 446-451.

[2] escherdroste.math.leidenuniv.nl, *Escher and the Droste Effect*, Leiden University, Leiden, The Netherlands

[3] Doris Schattschneider*, Visions of Symmetry,* W.H. Freeman, 1990.

[4] Bruno Ernst, *De toverspiegel van M.C. Eshcer*, Meulenhoff, Amsterdam, 1976; English translation by John E. Brigham: *The Magic Mirror of M.C. Escher*, Ballantine Books, New York, 1976.

[5] E. Thé (design), *The Magic of M.C. Escher,* Harry N. Abrams, New York and London, 2000.