Mathematical Models of Gothic Structures

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Abstract

The constructive characteristics of the Gothic style are unique in the history of Architecture. Gothic cathedrals pushed structure to the limit associating their structural elements to a linear frame that supports forces in a delicate balance. In order to understand this fragile equilibrium we must create a precise geometric model of the building, simple conceptually but substantially representative of the structural and constructive system. Each part of the building has specific constructive and structural characteristics that must be represented in the model. The mathematical model of the building will be the result of combining a geometric model with a mechanical model containing the physical properties obtained from experimental measurements.

1. Introduction

Showing the visitor a range of geometrical models, Spanish Architect Antonio Gaudi once remarked, with, with excitement in his eyes: *Wouldn't it be beautiful to learn geometry in this way?* Without any doubt, mathematical education for architecture students will be more effective and pleasant if all the theoretical knowledge were explained with the help of real architectural examples. The relationship between classic architecture and mathematics is well known. Architecture, unlike other scientific disciplines, can be used as a never ending source of numerical, algebraic, geometric, analytic and topologic problems, to name just a few fields of mathematics. A modern concept of architecture should necessarily include mathematics for its comprehension. Reciprocally, the teaching of mathematics in architecture should be based on the constructive event to be effective.

Interdisciplinary education provides a positive stimulus for both teachers and students, resulting in a much more persistent and interesting training. It is obvious that mathematical knowledge acquired inside an architectural environment is more likely to be applied by future architects after their university studies. As an example of this way of learning mathematics, in this paper we will show some ideas and mathematical concepts related to one of the more complex branches of architecture: restoration, repair and maintenance of Gothic buildings.

2. Why such interest in the Gothic Style?

The constructive characteristics of the Gothic style are unique in the history of Architecture. Gothic cathedrals pushed structure to the limit −soaring cross-vaulting, pointed arches, hollow walls and piers covered with tracery− and used the arch as an external brace, the flying buttress, to form one of the most beautiful stylistic elements of the gothic style.

Gothic buildings changed the pier concept inherited from the Romanesque tradition recovering the concept of the pier as a skeleton, the pillar instead the wall, the arch as vault centre instead of the arch as vault element, stress concentration points instead of stress lines. The whole system is a linear frame that supports forces in a delicate balance. This is the magnificence of the Gothic style.

This fragile equilibrium implies a permanent stress on all the elements. Any failure provokes other structural failures in a chain reaction threatening the integrity of the building. Obviously the technology of the thirteenth to sixteenth century technology could not introduce numerical structural calculation because it simply didn't exist. The progress came by trial and error, empirical methods, and the results were transmitted by traveling master builders and stonemasons.

Figure 1: *Main nave of the Church of Santa Maria la Real, Najera, Spain. A geometric analysis of the church shows the vaulting system to be completely distorted. Arches and vaults are deformed, the walls and columns are bent and cracks appear in the aisles. Geometry is the first test to verify the structural state of a Gothic construction.*

3. Why a Mathematic model?

A Gothic building is special in the sense that its construction usually takes a long period of time. The building process was very slow due to technical and economic problems. Besides, initial design errors were detected and corrected during the construction, modifying the original design.

The construction of a Gothic temple usually began with the apse and proceeded towards the end of the nave and aisles. This method implies that the structural equilibrium of each part of the building has to be solved independently. Consequently, the whole structure suffers from changing its equilibrium by absorbing settling deformations due to its provisional situation. Furthermore, the technological progress during the long construction process causes heterogeneity between parts finished with older procedures and materials and parts executed at later time. This is the reason why an accurate correlation between a mathematical model and the real building cannot actually be found. However, mathematical modelling is the only effective tool to estimate and understand the actual balance of a Gothic temple.

The study of gothic buildings must take into account the following aspects:

- Heterogeneity in materials due to long construction periods;
- Technical alterations introduced by craftsmen;
- Wide variety of stone and mortars;
- Different stages of vertical and horizontal development;
- Additions, substitutions and partial collapses during the construction
- Restorations and maintenance of the building.

These various aspects mean that, in many cases, it will be necessary to develop several mathematical models of the same building corresponding to different periods of time in order to determine the stress evolution of the building in a qualitative rather than a quantitative manner.

To find the effectiveness of a mathematical model, we must verify two criteria:

- Predicted deformations must agree qualitatively with building deformations.
- Deformations measured *in situ* in the building must be of the same order as those predicted by mathematical means.

Figure 2: *Photogrammetry and topography are two techniques used to take precise coordinates from existing conditions. Photogrammetry is based on the measurement of an object in two photographs taken from separated points towards the same focus. Topography is based on the geometry of an object that looks smaller the further away it is. A Romanesque style portico (left) rendered by means of photogrammetry and the nave of a gothic church (right) measured using topography. Photogrammetry is a technique usually used for detailed, ornamented, models while topography is generally used for larger, less detailed, structures.*

4. Geometry of a historic building

Describing the geometry of a building is the first phase of historic restoration work. A precise geometric model must be simple conceptually, but substantially representative of the structural and constructive system of the building. In order to create an accurate survey, there exist two basic techniques for acquiring the geometrical data that defines a historic building: photogrammetry and topography.

Photogrammetry is based on the principle of three-dimensional vision starting from two bidimensional images taken from points that are slightly separated. Just as we use our brains to comprehend spatial depth using two flat images taken from our left and right eyes, a computer can reconstruct a three dimensional environment using two photographs taken from separate points directed toward the same focus. A pair of photographs of this type is known as a stereoscopic image.

Figure 3: *Interior of the Bilbao Cathedral. The making of a computer model is similar to the physical creation of the Cathedral. First, the columns and arches are constructed, followed by the vaults, piers and flying buttresses, and finally the walls, windows and ornaments.*

The geometric basis of photogrammetry is the following: an object placed near the line of infinity appears in the same position in the bidimensional images, whilst the relative positions of objects near the observer vary between the two images. By measuring the displacement of the object in both images, we are able to deduce the distance from the observer in the real world.

Topography is a well-known process for obtaining the spatial coordinates of any construction. The geometrical theory is extremely simple: an object looks bigger or smaller to us depending on the distance from the viewer. With an especial device, the size of the object is measured and the distance of the point deduced. By tracing imaginary triangles from known coordinates, any visible point of the building can be measured with a high degree of precision.

Once we have a database with enough points from the building, we proceed to the elaboration of a three dimensional solid model. Several shapes are used to create the final model: ellipsoids, cylinders, paraboloids, hyperboloids, and other shapes that fit the coordinates from the database until the geometry is successfully completed.

5. Experimental measurements

Each part of a building has specific constructive and structural characteristics that must be represented in the model. Every element of the mathematical model needs to be associated with its mechanical properties.

The measurement of stresses and their directions is another important task. Very high stresses might fracture the stone-mortar ensemble. Stresses whose directions do not follow the vertical structural elements usually transmit forces outwards that should be opposed by the external buttress system. If this is not the case, traction forces will appear in the structure producing serious disorders in the building. Also, bending, cracks and deformation of structural elements might have occurred over time and should be measured for a representative period of time in order to achieve the best possible simulation of the building.

All these processes, known as monitoring, are completed by physical analyses of the soil and the materials used in the construction, mainly compression/deformation reports. The correlation of all the available reports, simulations, monitoring, weather conditions and soil prospecting may provide clues for detecting and solving the structural problems present in a building.

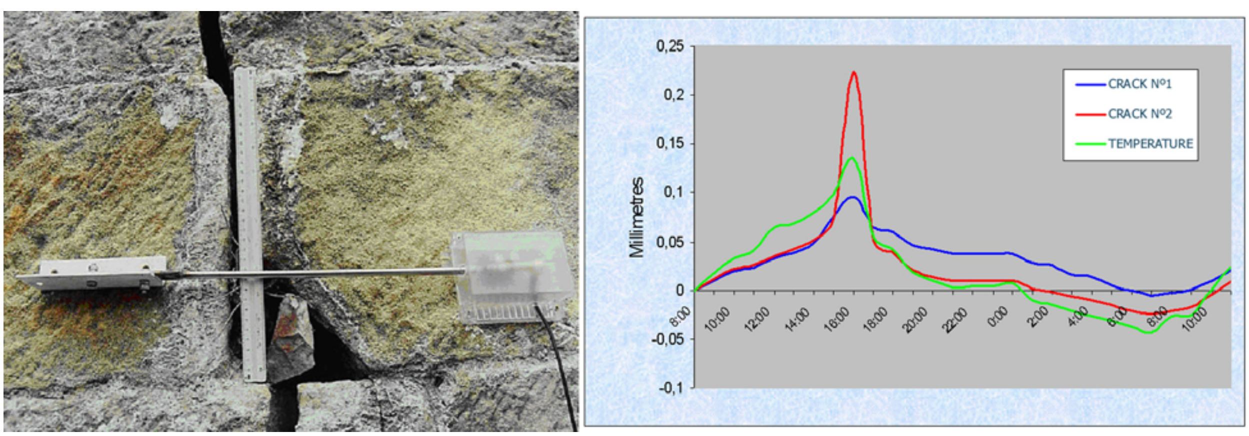

Figure 4: *An example of a device to monitor cracks in a building (left) and the graph representing the movement of two cracks during a 24 hours period (right). The graph also includes a third measurement indicating the external temperature of the building. It is obvious that there is a clear correlation between the temperature and the movement of the cracks, which work as expansion joints.*

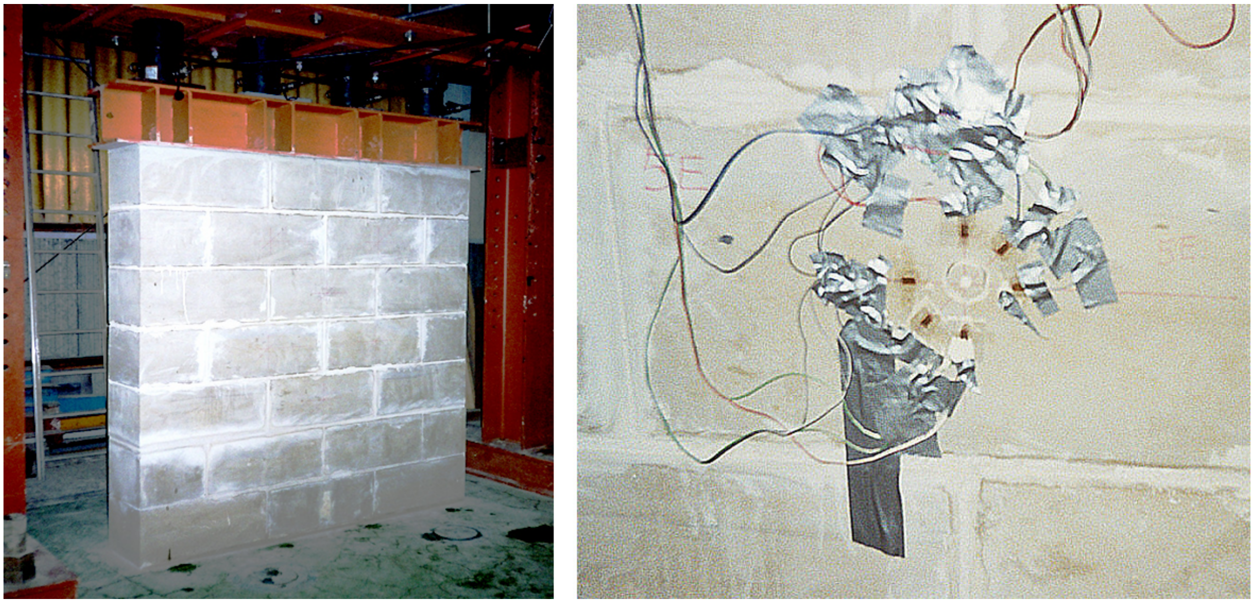

Figure 5: *An example of an experimental method to measure stresses, the Donostia Method, which was developed by the authors and has been widely used in several historic buildings. The left image shows a simulation of a masonry wall loaded in a laboratory. The image on the right shows the basic principle of the process: several electronic extensometric gauges placed around a hole drilled in the stone measure the deformation of the hole after the drilling. The dimension of the deformation, which of course is not perceptible by the naked eye, is transmitted from the gauges to an analog computer. Depending on the mechanical properties of the material, and the deformation measured, the stresses in all directions are calculated.*

6. The Finite Elements Method

The Finite Elements Method is the methodology that nowadays gives the most satisfactory results in the analysis of historic architectural structures, especially Gothic Cathedrals. The basis of this method consists in dividing a surface or volume into a reasonable number of small elements. The mechanical characteristics, as stress or displacement, of each and every element is calculated and transmitted to the neighbouring elements.

The input for a Finite Elements Method analysis is a computer model representing the geometry of the building. This model is not a simple drawing: each geometric element must be perfectly defined and assembled with their neighbours. Also, it must have appropriate contour conditions and physical properties, such as the Young and Poisson Modulus and density, amongst others.

Once the geometric, mathematical, model is completed, it is meshed into small elements. In space, we usually use tetrahedra (four nodes) or brick (eight nodes) elements, although these elements can also be implemented and extended to ten and twenty nodes respectively by including an extra node to each arista.

All this information is then translated to a system of equations in matrix form. The size of the matrix depends not only on the number of elements and nodes, but on the number of degrees of freedom. The degrees of freedom can be considered as the variables we want to solve for each node. A typical degree of freedom is six, which includes translation and rotation for each node along the three axes X, Y and Z.

After the matrix containing the model is solved, stresses and displacements for each node can be easily estimated. The results are usually represented graphically with colour maps that represent magnitudes. This technique is widely used, as it allows a quick visual understanding and interpretations of the model.

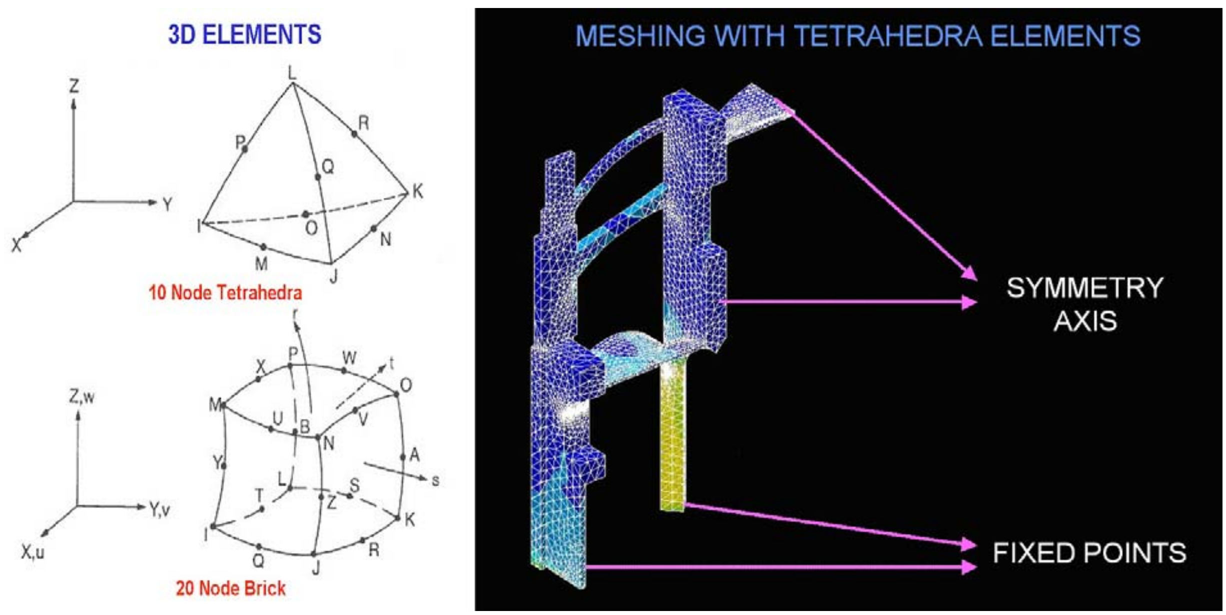

Figure 6: *Meshing a section of the external resistance system of a Gothic church (flying buttresses, vaults, pier and column). The basic element selected for the mesh was the 10-node tetrahedra. Tetrahedra fit perfectly on complex geometries, like the gothic, with a reasonable number of nodes.*

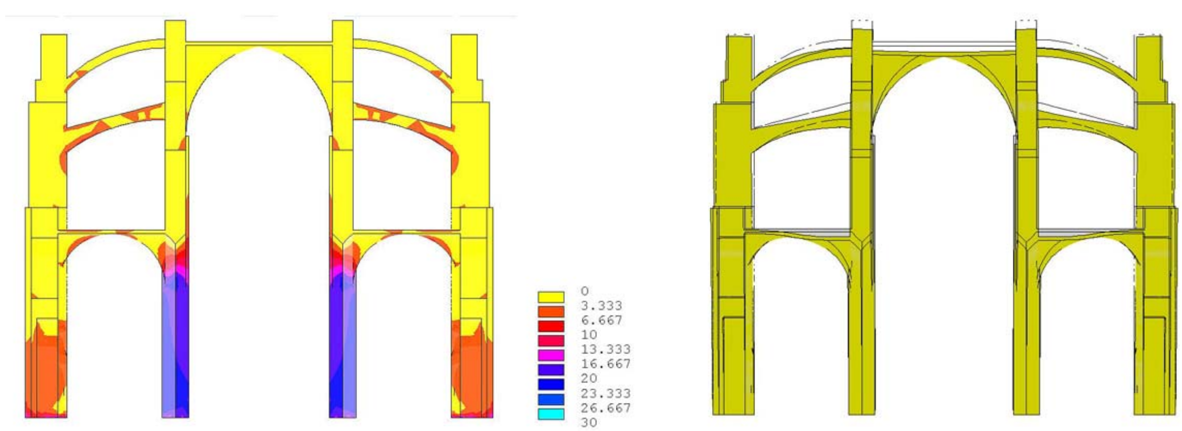

Figure 7: *Church of San Antonio Abad in Bilbao. The picture on the left shows the equivalent stresses on a section of the church. The picture on the right shows an estimation of the deformation suffered by the temple.*

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