Two Results Concerning the Zome Model of the 600-Cell

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Abstract

Two results concerning the Zome System are offered here. The first is a surprisingly simple model of Gosset's 8-dimensional semiregular polytope 4_{21}, built using the two "natural" sizes of Zome models of the 600-cell. Next we provide parts lists for Zome models of the 15 convex uniform polychora with the same symmetry as the 600-cell. For good reason, a few of these models have never been built before, but having the parts list available is a first step towards realizing all of them. We also offer some guidance on how to build these models.

1. Introduction

The Zome System has proved to be a useful tool for studying exceptional geometry. It would annihilate the idea to give a proper mathematical definition of "exceptional", although one can say generally that a geometrical system is exceptional if it seems to defy generalization. The Coxeter groups for H_4 and E_8 and all their associated geometries appear to do just this, and the Zome System, remarkably, is a great tool for studying their associated geometries. Moreover, generally speaking, exceptional objects often possess a high degree of symmetry. The group for H_4 has 14,400 elements, and the group for E_8 has an astounding

$$3! \cdot 4! \cdot 5! \cdot 8! = 696,729,600$$

elements. Thus, in this context, these objects have strong appeal to our aesthetic senses. In these notes we provide descriptions of many Zome models associated with H_4 and E_8. Zome models are generally very pretty, and these models illustrate this as well.

The first result is that one may use the Zome System to model Gosset's 8-dimensional semiregular polytope 4_{21}, [6]. The most efficient way to define this polytope, perhaps, is to say it is the convex hull of the 240 roots of the exceptional E_8 lattice. The Zome model is the result of the observation that, in any connected Zome model, the balls must lie in a quasicrystalline lattice related to the E_8 lattice, [3, 5]. The construction of the model is remarkably simple, assuming one can build the model of the 600-cell using the Zome System. Of course, skeletons of the other semiregular polytopes 3_{21} and 2_{21} corresponding to E_7 and E_6 are imbedded in 4_{21}, so one also obtains models of these automatically, [2].

Next, there are parts lists for the construction of all of the convex uniform polychora with the same symmetry of the 600-cell, also called "H_4-polychora". These can be described as analogues of the Archimedean polyhedra in 4 dimensions. These counts were obtained by means of a correspondence theorem concerning some "fundamental" strut families $\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$ and $\{\mathrm{R2}, \mathrm{R3}, \mathrm{Y3}, \mathrm{B3}\}$ and the conjugacy classes of the icosahedral group. Specific instructions on how to build these models are only outlined. It is assumed that the reader has already constructed Zome models of some of these polychora, especially the 600-cell or the 120-cell projections, and that s/he could probably figure out how to build the remaining models with some minimal guidance. Naturally, these 15 models, being mere edge skeletons, represent not only convex polychora, but whole regiments of uniform polychora with H_4 symmetry.

One must refer to the 600-cell repeatedly throughout all of these notes. In fact, to get started, the user should build a model of the 600-cell immediately, using the smaller strut family $\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$. One can find instructions for how to build this fundamental model in [7].

2. The Coxeter Graphs of Zome

In order to understand the geometry associated with these objects, it helps to be a little familiar with the theory of Coxeter groups. We don't have the space to go into great detail, but it will be helpful to be able to recognize the Coxeter graphs for H_4 and E_8:

Figure 1. The Coxeter Graph for H_4.

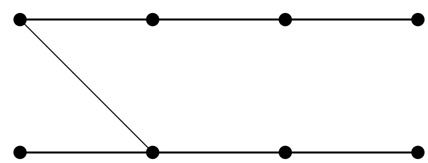


Figure 2. The Coxeter Graph for E_8.

A Coxeter graph contains all the information about how the group is presented by generators and relations. First, each vertex represents a mirror in n-dimensional space, where n is the number of vertices. In other words, each vertex corresponds to an element of order 2. Next, each unmarked edge designates that the corresponding pair of mirrors intersects at $60°$, the edge marked by '5' designates that the angle of intersection of these mirrors is $36°$, and a non-edge between two vertices designates that the angle between the mirrors is $90°$. Not surprisingly, this collection of n mirrors is called a "kaleidoscope". The classification of kaleidoscopes which yeild a finite group appears in [2].

These two Coxeter graphs are "exceptional" in their own way because they appear to defy generalization in the families where they are usually seen. The Coxeter group for H_4 is the symmetry group of the 600-cell (and all of the polychora we consider here), and the Coxeter group for E_8 is the symmetry group of Gosset's 8-dimensional figure 4_{21}.

3. Gosset's Figure in 8 Dimensions.

3.1. Introduction. Gosset's figure 4_{21} in 8 dimensions is a remarkable semiregular polytope having 240 vertices and 6720 edges. It is "semiregular" because it comes about as close to being regular as it can, considering the enormous constraints that "regularity" imposes. All of its 7-dimensional hyperfaces are regular polytopes. Its symmetry group, as we have already remarked, has nearly 700 million elements. Gosset's simpler semiregular polytopes 3_{21} with 56 vertices and 756 edges and 2_{21} with 27 vertices and 216 edges, also possessing a high degree of symmetry, exist embedded as 2-skeletons in Gosset's 8-dimensional figure, so this one object 4_{21} has a wealth of exceptional structure. If we had a way to see in 8 dimensions, surely we would be dazzled by its beauty.

Remarkably, it is possible to model Gosset's figure using the Zome System, and, just as remarkably, the model is incredibly simple, assuming one is moderately familiar with the Zome model of the 600-cell. One merely unites the two "natural" sizes of Zome models of the 600-cell which can be built using the commonly available parts. Thus, one starts by building the small version of the 600-cell, using the small strut family $\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$. Then use the larger strut family $\{\mathrm{R2}, \mathrm{R3}, \mathrm{Y3}, \mathrm{B3}\}$ to build the large version of the 600-cell with the small model serving as a core. Upon completion, one should have a model resembling that in the photograph.

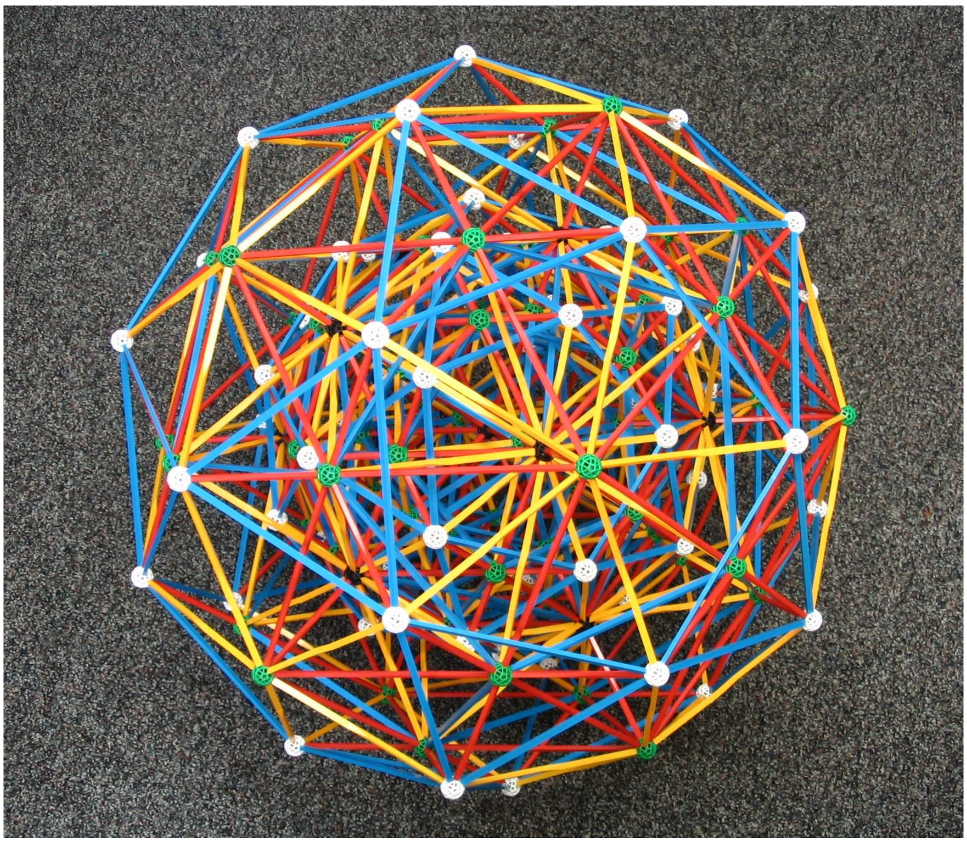


Figure 3. Zome Model of Gosset's 8-Dimensional Figure.

As we have mentioned, the symmetry group of 4_{21} is the Coxeter group for E_8. Notice that one can "collapse" the graph of E_8 onto the graph of H_4 so that three of the edges of the E_8 graph coincide with the edge marked by '5' in the graph for H_4. This 2-to-1 collapsing map between these two graphs coincides geometrically with the fact that the union of the concentric union of two Zome models of the 600-cell yields a faithful model of Gosset's figure. Said differently, it is possible to imagine the E_8 graph as the union of two copies of the H_4 graph and see some geometric significance to this, [3, 5].

The Zome model of Gosset's figure 4_{21} certainly has its faults. The most glaring is as follows: Whereas the balls of the Zome model faithfully represent the images of the 240 vertices under an orthogonal projection from 8-dimensional space to 3-dimensional space, the struts don't. Gosset's figure has 6,720 edges, and only a few of these are represented by Zome struts. Also, the struts on one of the Zome models of the 600-cell in the Zome model of Gosset's figure do not faithfully represent edges of Gosset's figure.

4. The H_4 Polychora.

4.1. Introduction. Recall that a polyhedron represents an Archimedean solid if (a) it is convex, (b) all the faces are regular polygons, and (c) there is only one congruence type of vertex figure. In four dimensions, one obtains a more robust family of polychora if one alters this definition just slightly before applying dimensional analogy. One says that a polychoron is uniform if (a) all of its hyperfaces are uniform polyhedra and (b) there is only one congruence type of vertex figure. Recall also that a polychoron has precisely two polyhedra sharing each two-dimensional face. We are interested here in the 15 convex uniform polychora having the same symmetry as the 600-cell. We refer to these briefly as the "H_4 polychora", although properly speaking this family contains many more than 15 polychora. As a matter of fact, since we are using the Zome System, our models will indeed represent whole regiments of polychora with H_4 symmetry.

One can imagine the H_4 polychrora as corresponding to the 7 Archimedean solids including the regular icosahedron, the icosidodecahedron, the regular dodecahedron, the truncated icosahedron, the rhombicosidodecahedron, the truncated icosahedron, and the rhombitruncated icosidodecahedron. In a very natural way, these 7 polyhedra correspond to the non-empty subsets of a 3-element set. Technically, for each Coxeter group G, there is associated a family of convex uniform polytopes having the symmetry group G, and each of these is determined by its Wythoff symbol. One may find a detailed description of this classification ansatz in [2]. While the technicalities behind the Wythoff are complicated, one should observe that it provides a convenient and efficient substitute for ungeneralizable verbose terms such as "rhombitruncated", to give one example. Thus, in the table appearing below, we refer to each of these 15 convex polychora merely by its associated Wythoff symbol.

All of the Zome models we consider here require a system of only four Zome lengths, either

$$\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$$

or the scaled-up versions

$$\{\mathrm{R2}, \mathrm{R3}, \mathrm{Y3}, \mathrm{B3}\}.$$

The 600-cell projection or the 120-cell projection should appear as obvious examples. As of this writing, only a few more than half of the Zome models of the H_4 polychora have ever been built by human hands, roughly those appearing in the first half of the table. Most of these models require hundreds if not thousands of Zome parts, so it is desirable to have a precise count on how many pieces are needed before one begins to make one of them. The first part of this section gives a parts count for all 15 of these polychora. The counts are based on some topological considerations and an interesting result which relates these numbers to the ratio $[1 : 12 : 12 : 20 : 15]$. Next in this section, we give some indication on how to put these together, once one has all the required pieces.

4.2. Brief Description of the Table. The first column shows the Wythoff symbol for each of the 15 polychora. The values B, R_1, R_2, Y_2, and B_2 give the required numbers of balls and R1, R2, Y2, and B2 struts. Notice that $R_1 = R_2$ for every polychoron. The last two columns ∂B and ∂B_2 require further explanation. Briefly, these are the numbers of balls and blue struts "on the boundary".

4.3. Some Details. Topologically, each of the polychora we consider here is homeomorphic to the hypersphere

$$S^3 = \{(w, x, y, z) : w^2 + x^2 + y^2 + z^2 = 1\},$$

an analogue of the common sphere S^2 in 3 dimensions. Every Zome model considered here represents the image of the projection

$$\pi : \begin{cases} \mathbb{R}^4 & \to & \mathbb{R}^3, \\ (w,x,y,z) & \mapsto & (x,y,z). \end{cases}$$

Thus, if e is any k-dimensional cell of a polychoron, then $\pi(e)$ is a cell of equal or lower dimension in $\mathbb{R}^3$. It is convenient to say that the image of π is actually a Zome model. Thus, if v is a vertex, then one may say that $\pi(v)$ is a ball and if e is an edge, then $\pi(e)$ is a strut (generally).

Consider the result of applying the projection π to S^3. One can check that the image is

$$\pi(S^3) = \{(x,y,z) : x^2 + y^2 + z^2 \leq 1\}.$$

In other words, $\pi(S^3)$ is a closed solid ball in $\mathbb{R}^3$. It is useful to partition $\pi(S^3)$ into two sets

$$\pi(S^3) = S^2 \amalg B^3,$$

where $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ is the ordinary 2-sphere and $B^3 = \{(x,y,z) : x^2+y^2+z^2 < 1\}$ is an open ball, the interior of $\pi(S^3)$. The significance of this partition is as follows: If $p \in B^3$, then the preimage $\pi^{-1}(p)$ consists of two points in S^3, while if $p \in S^2$, then the preimage is only one point in S^3. This observation is instrumental in obtaining the parts lists.

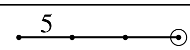
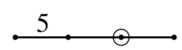
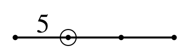
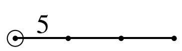
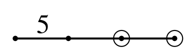
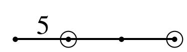
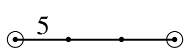
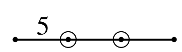
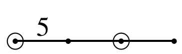
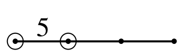
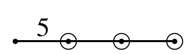
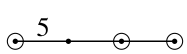
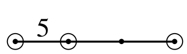
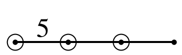
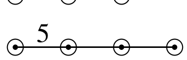
Wythoff Symbol	v	e	B	$R_1 = R_2$	Y_2	B_2	∂B	∂B_2
	120	720	75	72	120	120	30	60
	720	3600	396	360	600	480	72	60
	1200	3600	640	360	600	480	80	60
	600	1200	330	120	200	180	60	60
	1440	4320	780	432	720	600	120	120
	3600	10800	1860	1080	1800	1380	120	60
	2400	7200	1260	720	1200	960	120	120
	3600	7200	1860	720	1200	960	120	120
	3600	10800	1860	1080	1800	1380	120	60
	2400	4800	1260	480	800	660	120	120
	7200	14400	3660	1440	2400	1860	120	120
	7200	18000	3660	1800	3000	2280	120	60
	7200	18000	3660	1800	3000	2280	120	60
	7200	14400	3660	1440	2400	1860	120	120
	14400	28800	7200	2880	4800	3600	0	0

Table 1. Zome Parts List for H_4 Polychora.

Choose an H_4 polychoron X. The method for determining the required number B of balls to build $\pi(X)$ is perhaps the simplest. Let v be the number of vertices of X. From the topological property of π, one sees that the number of balls is $v/2$ plus some correction term. This correction term depends only on which balls lie on the boundary of the model. If ∂B denotes the number of balls on the boundary, then one evidently has

$$B = \frac{v + \partial B}{2}.$$

One may quickly determine ∂B by constructing part of the boundary of $\pi(X)$. This explains the "∂B" column.

The method for obtaining the required numbers of struts in $\pi(X)$ uses a "correspondence theorem" given (but not proven) below. First define an "enhanced" polychoron as follows: We know that the projection $\pi(X)$ may be constructed with a family of Zome struts, say $\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$. Mark each edge of X according to how it is mapped by π, whether it is collapsed to a point or mapped to one of the struts from $\{\mathrm{R1}, \mathrm{R2}, \mathrm{Y2}, \mathrm{B2}\}$. Let E denote the set of these marked edges, and call this set the "virtual struts" of X. The enhanced polychoron is the ordered pair (X, E), and the correspondence theorem reads:

Theorem. *Suppose X is a convex uniform polychoron with H_4 symmetry, n is the number of collapsed edges, r_1, r_2, y_2, and b_2 respectively are the numbers of virtual struts marked by R1, R2, Y2, and B2, and e is the total number of edges. Then*

$$(n, r_1, r_2, y_2, b_2) = \frac{e}{60} \cdot (1, 12, 12, 20, 15).$$

Given the correspondence theorem, it is a fairly simple matter to determine how many struts of various colors are required to make $\pi(X)$. As with the balls, the required numbers of R1 struts, R2 struts, Y2 struts, and B2 struts are given approximately by

$$\begin{array}{rcl} R_1 & \approx & r_1/2 = e/10, \\ R_2 & \approx & r_2/2 = e/10, \\ Y_2 & \approx & y_2/2 = e/6, \\ B_2 & \approx & b_2/2 = e/8, \end{array}$$

where the exact values are obtained by adding some correction terms. After constructing a partial Zome model of the boundary of $\pi(X)$, one discovers that a correction term is needed only for the B2 struts. With that, let ∂B_2 denote the number of B2 struts lying on the boundary. Then one has

$$\begin{array}{rcl} R_1 = R_2 & = & e/10, \\ Y_2 & = & e/6, \\ B_2 & = & e/8 + \partial B_2/2. \end{array}$$

Again, the data for ∂B_2 are in the table.

4.4. Assembling the models. Given that one may obtain all the parts required to make any one of these models, one is still faced with the problem of putting all the pieces together. We do not have the space to detail the construction of every Zome model described here, so we offer some general guidelines with references to a few well-known specific examples. First off, observe that the models in the table can be grouped into four sets, according to the number of circled vertices in the Wythoff symbol. The first four polychora, having only one circled vertex and including the 600-cell and 120-cell, are the simplest. Although it may seem presumptuous to say, generally speaking, if one can figure out how to build these first four, it does not take much more effort to see how the remaining 11 are put together. Although it was suggested earlier, perhaps it deserves repeating that in order to follow this, one should build the simpler models first, especially the 120-cell and the 600-cell models.

Icosahedral Symmetry. All of the Zome models we consider here have icosahedral symmetry. This fact has profound implications on how one must proceed to build any one of these models. Enforcing the presence of icosahedral symmetry goes as follows: If at any time during the construction we determine the correct way that a Zome part must be attached, then we must "complete the stage" by attaching corresponding Zome parts to the model so that the whole assembly has icosahedral symmetry. In this regard, it helps to be familiar with some data associated to the icosahedral group. One should notice that the numbers $\{12, 20, 30, 60, 120\}$ make frequent appearances. For example, the Zome model of the 600-cell has 7 layers comprised of

$$20 + 20 + 20 + 30 + 60 + 60 + 60 = 270$$

solid tetrahedra. These sorts of observations pervade the construction of any one Zome model of the H_4 polychora.

Cellular Structure. If one desires to make one of these 15 Zome models, one must acquaint oneself with the cellular structure of its underlying polychoron. This means that one must know what types of Archimedian solids make up the polychoron, and how they are arranged around every vertex and every edge. Since these polychora are uniform, there is only one vertex configuration. The number of circled vertices in the Wythoff symbol is equal to the number different types of edge configurations. For example, the truncated 600-cell, having the two right-most vertices circled, has two types of edges, one where an icosahedron and two truncated tetrahedra meet, and the other where 5 truncated tetrahedra meet.

Once a firm grasp of the cellular structure of the polychoron is established, one follows a sort of "analytic continuation" based on the Coxeter group for H_4. It is a stated assumption that this group must serve as a set of symmetries for each of these polychora, so each of these polychora is extraordinarily homogeneous with respect to H_4. Each polychoron can be considered as an assembly of precisely 14,400 parts all having the exact same structure. To give a couple examples, one may partition each tetrahedron of the 600-cell into 24 congruent tetrahedra or partition each dodecahedron of the 120-cell into 120 congruent tetrahedra. In both cases one obtains a total of 14,400 congruent tetrahedra. One obtains similar partitions for all of these 15 polychora, always finding that there are precisely 14,400 congruent parts. Technically, one says that the action of the natural action of the Coxeter group on 4-dimensional space has 14,400 fundamental regions.

The Squashing Phenomenon. One must remember that each of these Zome models accurately represents an orthogonal projection of one of these 4-dimensional figures into 3-dimensional space. With this projection, some distortion of the cells is inevitable. This is apparent in the Zome model of the 120-cell, which is comprised of 120 regular dodecahedra. One notices that the dodecahedra near the center of the model are more rounded than those appearing near the edge. In fact, the Zome model has precisely 30 dodecahedra which have been completely flattened by the projection into 3-space. These appear on the boundary of the model as irregular hexagons filled with 4 distorted pentagons each. Every Zome model of an H_4-polychoron, as described here, must possess this property of having distorted cells near the outer boundary and more rounded cells near the center.

Engineering. The stability of one of these models depends generally on two factors, the overall weight of all the pieces, and the number of struts which must connect to each ball. Obviously the heavier models are more unstable. Moreover, it can be argued that the stability increases roughly with the number of struts at every ball. The Zome model of the 600-cell is by far the most stable, requiring relatively few pieces and having a total of 12 edges at every vertex. Most of the models corresponding to the last five entries of the table have only 4 edges per vertex and must be relatively heavy.

To address these problems, there are several obvious solutions. Some Zome builders have used ingenious Zome support systems, connected in various key points close to the contact point between the model and the floor. In this regard, it also helps to inspect the model for which part of the boundary is best suited to handle the pressure from above; one generally looks for a large part of the boundary which is completely flat so that one may distribute the weight over a large area. For example, if one wishes to build a Zome model of the omnitruncated 120/600-cell, corresponding to the last entry in the table, one may notice that the boundary has precisely 30 rhombitruncated icosidodecahedra which have been completely flattened by the projection. Since these flat parts of the boundary have so much area, it is natural to rest the model on this surface, and build from the ground up. Finally, another solution is to build the larger models using a strut family which, as of this writing, is not yet available, say $\{\mathrm{R0}, \mathrm{R1}, \mathrm{Y1}, \mathrm{B1}\}$. Naturally, this would reduce the total weight by $1/\tau$, where τ is the Golden Ratio, and thus yield a more stable model. Moreover, smaller struts are less flexible in proportion to their weight, so there is less total strain on the pieces when one uses smaller struts.

References

[1] Michael Artin. *Algebra.* Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[2] H. S. M. Coxeter. *Regular Polytopes.* 3rd ed. Dover Publications Inc., New York, 1973.

[3] John H. Conway and Neil J. A. Sloane. *Sphere Packings, Lattices, and Groups.* 3rd ed. Springer-Verlag, Berlin, 1999.

[4] P. Du Val. *Homographies, Quaternions and Rotations.* Oxford University Press, 1964.

[5] Veit Elser and Neil J. A. Sloane. A highly symmetric four dimensional quasicrystal. *J. Phys. A: Math. Gen.* **20** (1987), 6161-6167.

[6] Thorold Gosset. On the regular and semi-regular figures in space of n dimensions. *Messenger of Math.* **29** (1900), 43-48.

[7] George W. Hart and Henri Picciotto. *Zome Geometry: Hands on Learning With Zome Models.* Key Curriculum Press, Emeryville, California, 2001.

[8] James E. Humphries. *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990

[9] Robert V. Moody and Jiri Patera. Quasicrystals and icosians. *J. Phys. A: Math. Gen.* **26** (1993), 2829-2853.

[10] Reza Sarhangi and Slavik Jablan (editors). Conference Proceedings of *Bridges: Mathematical Connections in Art, Music, and Science.* 2001.