Fractal Tilings Based on Dissections of Polyhexes

Robert W. Fathauer
Tessellations Company
3913 E. Bronco Trail
Phoenix, AZ 85044, USA
E-mail: tessellations@cox.net

Abstract

Polyhexes, shapes made up of regular hexagons connected edge-to-edge, provide a rich source of prototiles for edge-to-edge fractal tilings. Numerous examples are given of fractal tilings with 2-fold and 3-fold rotational symmetry based on prototiles derived by dissecting polyhexes with 2-fold and 3-fold rotational symmetry, respectively. A systematic analysis is made of candidate prototiles based on lower-order polyhexes.

1. Introduction

Fractals and tiling are two fields in mathematics with considerable esthetic appeal to people from all walks of life. They can be combined to form a variety of visually appealing constructs that possess fractal character and at the same time obey many of the properties of tilings. Previously, we described families of fractal tilings based on kite- and dart-shaped quadrilateral prototiles [1], v-shaped prototiles [2], and prototiles that are segments of regular polygons [3]. Many of these constructs may be viewed online [4]. These papers appear to be the first attempts at a systematic treatment of this topic.

A few tilings of this sort were explored by the Dutch graphic artist M.C. Escher; e.g., see his "Square Limit" print of 1964 [5]. Escher desired strongly to depict infinite tilings in a finite print. His well-known "Circle Limit" prints are another solution to this problem, making use of hyperbolic geometry [5]. These are based on an illustration he found in H.S.M. Coxeter's writings. When Coxeter saw "Circle Limit III, he wrote to Escher, who described the correspondence in 1960: "I had an enthusiastic letter from Coxeter about my coloured fish, which I sent him. Three pages of explanation of what I actually did …" [6]. The fractal tilings described here are additional solutions to Escher's quest for depicting infinity.

In Grünbaum and Shephard's book *Tilings and Patterns* [7], a tiling is defined as a countable family of closed sets (tiles) that cover the plane without gaps or overlaps. The constructs described in this paper do not for the most part cover the entire Euclidean plane; however, they do obey the restrictions on gaps and overlaps. To avoid confusion with the standard definition of a tiling, these constructs will be referred to as "*f*-tilings", for fractal tilings.

The tiles used here are "well behaved" by the criterion of Grünbaum and Sheppard; namely, each tile is a (closed) topological disk. These *f*-tilings are also edge-to-edge; i.e., the corners and sides of the tiles coincide with the vertices and edges of the tilings. However, they are not "well behaved" by the criteria of normal tilings; namely, they contain singular points, defined as follows. Every circular disk, however small, centered at a singular point meets an infinite number of tiles. Since any *f*-tiling of the general sort described here will contain singular points, we will not consider singular points as a property that prevents an *f*-tiling from being described as "well-behaved".

The author's primary interest in these objects is recreational and esthetic. The fractal constructs described here are of interest because they are distinct from other fractal constructs, as we shall describe.

The prototiles considered in this paper are derived by dissecting polyhexes. A polyhex is a shape made by connecting regular hexagons in edge-to-edge fashion. We will call it an *n*-hex if it is a union of n hexagons. For a discussion of different types of polyhexes and conventional tilings using polyhexes, see Ref. 7.

Each prototile has one or more long edges and two or more short edges. The *f*-tilings are constructed by first matching the long edges of identical prototiles, then cloning the prototile, reducing its size by the ratio of the length of a short edge to that of a long edge, fitting multiple copies of these smaller tiles around the first generation of tiles according to a matching rule, and finally iterating this process an infinite number of times. The group of first-generation tiles forms the dissected polyhex. In practice, the appearance of the overall tiling changes little after 4-7 generations, since the tiles become extremely small relative to the first-generation tiles. The figures shown here are constructed by iterating until the individual tiles become very small on the scale of the page size. For a given *f*-tiling, every tile is similar to the single prototile, though in some cases mirrored variants are used. *f*-tilings may also be constructed that possess more than one prototile, but these will not be considered here.

2. Candidate Prototiles

Two requirements simplify the search for polyhex-based prototiles that allow *f* –tilings.
1. The generating polyhex must have 2-fold or 3-fold rotational symmetry. The reason 6-fold rotational symmetry is not allowed is discussed in Section 3, and other rotational symmetries do not occur in polyhexes. While polyhexes without rotational symmetry may be used, they generate no new prototiles, and therefore they do not need to be considered. Mirrored variants of polyhexes are not considered to generate distinct prototiles, as they would result in *f*-tilings that are an overall mirror of the *f*-tilings constructed from non-mirrored variants.
2. For a prototile generated by bisecting a polyhex, each dissecting line, which will form the long edge of the prototile, must originate and terminate at vertices of the polyhex and pass through the centroid of the polyhex. For a prototile generated by trisecting a polyhex, each dissecting line, which will form one long edge of the prototile, must run from the centroid of the polyhex to a vertex. The three trisecting lines are related to each other by rotations of 120° about the centroid. The bisecting and trisecting lines (long edges of the prototile) must be longer than the short edges of the polyhex (short edges of the prototile).

Figure 1 shows candidate polyhexes made up of 1 to 4 hexagons and prototiles that meet these criteria. Note that no *n*-hexes with 3-fold rotational symmetry exist for n = 2, 5, 8, 11, … While higher-order polyhexes are not shown in Figure 1, an example of an *f*-tiling based on a prototile derived from a 9-hex is given in Section 3.

3. *f*–tilings Based on Dissected Polyhexes

3.1. Bounded *f*-tilings. In this section, we give a number of examples of *f*–tilings constructed from prototiles derived by dissecting polyhexes. Our first examples have overall 2-fold rotational symmetry. There are two general construction options explored for these. In each, the starting point is a pair of tiles that form the generating polyhex. These are surrounded by smaller tiles that are either mirrored or non-mirrored copies of the larger tiles. The construction process proceeds iteratively, with each successive generation of smaller tiles either mirrored or unmirrored, as dictated by the earlier choice.

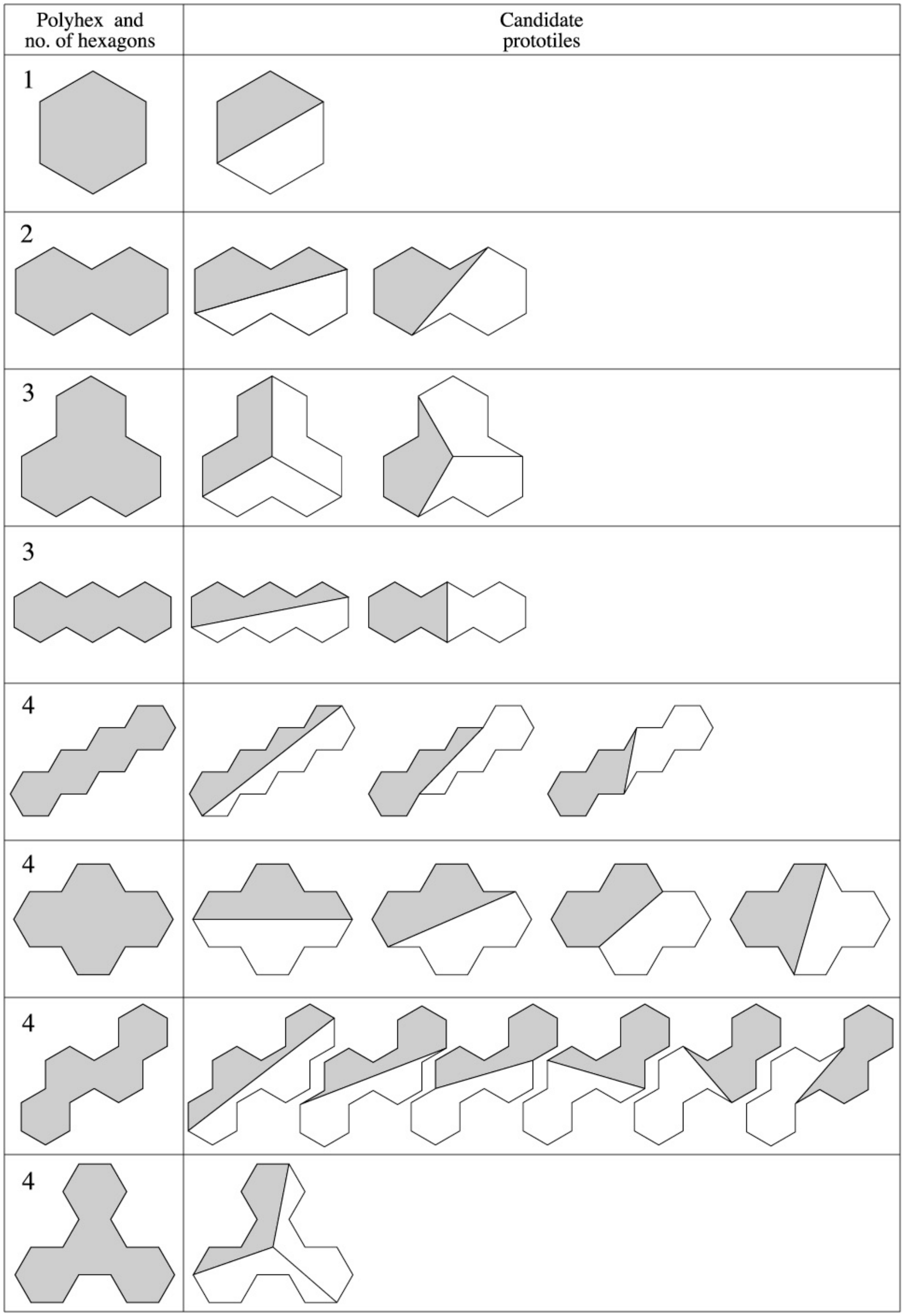

Figure 1: *Candidate prototiles (shaded) for polyhexes made up of 1 to 4 hexagons.*

In Figure 2, we show two examples of *f*-tilings of this sort. In Figure 3, we show an example of a candidate prototile that does not allow an *f*-tiling. In general, the boundaries of *f*-tilings are fractal curves, though there are cases in which the boundaries are non-fractal polygons. The boundaries are similar to Koch islands and related constructs, in which a line segment is distorted into multiple smaller line segments, which are in turn distorted according to the same rule, etc. [8].

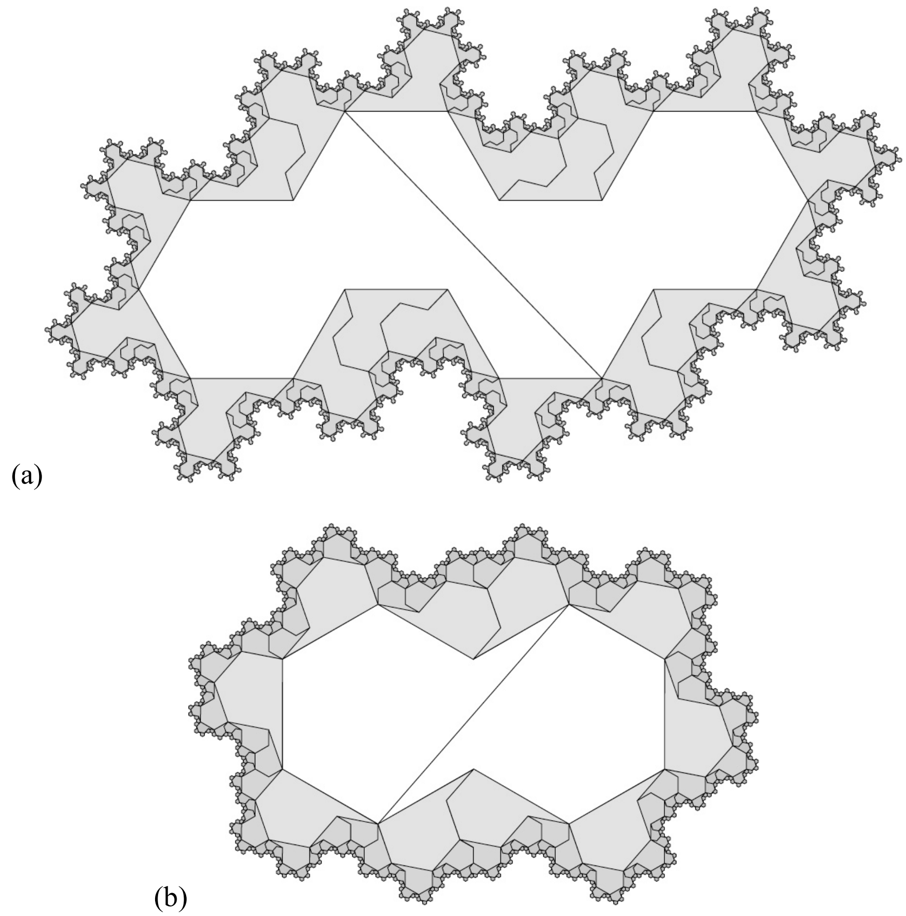

Figure 2: *(a) An f–tiling generated from a prototile based on a 4-hex, in which the scaling factor between successive generations of prototiles is approximately 0.2773.* (b) *An f–tiling generated from a prototile based on a 2-hex, in which the tiles are scaled by approximately 0.3780 and also mirrored between successive generations.*

Next we show some examples with overall 3-fold rotational symmetry. For these prototiles, there are two choices of matching rules for arranging tiles of a given generation around tiles of the next larger generation. This is illustrated in Fig. 4. In addition, the option of mirroring between generations also exists, so that there are four options overall. Another 3-fold rotationally-symmetric example is shown in Figure 5. In addition, the fifth prototile in Figure 1, derived by trisecting a 3-hex, has recently been produced as a physical tiling puzzle [9].

There are no *f*-tilings for prototiles of this sort that have 6-fold rotational symmetry. This occurs because any prototile derived by dissecting a polyhex into sixths will have two long edges and an odd number of short edges. Arranging the second generation tiles around the first generation tiles in a consistent fashion therefore leads to unmatched long edges for some second-generation tiles. This is illustrated in Figure 6.

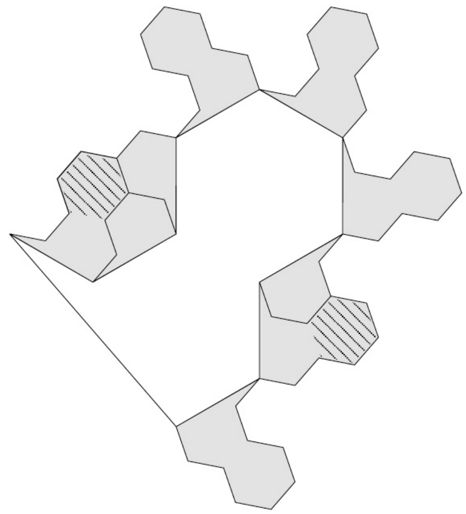

Figure 3: *An example of a prototile that does not allow an f-tiling. Note that the second generation tiles overlap in the shaded regions.*

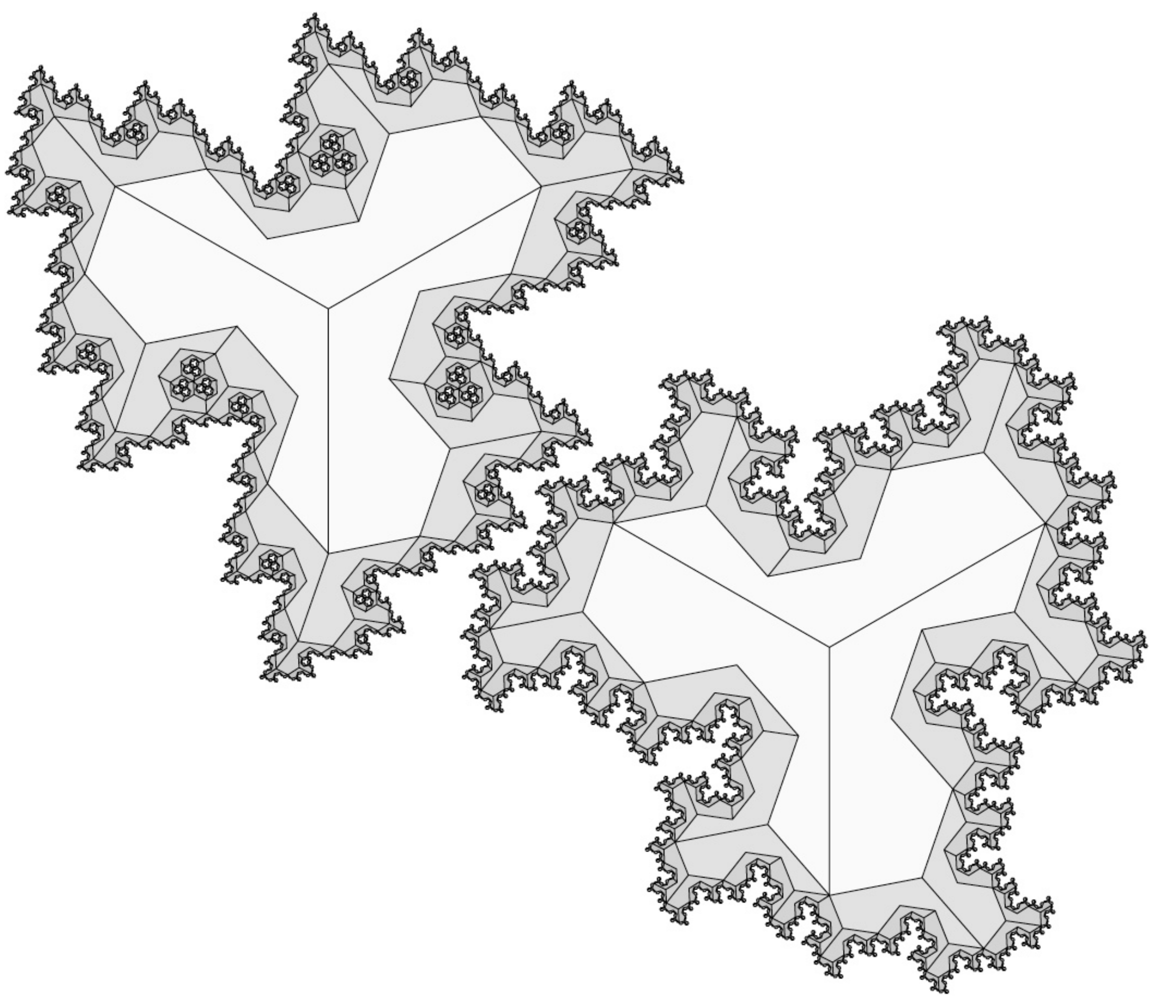

Figure 4: *Two f–tilings generated from a prototile based on a 4-hex, in which the scaling factor between successive generations of prototiles is approximately 0.3780. Different matching rules are used for the two.*

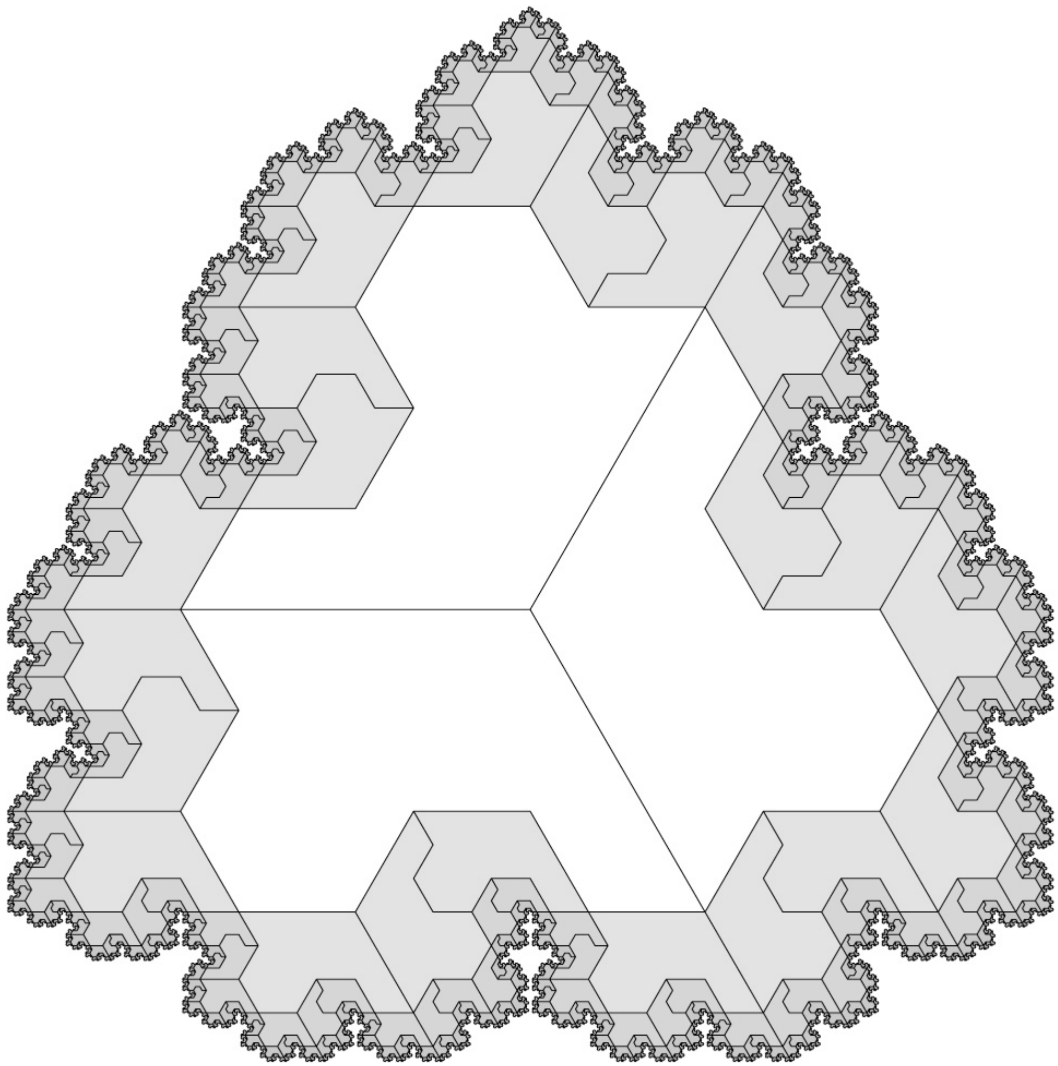

Figure 5: *An f–tiling generated from a prototile based on a 9-hex, in which the scaling factor between successive generations of prototiles is 1/3.*

3.2. Plane-filling *f*-tilings. In some cases, an entire *f*-tiling constitutes a supertile that tiles the plane; both the *f*-tilings in Figure 2 possess this property. In other cases, an *f*-tiling that covers the plane may be constructed by first constructing a non-fractal supertile that tiles periodically. Figure 7 shows an example in which the supertile is regular hexagon. A supertile could be any polyhex that tiles the plane. Figure 8 shows an example in which the supertile is an infinite strip rather than a polyhex.

4. Conclusions

We have presented several examples of fractal tilings (*f*-tilings) based on prototiles derived by dissecting polyhexes with 2-fold and 3-fold rotational symmetry. There are an infinite number of polyhexes and an infinite number of *f*-tilings of this sort. However, in general they become increasingly less interesting for higher-order polyhexes due to the fact that the scaling factor between successive generations becomes more extreme. Polyominoes and polyiamonds [7] may also be used to generate prototiles for *f*-tilings; some examples may be seen online [4].

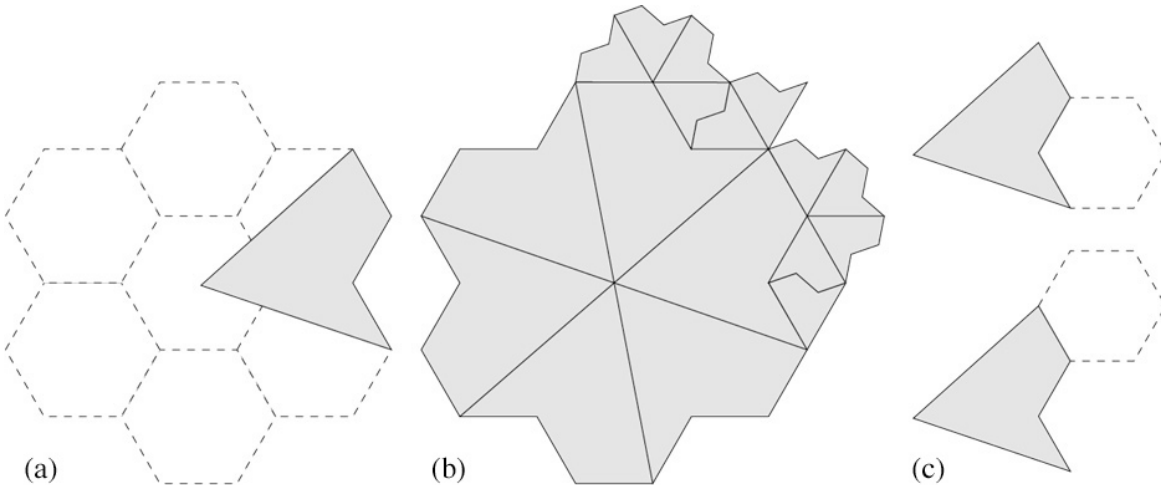

Figure 6: *(a) The prototile generated by dissecting the smallest 6-fold rotationally symmetric polyhex (excluding a single hexagon) with 6-fold rotational symmetry into sixths. Note that the prototile has 2 long edges and 3 short edges. (b) Consistent arrangement of second-generation tiles results in some unmatched long edges. (c) Prototiles generated from higher-order 6-fold rotationally-symmetric polyhexes will all have an odd number of short edges. Attempting to produce f-tilings based on such prototiles will always have the problem of unmatched long edges in the second generation.*

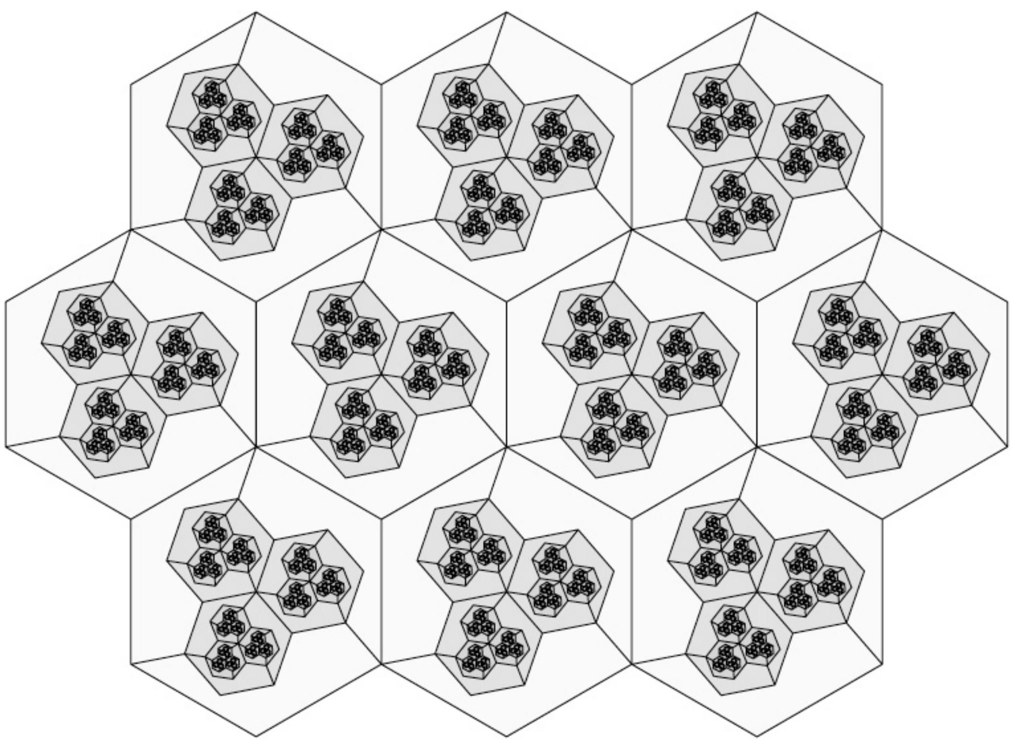

Figure 7: *A plane-filling f–tiling in which the individual tiles are arranged in regular-hexagon supertiles, each of which contains an infinite number of tiles. The prototile is the same as that used in Figure 4.*

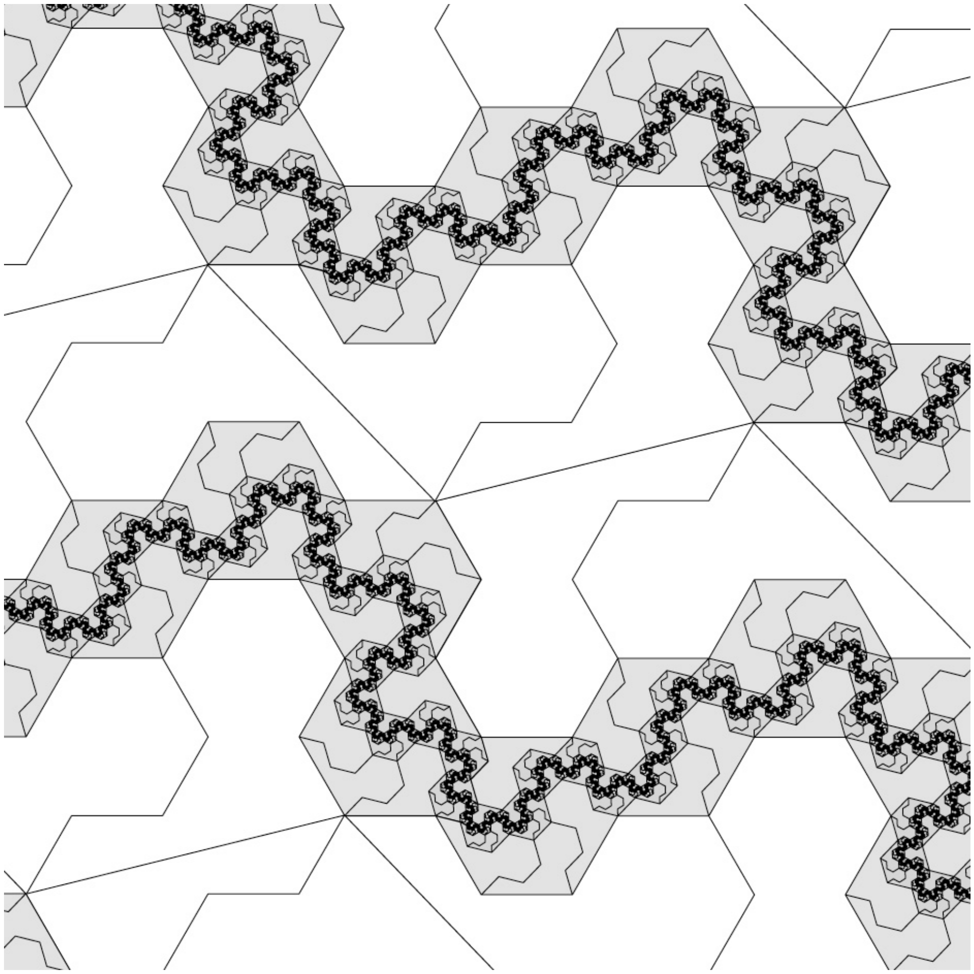

Figure 8: *A plane-filling f–tiling for which the supertile is an infinite strip. The prototile is the same as that used in Figure 2a.*

References

[1] Robert W. Fathauer, *Fractal tilings based on kite- and dart-shaped prototiles*, Computers & Graphics, Vol. 25, pp. 323-331, 2001.

[2] Robert W. Fathauer, *Fractal tilings based on v-shaped prototiles*, Computers & Graphics, Vol. 26, pp. 635-643, 2002.

[3] Robert W. Fathauer, *Self-similar Tilings Based on Prototiles Constructed from Segments of Regular Polygons*, in Proceedings of the 2000 Bridges Conference, edited by Reza Sarhangi, pp. 285-292, 2000.

[4] Robert W. Fathauer, http://members.cox.net/fractalenc/encyclopedia.html.

[5] Bruno Ernst, *The Magic Mirror of M.C. Escher*, Ballantine Books, New York, 1976.

[6] F.H. Bool, J.R. Kist, J.L. Locher, and F. Wierda, *M.C. Escher – His Life and Complete Graphic Work*, Abrams, New York, 1982.

[7] Branko Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, New York, 1987.

[8] H.-O. Peitgen, H. Jürgens, and D. Saupe, *Fractals for the Classroom – Part One*, Springer-Verlag, New York, 1992.

[9] This puzzle, known as *HexaPlex*, may be seen at http://www.tessellations.com.