Golden Fields, Generalized Fibonacci Sequences, and Chaotic Matrices

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Abstract

The diagonals of regular n-gons for odd n are shown to form algebraic fields with the diagonals serving as the basis vectors. The diagonals are determined as the ratio of successive terms of generalized Fibonacci sequences. The sequences are determined from a family of triangular matrices with elements either 0 or 1. The eigenvalues of these matrices are ratios of the diagonals of the n-gons, and the matrices are part of a larger family of matrices that form periodic trajectories when operated on by the Mandelbrot operator. Generalized Mandelbrot operators are related to the Lucas polynomials have similar periodic properties.

1. Introduction

It is well known that the ratio of successive terms of the Fibonacci sequence approaches the golden mean, $\tau = \frac{1+\sqrt{5}}{2}$, in the limit and that the diagonal of a regular pentagon with unit edge has length τ. We show in a more detailed version of this paper to appear in the Japanese journal FORMA that the Fibonacci sequence can be generalized to characterizing all of the diagonals of regular n-gons for n an odd integer. In this paper we restrict ourselves to the cases of n = 5 and 7. Furthermore, a geometric sequence in τ is also a Fibonacci sequence and shares all of the algebraic properties inherent in the integer Fibonacci sequence. Similar sequences involving the diagonals of higher order n-gons also have algebraic properties. In fact we shall show that they form a field in which the basis vectors are the diagonals. We shall call these as Steinbach [1] did, "golden fields " Products and quotients of the diagonals of an n-gon can be expressed as a linear combination of the diagonals.

The results depend strongly on a set of polynomials related to the Fibonacci numbers, and the Lucas polynomials, both of which are related to the Chebyshev polynomials. All of the roots of the Fibonacci polynomials are of the form x = 2cos kπ/n while the Lucas polynomials map $2\cos A$? $2\cos mA$. As a result, we show that a family of matrices with 0,1,-1 elements form periodic trajectories when operated on by matrix forms of the Lucas polynomials. We refer to these as Mandelbrot Matrix Operators since the Lucas polynomial $L_2(x)$ corresponds to the Mandelbrot operator at the extreme left hand point on the real axis, a point of full-blown chaos. Kappraff and Adamson [2] have shown in a previous paper that the higher order Lucas equations lead to generalized Mandelbrot sets.

2. Preliminaries

Our work is based on the Diagonal Product Formula (DPF) of Steinbach [1].

Proposition: Diagonal Product Formula:
Consider a regular n-gon (Figure 1) for odd n and let ρ_0 be the length of a side and ρ_k the length of the kth diagonal with k $\leq \frac{n-3}{2}$. Then

$$\begin{aligned}
&\rho_0\ \rho_k = \rho_k \\
&\rho_1\ \rho_k = \rho_{k-1} + \rho_{k+1} \qquad (1)\\
&\rho_2\ \rho_k = \rho_{k-2} + \rho_k + \rho_{k+2} \\
&\rho_3\ \rho_k = \rho_{k-3} + \rho_{k-1} + \rho_{k+1} + \rho_{k+3} \\
&\quad . \\
&\quad . \\
&\rho_h\ \rho_k = \sum_{i=0}^{h} \rho_{k-h+2i}
\end{aligned}$$

In what follows we shall let ρ_0 = 1.

Using a chain of substitutions in the DPF, Steinbach [1] derived for the regular n-gon, the following formula basic to the combinatorics of polygons,

$$C(k,0)x^k - C(k-1,1)x^{k-2} + C(k-2,2)\ x^{k-4} - \ldots = -C(k-1,0)x^{k-1} - (k-2,1)x^{k-3} - \ldots \qquad (2)$$

where k = $\frac{n-1}{2}$ and C(i,j) = $\frac{i!}{j!(i-j)!}$.

If we write Equation 2 as $P_k(x) = 0$, $P_k(x)$ has the recurrence relation,
$P_{k+1}(x)= xP_k(x) - P_{k-1}(x)$ where $P_{-1} = 1$ and $P_0 = 1$. $P_k(x)$ is referred to as the DPF polynomials and can be expressed in terms of the derivatives of the Chebyshev polynomials.

Consider the following identity, the proof of which is given in the Appendix A:

$$\text{Sin } 2nA / \text{Sin } 2A = K_n(x) = P_k(x)P_k(-x). \qquad (3a)$$

where x = 2 cos 2A, k = $\frac{n-1}{2}$ and $K_n(x)$ is the sequence of polynomials which we have referred to as *Fibonacci-Pascal polynomials* (see Appendix A) with alternating signs since the absolute values of the coefficients of K_n can be found along a diagonal of Pascal's triangle and sum to the n-th Fibonacci number [3]. They are generated by the recursion,

$$K_{k+1}(x)= xK_k(x) - K_{k-1}(x) \text{ where } K_1 = 1 \text{ and } K_2 = x.$$

The first four Fibonacci-Pascal Polynomials for odd n are,

$$\begin{aligned}
K_1 &= 1 \\
K_3 &= x^2 - 1 = (x-1)(x+1) \qquad (3b)\\
K_5 &= x^4 - 3x^2 + 1 = (x^2 - x - 1)(x^2 + x - 1)
\end{aligned}$$

$$K_7 = x^6 - 5x^4 + 6x^2 - 1 = (x^3 - x^2 - 2x + 1)(x^3 + x^2 - 2x - 1)$$

Note that the sum of the absolute values of the coefficients of K_n is the nth Fibonacci number. If A $= \frac{j\pi}{2n}$, it follows from Equation 3a and 3b that $K_n(x) = 0$ and that $\pm 2\cos\frac{j\pi}{7}$ are roots of $P_3(x)$ and $P_3(-x)$. For example,

sin 14A / sin 2A = $K_7(x) = P_3(x)P_3(-x)$.

where x = 2 cos 2A and,

P(x) = cos7A/cosA and P(-x) = sin7A/sinA

Note that the sum of the absolute values of the coefficients of K_7 is 13, the 7th Fibonacci number. If A $= \frac{j\pi}{14}$, it follows that $K_7(x) = 0$ and that $\pm 2\cos\frac{j\pi}{7}$ are roots of $P_3(x)$ and $P_3(-x)$.

A general formula for the j-th diagonal of an n-gon with unit edge from KAPPRAFF (2002) is,

$$\rho_j = \frac{\sin\frac{(j+1)\pi}{n}}{\sin\frac{\pi}{n}} \quad \text{for} \quad 0 \le j \le \frac{n-3}{2} \tag{4}$$

where ρ_0 is the edge of the n-gon.

3. The Pentagon

We begin with a statement of the case for n = 5, the pentagon. The standard Fibonacci sequence, $F_5^{(1)}$ is,

$$a_1 \ a_2 \ a_3 \ \dots a_k \ a_{k+1} \ \dots \ = 1 \ 1 \ 2 \ 3 \ 5 \ 8 \ 13 \ 21 \ \dots \tag{5}$$

where $\lim \frac{a_{k+1}}{a_k} = \tau$

The following τ -sequence has identical algebraic properties as the integer sequence,

$$1 \quad \tau \quad \tau^2 \quad \tau^3 \quad \tau^4 \ \dots \tau^k \dots \tag{6}$$

i.e., it is also a Fibonacci sequence where,

$$1 + \tau = \tau^2 \tag{6a}$$

Since the diagonal of the pentagon with unit edge has length τ , we shall refer to this as a ρ_1-sequence, where $\rho_1 = \tau$.

Equation 6a satisfies the PDF for n = 5. We present this in Table 1 as a multiplication table expressed as left x top.

Table 1

x	1	ρ_1
1	1	ρ_1
ρ_1	ρ_1	$1+\rho_1$

From this relation we can derive a generating matrix for the ρ_1-sequence by considering successive pairs of elements from the sequence to be a vector, i.e.,

$$\overrightarrow{v_1} = (\rho_1, 1)^T,\ \overrightarrow{v_2} = (\rho_1^2, \rho_1)^T,\ \overrightarrow{v_3} = (\rho_1^3, \rho_1^2)^T, \ldots$$

Consider the matrices,

$$M_5^{(1)} = \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \quad \text{and} \quad M_5^{(1)-1} = \begin{matrix} 0 & 1 \\ 1 & -1 \end{matrix} \qquad \text{(7a and b)}$$

where $M_5^{(1)}\overrightarrow{v_n} = \overrightarrow{v_{n+1}}$. Therefore, $M_5^{(1)}\ (\rho_1, 1)^T = (\rho_1^2, \rho_1)^T = (1+\rho_1, \rho_1)^T$. The notation $M_5^{(1)}$ refers to the fact that the matrix generates the ρ_1-sequence for the 5-gon.

The same matrix also generates the Fibonacci sequence where, $M_5^{(1)}\overrightarrow{u_n} = \overrightarrow{u_{n+1}}$ where $\overrightarrow{u_1} = (1,1)^T$, $\overrightarrow{u_2} = (2,1)^T$, $\overrightarrow{u_3} = (3,2)^T$, …

The eigenvalues of the inverse matrix $M_5^{(1)-1}$ in order of decreasing absolute values are

$$\lambda_1 = -2\cos\frac{\pi}{5}, \lambda_2 = 2\cos\frac{2\pi}{5}, \qquad (8)$$

obtained as the zeros of the irreducible characteristic polynomial,

$$P_2(-x) = x^2 + x - 1 \qquad (9)$$

where $P_2(x)$ is the generating polynomial of Equation 2 for n = 5. That eigenvalues 8 are the zeros of Polynomial 9 follows from Equation 3a. The eigenvalues can also be written as the ratio of diagonals,

$$\lambda_1 = -\frac{\rho_1}{\rho_0},\ \lambda_2 = \frac{\rho_0}{\rho_1}\ . \qquad (10)$$

Furthermore, it follows from the DPF that, in general, when n is prime, quotients of the diagonals can be written as a linear combination of diagonals (including edge 1) with coefficients 0, 1, -1. For n = 5, Table 3 presents the ratio of diagonals, expressed in terms of left ÷ top.

Table 2

÷	1	ρ_1
1	1	$\rho_1 - 1$
ρ_1	ρ_1	1

Thus the diagonals of a pentagon form a golden field with basis vectors: 1, ρ_1.

4. The Heptagon

Denote the two diagonals of a heptagon by ρ_1 and ρ_2 (ρ_0 =1). From Equation 4,

$$\rho_1 = 1.801\ldots \quad \text{and} \quad \rho_2 = 2.24\ldots$$

From Equations 1, the DPF, the product of diagonals are given by Table 4 expressed as left x top.

Table 3

x	1	ρ_1	ρ_2
1	1	ρ_1	ρ_2
ρ_1	ρ_1	$1 + \rho_2$	$\rho_1 + \rho_2$
ρ_2	ρ_2	$\rho_1 + \rho_2$	$1 + \rho_1 + \rho_2$

Consider the ρ_2 -sequence,

$$1 \quad \rho_1 \quad \rho_2 \quad \rho_1\rho_2 \quad \rho_2^2 \quad \rho_1\rho_2^2 \quad \rho_2^3 \quad \rho_1\rho_2^3 \;\cdots \qquad (11)$$

and the vectors,

$$\overrightarrow{v_1} = (\rho_2 ,\ \rho_1 ,\ 1)^{\mathrm{T}},\ \overrightarrow{v_2} = (\rho_2^2 , \rho_1\rho_2 , \rho_2)^{\mathrm{T}},\ \overrightarrow{v_3} = (\rho_2^3 , \rho_1\rho_2^2 , \rho_2^2)^{\mathrm{T}},\ldots$$

Using the relationships in Table 3, we define the matrix,

$$M_7^{(2)} = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \quad \text{and} \quad M_7^{(2)-1} = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{matrix} \qquad \text{(12a and b)}$$

where, $M_7^{(2)}\ (\rho_2,\ \rho_1,\ 1)^{\mathrm{T}} = (\rho_2^2, \rho_1\rho_2, \rho_2)^{\mathrm{T}} = (1 + \rho_1 + \rho_2,\ \rho_1 + \rho_2,\ \rho_2)^{\mathrm{T}}$. Matrix $M_7^{(2)}$ generates the ρ_2-sequence for the 7-gon and will be referred to as the principal matix,

Likewise, $\quad M_7^{(2)}\,\overrightarrow{u_n} = \overrightarrow{u_{n+1}}\quad$ where,

$$\overrightarrow{u_1} = (1, 1, 1)^{\mathrm{T}},\ \overrightarrow{u_2} = (3, 2, 1)^{\mathrm{T}},\ \overrightarrow{u_3} = (6, 5, 3)^{\mathrm{T}}, \ldots$$

results in the generalized Fibonacci sequence, $F_7^{(2)}$,

$$a_1\ a_2\ a_3\ \ldots a_k \ldots = 1\ 1\ 1\ 2\ 3\ 5\ 6\ 11\ 14\ 25\ 31 \ldots \tag{13}$$

where $\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{11}{6}, \frac{25}{14}, \ldots \rightarrow \lim \frac{a_{2k}}{a_{2k-1}} = \rho_1$, and

$\frac{1}{1}, \frac{3}{1}, \frac{6}{3}, \frac{14}{6}, \frac{31}{14}, \ldots \rightarrow \lim \frac{a_{2k+1}}{a_{2k-1}} = \rho_2$

The irreducible characteristic polynomial of the inverse matrix $M_7^{(2)-1}$ is,

$$P_3(x) = x^3 - x^2 - 2x + 1 \tag{14}$$

which can be derived from Equation 2 for n = 7. As a result of Equation 3a, its roots are the eigenvalues,

$$\lambda_1 = 2\cos\frac{\pi}{7},\ \lambda_2 = -2\cos\frac{2\pi}{7},\ \text{and}\ \lambda_3 = 2\cos\frac{3\pi}{7} \tag{15}$$

where,

$$\lambda_1 = \frac{\rho_1}{\rho_0},\ \lambda_2 = -\frac{\rho_2}{\rho_1},\ \text{and}\ \lambda_3 = \frac{\rho_0}{\rho_2}. \tag{16}$$

Table 4 lists the quotients of the diagonals as sums of diagonals expressed as left ÷ top.

Table 4.

÷	1	ρ_1	ρ_2
1	1	$1+\rho_1-\rho_2$	$\rho_2-\rho_1$
ρ_1	ρ_1	1	ρ_1-1
ρ_2	ρ_2	ρ_2-1	1

Therefore the diagonals of a 7-gon form a golden field with basis vectors 1, ρ_1, ρ_2 and coefficients 0,1,-1.

5. The General Case

An n-gon for n odd has $\frac{n-3}{2}$ diagonals denoted by,

$$\rho_1,\ \rho_1, \ldots \rho_m \quad \text{where } m = \frac{n-3}{2}.$$

The ρ_m-sequence is,

$$1 \quad \rho_1 \quad \rho_2 \ldots \rho_m \quad \rho_1\rho_m \quad \rho_2\rho_m \ldots \rho_{m-1}\rho_m \quad \rho_m^2 \quad \rho_1\rho_m^2 \quad \rho_2\rho_m^2 \ldots \rho_{m-1}\rho_m^2 \quad \rho_m^3 \quad \rho_1\rho_m^3 \ldots \quad (17)$$

The matrix corresponding to the DPF relationships is again upper triangular. The eignevalues are determined from the characteristic equation of the inverse matrix. From Equation 2, the characteristic polynomial, where k = $\frac{n-1}{2}$ is,

$P_k(x)$ for k odd, and
$P_k(-x)$ for k even.

The characteristic polynomials are irreducible when n is prime. If n_1 is a factor of n then the characteristic polynomial is factorable, and either $P_{k_1}(x)$ or $P_{k_1}(-x)$, corresponding to the inscribed n_1-gon is a factor of $P_k(x)$ or $P_k(-x)$. The eigenvalues can be expressed by the following concise formula,

$$\lambda_j = 2\cos(2(k-j)+1)k\frac{\pi}{n} \qquad (18)$$

where,

$$|\lambda_j| = \frac{\rho_{2(k-j)}}{\rho_{j-1}} \qquad (19)$$

for $j = 1,2,\ldots,k$ and k = $\frac{n-1}{2}$.

Note that in Equation 34, $\rho_{2i} = \rho_{2(k-i)+1}$ for $\frac{n-1}{2} \le i \le n-3$.

6. Polygons and Chaos

Consider the sequence of Lucas polynomials, L_m generated by the recursion,

$$L_{m+1} = L_m - L_{m-1}$$

where $L_1 = 2$ and $L_2 = x$. The Lucas polynomials are related to the Chebyshev polynomials of the second kind and have the defining property described by Kappraff and Adamson [2], and Kappraff [3],

$$L_m(2\cos\theta) = 2\cos m\theta. \qquad (20)$$

In particular, $L_2 = x^2 - 2$ is a special case of the operator that generates the Mandelbrot set,

$$z ? \; z^2 + c$$

for c = -2, the leftmost point on the real axis of the Mandelbrot set. Beginning with $x = x_0$, the recursion,

$$x ? \; x^2 - 2 \qquad \text{(21a)}$$

generates the trajectory: $x_0, x_1, x_2, \ldots x_k \ldots$ where $x_0 \to x_1 \to x_2 \to \ldots \to x_k \to \ldots$ If $x_p = x_0$ the trajectory is periodic with period p.

Next consider x to be the n x n diagonalizable matrix X, and rewrite Equation 21a as,

$$X ? \; X^2 - 2I \qquad \text{(21b)}$$

where I is the n x n identity matrix. We refer to Equation 21b as the "Mandelbrot matrix operator." We claim that for each n-gon for odd n, setting either $X_0 = -M_n^{(m)^{-1}}$ or $X_0 = -M_n^{(m)^{-1}}$ (see Equation 21b) results in a periodic trajectory of period p depending only on the value of n, with the same values of p [1], i.e., p is the smallest positive integer such that,

$$2^p \equiv \pm 1 (\mathrm{mod}\, n) \qquad \text{(22)}$$

For example, for the pentagon, n = 5, using Equation 7b,

$$X_0 = M_5^{(1)^{-1}} \to X_1 = \begin{matrix} 0 & 1 \\ 1 & -1 \end{matrix} \to X_2 = M_5^{(1)^{-1}}$$

so that $M_5^{(1)^{-1}}$ repeats with period 2. For the hexagon, n = 7, using Equation 12b,

$$X_0 = -M_7^{(2)^{-1}} \to X_1 = \begin{matrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{matrix} \to X_2 = \begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{matrix} \to X_3 = -M_7^{(2)^{-1}}$$

so that $-M_7^{(2)^{-1}}$ has period p = 3.

We state this result as a Theorem.

Theorem: If X is an n x n diagonalizable matrix and either $X_0 = -M_n^{(m)^{-1}}$ or $X_0 = -M_n^{(m)^{-1}}$, depending on n, the Mandelbrot matrix operator, $X ? \; X^2 - 2I$ has a periodic trajectory with period p.

Proof: We shall demonstrate this for the case n = 5 and n = 7. The proof for general n follows in a similar manner.

Since X is diagonalizable, there exists a matrix of eigenvectors P such that,

$$X = P^{-1}\Lambda P \tag{23}$$

where, Λ is the matrix of eigenvalues, $\Lambda = \lambda_i \delta_{ij}$ (no summation on i) and δ_{ij} is the Kronecker delta. Replacing X into Operator 21b yields,

$$X^2 - 2I = P^{-1}((\lambda_i^2 - 2)\delta_{ij})P \,. \tag{24}$$

If λ_i or $-\lambda_i$ is given by Equation 32 then $X = M_{n_m}^{-1}$ or $X = -M_{n_m}^{-1}$ and the result follows by replacing λ_i or $-\lambda_i$ with its value given by Equation 15 into Equation 24 and using Equation 20 for m = 2. We shall demonstrate this for n = 5 and n = 7.

If n = 5, using Equation 20,

$$\lambda_1 = -2\cos\frac{\pi}{5} \to 2\cos\frac{2\pi}{5} \to 2\cos\frac{4\pi}{5} = 2\cos\frac{(5-1)\pi}{5} = -2\cos\frac{\pi}{5}$$

We abbreviate this sequence by considering the coefficients of the numerator of the arguments, i.e.,

$$\lambda_1 = -2\cos\frac{\pi}{5} \equiv 1 \to 2 \to 4 = 2\cos\frac{4\pi}{5} \equiv -2\cos\frac{\pi}{5}$$

In a similar manner,

$$\lambda_2 = 2\cos\frac{2\pi}{5} \equiv 2 \to 4 \to 8 \equiv 2\cos\frac{8\pi}{5} = 2\cos\frac{2\pi}{5}$$

Thus we have demonstrated that $M_{5_1}^{-1}$ has period 2.

If n = 7, using Equation 20,

$$-\lambda_1 = -2\cos\frac{\pi}{7} \equiv 1 \to 2 \to 4 \to 8 \equiv 2\cos\frac{8\pi}{7} = -2\cos\frac{\pi}{7}$$

$$-\lambda_2 = -2\cos\frac{2\pi}{7} \equiv 2 \to 4 \to 8 \to 16 \equiv 2\cos\frac{16\pi}{7} = -2\cos\frac{2\pi}{7}$$

$$-\lambda_3 = -2\cos\frac{3\pi}{7} \equiv 3 \to 6 \to 12 \to 24 \equiv 2\cos\frac{24\pi}{7} = -2\cos\frac{3\pi}{5}$$

Thus we have demonstrated that $-M_7^{(2)^{-1}}$ has period 3.

In a similar manner, as demonstrated by Kappraff and Adamson [1], using Equation 20, this result continues to hold for the generalized Mandelbrot matrix operators, X ? $L_m(X)$, with periods given by the smallest positive integer, p, such that,

$$m^p \equiv \pm 1 (\bmod n) \ .$$

where,

$$L_3(X) = X^3 - 3X$$
$$L_4(X) = X^4 - 4X^2 + 2$$
$$L_5(X) = X^5 - 5X^3 + 5X$$

7. Reflected Waves

Consider light rays incident to two slabs of glass as shown in Figure 2. There is one wave with no reflections, 2 waves with 1 reflection, and 3 waves with 2 reflections. In fact for the number of waves, N_k, with k reflections, $N_k = a_{k+1}$ from the $F_5^{(2)}$ - sequence (the standard Fibonacci sequence): 1,2,3,5,8,…,. [4].

Next consider three slabs of glass. It has been shown that $N_k = a_{2k+1}$, a subsequence: 1,3,6,14,31,… of $F_7^{(2)}$ (see sequence 13), the generalized Fibonacci sequence associated with the heptagon [5],[6]..

Likewise, for m planes of glass, $N_k = a_{(m-1)k+1}$, a subsequence of the generalized Fibonacci sequence $F_{(2m+1)}^{(m-1)}$.

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