# Symmetries and Design Science
# Two Graduate Courses for a Mathematics Education Program

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### Abstract

Two mathematics courses have been developed for the Master's program in Mathematics Education at Towson University. These two courses are very related to the ideas of (i) teaching theory of symmetries, (ii) teaching mathematics of designs and patterns, and (iii) integrating mathematics with other disciplines.

## 1. Introduction

The Master of Science in Mathematics Education program at Towson University provides mathematics teachers with advanced study in mathematics, mathematics education and general education. The program offers teachers additional experience in higher-level mathematics to enhance their teaching with additional depth and breadth of content. At the same time, it strengthens their background in the school mathematics curriculum, instructional practices, assessment and technology. It also provides them a relevant way of satisfying their in-service requirements for professional advancement.

This program was developed and established between 1997 and 1999. Much of the original rationale was drawn from contemporary mission statements and national reports on mathematics education that called for additional higher educational opportunities for mathematics teachers. Since Towson University first started as a normal school (an old name for a teacher training school), the improvement of the education of teachers has always been a critical part of its mission. In the University's Strategic Plan (1996), the goals include developing graduate programs "serving an adult and working population" such as in-service teachers, and to "expand outreach efforts to be responsive to regional economic goals and more closely integrated into the...educational...life of the Baltimore metropolitan community."

For the mathematics background component of the program in the Secondary track, a student should take four graduate courses in the three areas of "Algebra and Analysis", "Geometry", and "Probability and Statistics". Students must take one course from each area and one extra course from any of these areas. Even though there are standard graduate courses that are offered by the department, it seemed logical to design courses which are specifically tailored for these students that are mainly in-service high school teachers. Two of these courses are very related to the ideas of symmetries and mathematics connections with other disciplines: *The Algebra of Symmetries* and *Patterns in Mathematical Designs*.

The goal of this article is to outline the mentioned courses and present some examples. The hope is to strengthen these courses by useful suggestions and comments received by the readers of this article and also open a dialog for interested mathematics educators who are in process of developing such courses in their schools.

## 2. Teaching an Abstract Algebra Course to Teachers

Abstract Algebra is a difficult subject to teach because it is ABSTRACT. Most books take an axiomatic approach and begin with group theory and progress to rings and fields. A small number of books begin with rings under the assumption that the students have more intuition with rings because of their familiarity with the integers and the rational numbers. These books must develop group theory toward the end and have the same problems as the other books.

Teaching abstract algebra using the axiomatic approach has one major flaw. The examples of a group that are routinely given do not give the student any idea why the axioms were chosen. The better students will understand that the examples really are models of the axioms, but they will still not see more of the connections between axioms and examples.

A course called "The Algebra of Symmetries" has been designed to remedy this situation. This is a graduate course for in-service high school teachers. Its purpose is to teach the ideas of group theory so that they follow logically from a study of symmetries.

## 3. The Algebra of Symmetries

We use the book "Numbers and Symmetry, An Introduction to Algebra" by Johnston and Richman [3]. A number of activities have been added to the material in the book. In each case, an effort is made to connect the algebra and geometry of the situation.

We begin with a look at some numbers and how we can use geometry to investigate them. In keeping with Johnston and Richman, we look at the Gaussian integers and the division algorithm. The polar representation of complex numbers is explained and then the geometry of multiplication. The Gaussian integers form a two dimensional lattice and the multiples of the divisor form a sublattice. From Figure 1, it is easy to see exactly how the division algorithm works geometrically.

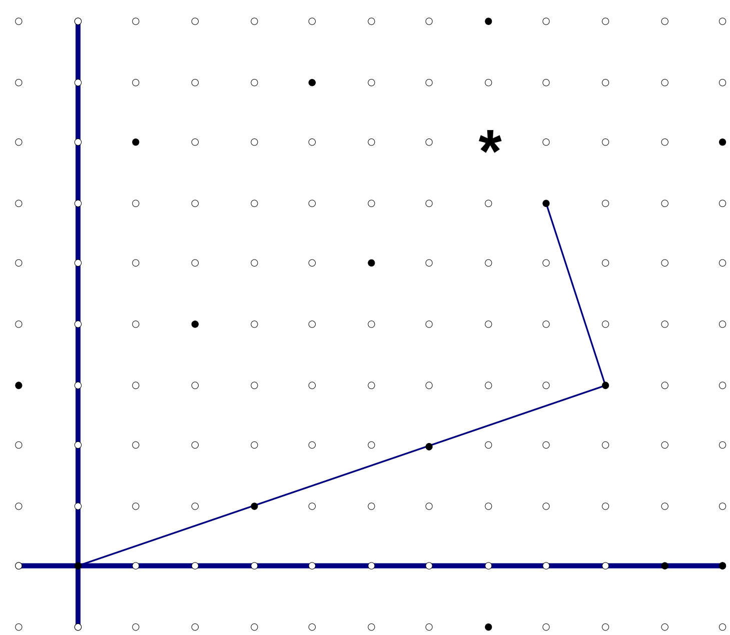


**Figure 1:** $7 + 7i = (3 + 3i)\,(3 + 3i) + (-1 + i)$

The Gaussian integers are represented in Figure 1 by either plus signs or small circles. They form a lattice in the plane. The sublattice of small circles is the set of all multiples of (3 + i). The norm is the square of the distance from the origin. We begin by picking a Gaussian integer, such as (7 + 7i), represented by the bold circle. It is surrounded by at most 4 small circles. Pick the nearest small circle; this is (8 + 6i) in the case illustrated. This small circle is at the end of an L shape with 3 lengths in the long part and 1 length on the short part of the L. So this circle is (8 + 6i) = (3 + i)(3 + i) and the quotient of (7 + 7i) divided by (3 + i) is (3 + i). The remainder is the number (-1 + i) and its norm is smaller than the norm of (3 + i). Division in the Gaussian integers is not unique because there are up to 4 small circles, each of which gives a different quotient and remainder and the remainder still has smaller norm than the norm of the dividend. This application gets the students thinking about the relationship between algebra and geometry. We investigate the geometry of numbers for 3 weeks before moving on to symmetries.

We begin the study of symmetries with the definition of the symmetry of a geometric figure. For two-dimensional patterns, there are three categories of symmetry groups: *rosette groups*, *frieze groups*, and *wallpaper patterns*. Rosette groups consist of finite cyclic groups and dihedral groups; cyclic groups are based on rotational symmetry and dihedral groups are based on both rotational and reflectional symmetries. Using a cardboard equilateral triangle, the students deduce that it has 6 symmetries. It is important to use the manipulative (in this case the cardboard triangle) at first. The instructor assigns a letter to each symmetry and the students and instructor construct the multiplication table satisfied by the group of symmetries.

The students are divided into groups and they repeat the same exercise with the square. After this exercise, we point out that all symmetries are reversible and composing two symmetries gives another symmetry. This leads naturally to the definition that a group of symmetries is a set of motions that includes the trivial motion and is closed under composition and inverses. Any symmetry, if performed enough times, is the same as not moving the shape at all (the trivial motion). The smallest number of times that is the same as the trivial motion is called the order of the element. It is also natural to look at subsets that contain the trivial motion and are closed under composition and inverses (a subgroup). Finally, we note that the number of elements in any subgroup divides the number in the entire group.

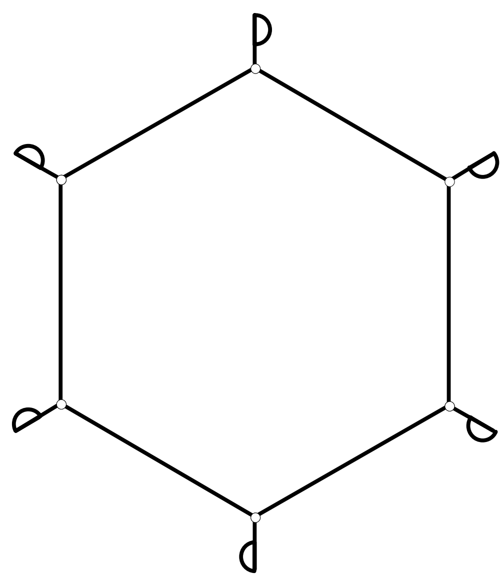

**Figure 2:** *The hexagon with flags*

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The symmetry group of the equilateral triangle is the symmetric group, $S_3$, which is non-abelian. The students are asked if the order of motions matters, and using the cardboard equilateral triangle quickly decide that it does. In addition, we look at the group of symmetries of a hexagon with flags on it, so that it has no reflections, and construct its multiplication table. Such a hexagon is displayed in figure 2. It also has only 6 symmetries and its symmetry group is the cyclic group $C_6$. The students are asked if the equilateral triangle and the hexagon with flags have the same "symmetry", since they both have 6 symmetries. They quickly see that these groups are different and we begin to develop the concept of an isomorphism between groups. We also explore the subgroup structure of each of these groups and construct its subgroup lattice. The symmetry group of the hexagon with flags is also used to develop the concept of a cyclic group. These explorations take time, but the increase in the student's intuition about the basic concepts of group theory make it time well spent. Additional exercises using the software below may be designed to reinforce the geometric aspects of the rosette groups, if time permits. This is done in the *Patterns in Mathematical Designs* course.

*Java Kali* (www.geom.umn.edu/java/Kali/program.html) is a software utility which was created based on symmetries. The software presents a set of buttons for each set of Rosette groups and also a board that allows you to create your design based on combinations of symmetries. The following figure presents two examples that are created using this utility based on the mentioned dihedral group $S_3$ and the cyclic group $C_6$.

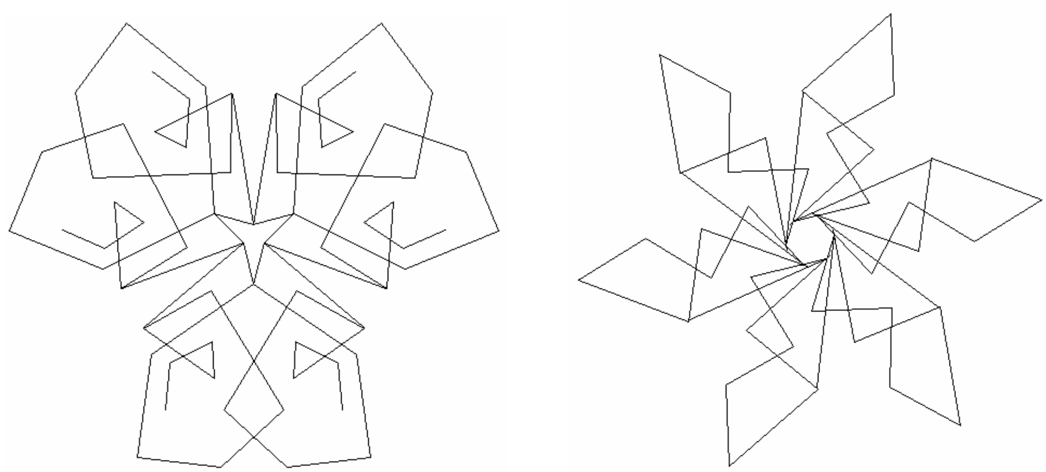

**Figure 3:** *Kali designs based on the symmetries of the equilateral triangle and the hexagon with flags*

The *Algebra of Symmetries* course continues by looking at plane figures such as a square. We note that the rotations can be performed inside the plane (planar motion), but any reflection can only be realized as a motion through three dimensions. The concept of the rotation group is developed at this point. The set of all planar motions is a subgroup and the composite of any reflection and a planar motion cannot be a planar motion. This leads directly to the concept of a coset and also foreshadows Lagrange's Theorem. Later, we make the same observation concerning the physically realizable motions of a three dimensional object and the non-physically realizable symmetries of the same object. These geometric ideas about cosets must be related to the algebraic definitions of cosets. In particular, the multiplication table of various groups is used to define cosets. The old DOS program, "Exploring Small Groups" by Ladnor Geissinger [2] or similar programs utilizing GAP (http://www.gap-system.org/), the free computer algebra system, can be utilized to color the cosets of a subgroup in the multiplication table. This graphically illustrates that a subgroup and its cosets have the same number of elements. In some cases, the cosets themselves form a group, called the quotient group.

At this point, the use of letters to represent symmetries is getting very cumbersome and so we introduce permutations of the vertices of the geometric figure. The various ideas involving the cycle structure of a permutation group, even and odd permutations and the symmetric and alternating groups follow. We finish this section by having the students (working in groups) list all of the symmetries of a regular tetrahedron and of the cube and give geometric descriptions of the physically realizable motions. At this point, the students must have a grasp of cosets to list all of the non-physically realizable motions of these three dimensional objects. In particular, we reinforce this when the instructor goes over the list of all symmetries. There is also a very good video, "Finite Symmetry Groups in Three Dimensions" [1] which describes the symmetry groups of the Platonic solids and of all solids with a finite number of symmetries.

At this point, we shift from thinking of moving the object to moving the space that the object is embedded inside. We develop the equations that give the plane coordinate transformations for any rotation around the origin or any reflection through a line through the origin. We show how these transformations may be written as $2 \times 2$ matrices and we are lead to the definition of matrix multiplication because of the way matrices represent symmetries. The multiplication table of matrices which represent the symmetries of a square is constructed and the students are asked to show that this is the same multiplication table that they have previously constructed. This involves assigning a matrix to each letter that they previously used in the multiplication table. Suppose that this assignment is done by the function f. Showing that these tables are the same naturally involves showing that $f(x \cdot y) = f(x) \cdot f(y)$ and we are lead to the definition of an isomorphism again. I also mention that representing finite groups as groups of matrices is a mathematical discipline, called Group Representation Theory. Finally, we spend some time looking at rings of matrices and relate this material to our study of numbers at the beginning of the course. Our assessment of this course indicates that the students find this way of looking at matrices very useful in their own teaching.

Up to this point, we have spent a lot of time on various groups of symmetries. Now we define a group abstractly. It makes sense why we use the group axioms and the students have plenty of experience with various groups of symmetries. The concepts are familiar as are most of the results that we will prove. The proofs still take some work, but the students find it easier than in a more abstract course. We also spend time defining normal subgroups and quotient groups. This section goes rather quickly.

We finish our study of groups by looking at the Euclidean groups $E(1)$, $E(2)$ and $E(3)$ where $E(n)$ is the group of all isometries of n dimensional Euclidean space. We prove that each Euclidean transformation can be uniquely written as an orthogonal transformation followed by a translation. We look closely at the subgroup of all orthogonal transformations which we can see is a group of matrices. The Seitz symbols are defined and associated to every element of $\mathrm{E}(n)$ and we see how to multiply them.

Now that we have the necessary group theory, we study and classify the seven Frieze patterns. The group theory needed to show that there are only seven Frieze patterns can be done in detail and the students seem to follow it without difficulty. We construct the different plane lattices and *Java Kali* can be used dynamically to enforce the mathematics involved in these patterns.

The simplest pattern is a frieze that consists of only one translation. In the following figure, **LLLLLL**, presents this group visually. It was also selected for the related button in *Java Kali*. Let **T** denote a translation along the $x$-axis, then the symmetry group of this pattern can be written as $\{\mathbf{T}^n \mid n \in \mathbf{Z}\}$. The group for the next pattern is made by a glide-reflection. If **G** denotes a glide reflection, we may write the symmetry group of this pattern as $\{\mathbf{G}^n \mid n \in \mathbf{Z}\}$. The symmetry group for the third pattern is generated by a translation **T** and a reflection **V** across the vertical line to the $x$-axis, $\{\mathbf{T}^n \mathbf{V}^m \mid n \in \mathbf{Z}, m = 0 \text{ or } 1\}$. The fourth pattern is the symmetry group generated by a translation **T** and a rotation **R** of 180°

(half-turn) about a point midway between consecutive L's. The symmetry group of this pattern can be written as $\{\mathbf{T}^n \mathbf{R}^m \mid n \in \mathbf{Z}, m = 0 \text{ or } 1\}$. A glide-reflection **G** and a half-turn **R** generate the pattern in the fifth design. The symmetry group for this pattern is $\{\mathbf{G}^n \mathbf{R}^m \mid n \in \mathbf{Z}, m = 0 \text{ or } 1\}$. A translation **T** and a horizontal reflection **H** generate the symmetry group for the sixth pattern. The symmetry group for this pattern is $\{\mathbf{T}^n \mathbf{H}^m \mid n \in \mathbf{Z}, m = 0 \text{ or } 1\}$. The symmetry group of the pattern in the last design is generated by a translation **T**, a horizontal reflection **H**, and a vertical reflection **V**. It can be expressed by $\{\mathbf{T}^n \mathbf{H}^m \mathbf{V}^k \mid n \in \mathbf{Z}, m = 0 \text{ or } 1, k = 0 \text{ or } 1\}$.

The students were asked to fill out an assessment on the course at the end. The consensus seems to be that they were between very satisfied and somewhat satisfied with the course, the material was reasonably well integrated and just a little bit more difficult then they would like. They believed that the material on the geometric representation of complex numbers and on matrices would help them with their teaching the most. A number of them felt that they needed more work on the idea of a coset. Finally, they all wished they had more time to study Wallpaper patterns at the end of the course.

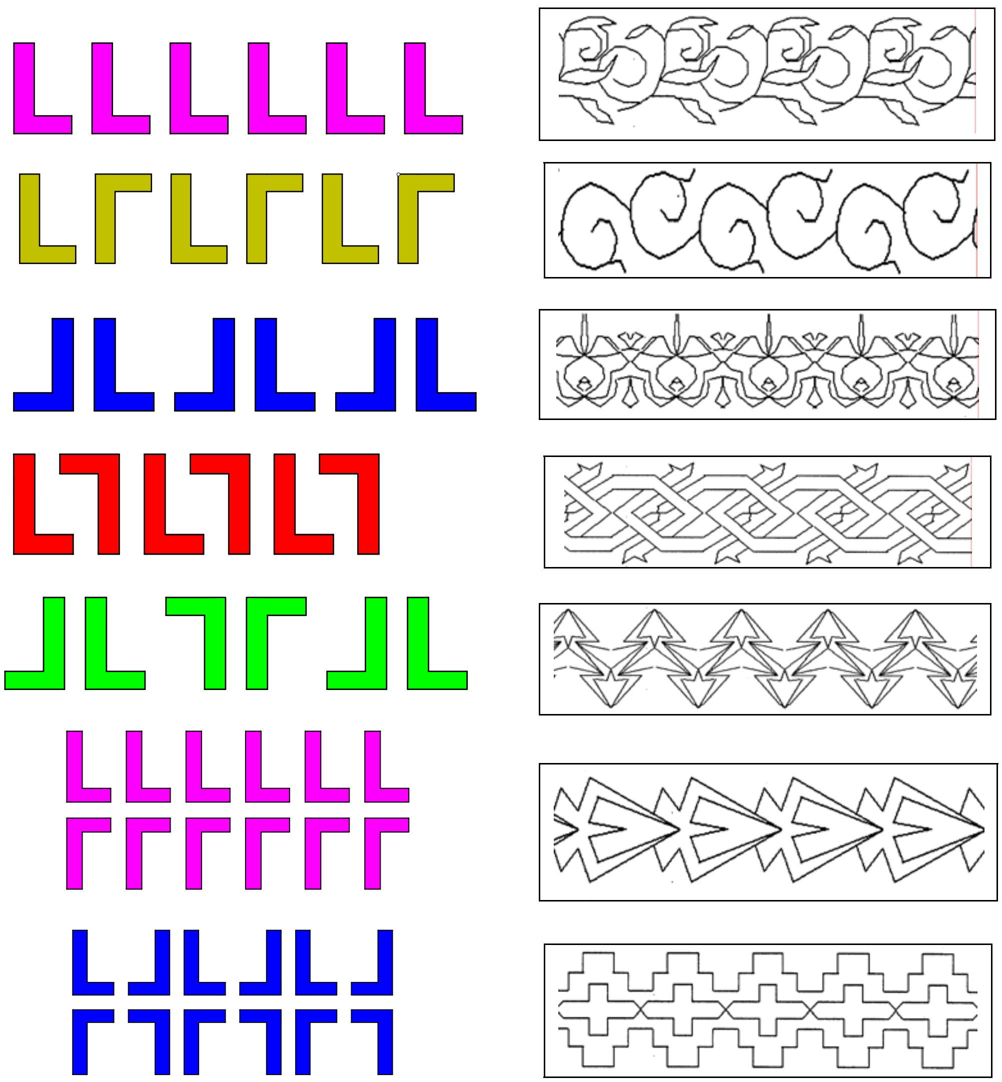

***Figure 4:*** *The "L" presentation and the Java Kali construction for frieze patterns.*

## 4. A Geometry Course from Different Perspective

The students who participate in this graduate program usually have experienced a single geometry course as their undergraduate background. This single course mostly is based on the review of synthetic Euclidean geometry, finite geometries, systems of axioms, classical theorems, and elementary transformations. If time allows, there is a quick review of the non-Euclidean geometries. Even though it is expected, because of time constraints, students in this undergraduate course hardly have a chance to deeply gain the knowledge of modern geometry including both Euclidean and non-Euclidean geometries, and the knowledge of the historical development of both Euclidean and non-Euclidean geometries. In addition, there are geometric ideas such as fractal geometry, geometry of solids, and tessellations that students never have chance to study in their undergraduate geometry. Moreover, students never experienced spherical and hyperbolic geometries as something that may exist through their models other than some axioms and properties.

## 5. Patterns in Mathematical Designs

*Patterns in Mathematical Designs* is a course that has advanced for mathematics teachers from two perspectives: the designers who are concern about the aesthetic and the constructions of a design, and the mathematicians who are concern about the relationships and properties of the components and the whole. This course exemplifies a geometric bridge between art and science. In this course, after presenting basic mathematics concepts and geometric theorems and properties, we isolate specific instances of the pattern of a general, either presented in a culture, or existed in nature, to formulate and generalize its properties. The spatial structures, whether crystalline, architectural, or choreographic, have parameters such as symmetry, proportion, connectivity, stability, etc. The purpose of this course is to explore these parameters with the objective to show, by way of demonstration, that there exist mathematical connections among the subjects of arts, architecture, chemistry, biology, engineering, and computer graphics. To be an effective educator, a teacher must be aware of these connections in order to utilize them for activities that promote innovative and creative approaches to teaching mathematics.

The course covers topics such as the systems of proportion in mathematics, art, architecture, and in nature; the Golden Mean, Fibonacci Series, Archimedes and Logarithmic Spirals, growth and similarity in nature; graphs and maps on the Euclidean plane and on a sphere, on a torus, and map coloring; Periodic and Non-periodic Tilings, Duality and the modules of semiregular tilings; polyhedra and Platonic Solids and their duality and combinatorial and space-filling properties. In addition to its importance in its applications to the real world, geometry is one of the three content areas that the mentioned master's degree majors in mathematics education should be competent.

The textbook for the course is "*Connections: The Geometric Bridge between Art and Science"* by Jay Kappraff [4]. Moreover, several pre-selected articles from the proceedings of the Bridges Conference series are included. Through the course the students have chance to review more articles from these proceedings for their individual projects. The projects will take the form of research papers, PowerPoint presentations, and/or lesson plans.

Through the following examples, based on a student's projects that resulted in two joint articles with her instructor [5, 6], we try to give an idea the types of topics and activities that can be covered in such a course.

**5.1. Spherical Geometry and Design.** There are mathematics educators that advocate for the inclusion of spherical geometry in a secondary school geometry curriculum. They believe that since spherical geometry is so much a part of our everyday experiences (after all, we live on a sphere), students should be afforded an opportunity to study spherical geometry alongside Euclidean geometry. However, their mathematics teachers never experienced such a geometry through their course of study to prepare them to teach this geometry. One venue to remedy this problem is to cover interesting topics from this geometry in the "Patterns of Mathematical Designs" course.

The spherical geometry that we will introduce to the students is in nature elliptic geometry (all great circles meet each other so there are no parallel lines), but we will not get involved with the abstraction and the steps of making a consistent system in this course.

Our approach, rather, in this course would be to experience the spherical geometry through some theorems and related activities on actual sphere in classroom.

The following activity leads to a formula for the area of a triangle on the sphere. In order to understand how the formula is derived, we need to draw triangles on the sphere and use colors to identify the different triangles under discussion. This formula is commonly known as Girard's Theorem. We draw a triangle on the sphere and label the angles $\angle \alpha$, $\angle \beta$, and $\angle \gamma$ as shown in the diagram.

We draw the α-lune and notice that there is a congruent α-lune on the back of the sphere. We repeat this for the β-lune and the γ-lune and notice that the triangle $\Delta ABC$ appears in each of the lunes. Notice also, that there is a copy of triangle $\Delta ABC$ in each of the lunes on the back of the sphere. If we now wished to get an expression for the area of the sphere in terms of the area of the lunes, we would get the following ($lune_\alpha$ = area of lune α)

Area of sphere = 2 $lune_\alpha$ + 2 $lune_\beta$ + 2 $lune_\gamma$ – 4 Area ($\Delta ABC$).

$$4\pi r^2 = 2\,\frac{\alpha}{360}(4\pi r^2) + 2\,\frac{\beta}{360}(4\pi r^2) + 2\,\frac{\gamma}{360}(4\pi r^2) - 4\,\text{Area}(\Delta ABC),$$

$$\text{Area}(\Delta ABC) = 2\,\frac{\alpha}{360}(\pi r^2) + 2\,\frac{\beta}{360}(\pi r^2) + 2\,\frac{\gamma}{360}(\pi r^2) - \pi r^2,$$

$$\text{Area}(\Delta ABC) = \pi r^2\left(\frac{\alpha+\beta+\gamma-180}{180}\right).$$

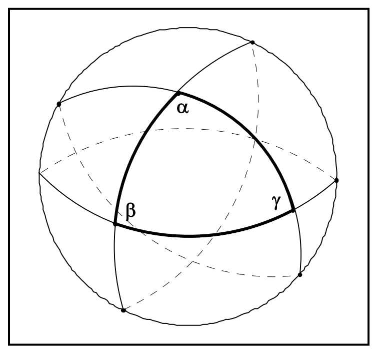


**Figure 5**

This formula has a very interesting consequence for the area of a triangle on the sphere. It states that the area of a triangle on the sphere is directly related to the angles of the triangle (and therefore, all similar triangles on sphere are congruent and we have no non-congruent similar objects on the sphere!).

This formula for the area of a triangle on sphere presents the *Girard's Theorem*, and the quantity

$$\alpha + \beta + \gamma - 180^\circ$$

is called the *spherical excess of the triangle.*

Spherical geometry was studied and known to the Babylonians, Indians, and Greeks more than 2000 years ago. Buzjani is a Persian mathematician who was born in the town of Buzjan in Khorasan Province in the tenth century. He moved to Baghdad and taught geometry and wrote several books on this subject and other branches of mathematics such as number theory and algebra. His most profound work is the completion of a spherical geometry system.

In a treatise written by Buzjani, On Those Parts of Geometry Needed by Craftsmen, the images of constructions on a sphere have been illustrated flat. Chapter 12 of the manuscript is about geometric constructions on a sphere. This chapter begins with the construction of the great circle of a sphere and then presents some spherical Platonic and Archimedean solids. The spherical tiling with twenty equilateral triangles (icosahedron) has been performed in problem numbered 180 as follows:

Let $E$ and $F$ be the two poles of a sphere (only $E$ (ه) can be seen in the following figure) and the great circle $C$ is perpendicular to $EF$. Divide circle $C$ to ten equal arcs. $C_iC_{i+1}$, where $i$ =1…9 (these divisions in the figure have been labeled as الف, ب, ج, … ).

With the centers of $C_1$ and $C_2$, and with the radius of $C_1$ $C_2$ we construct two arcs to meet at $A_1$ on $E$ side of the great circle $C$ (in the figure all $A_i$ ‘s are labeled as ص). With the centers of $C_2$ and $C_3$, and with the radius of $C_2C_3$ we construct two arcs to meet at $B_1$ on F side of the great circle $C$ (in the figure all $B_i$ ‘s are labeled as ق). We repeat this procedure to find five points $A_i$ on the $E$ side and five points $B_i$ on the $F$ side of the great circle. Then using great circles passing through $A_1$and $B_1$, $B_1$ and $B_2$, and $B_2$ and $A_1$ respectively we will construct the equilateral triangle $B_1A_1B_2$. With this procedure we will construct ten equilateral triangles of $B_iA_iB_{i+1}$, $i$ =1…4, $B_5A_5B_1$, $A_iB_iA_{i+1}$, $i$ =1…4, $A_5B_5A_1$. Now what is left in $E$ side is a spherical regular pentagon of $A_1$ $A_2$ $A_3$ $A_4$ $A_5$ and on the $F$ side is the spherical regular pentagon $B_1$ $B_2$ $B_3$ $B_4$ $B_5$. Using great circles that pass through $E$ and $A_i$, $i$ =1…5, we are able to divide this pentagon to five congruent equilateral triangles. We do the same on the other side of the great circle. This will conclude our construction of spherical icosahedron.

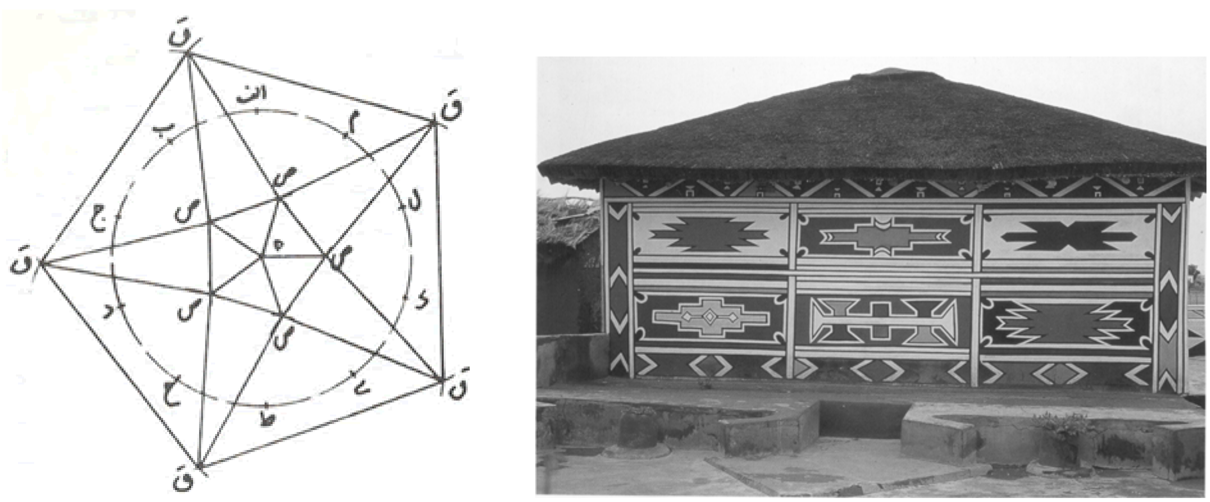

**Figure 6:** (a) *The spherical icosahedron construction,* (b) *A Ndebele tribe hut* [6].

**5.2. A Fractal Game Using African Patterns.** The peoples of Africa, south of the Sahara desert, constitute a vibrant cultural mosaic, extremely rich in its diversity. It is on the southern tip of this continent that we find the Ndebele tribe. An interesting introduction to fractal geometry can be found in some panels in the house decoration of Figure 6.b belong to this tribe.

The design can be seen as originating from the form in Figure7.a. The area of this figure is $A = 4bh - \frac{1}{2}h\sqrt{a^2 - h^2}$ . After the first iteration, the diagram becomes as in Figure 7.b. The area is $A + 4(\frac{1}{4}bh) = A + bh$ . On the next iteration, the diagram becomes as in Figure 7.c.

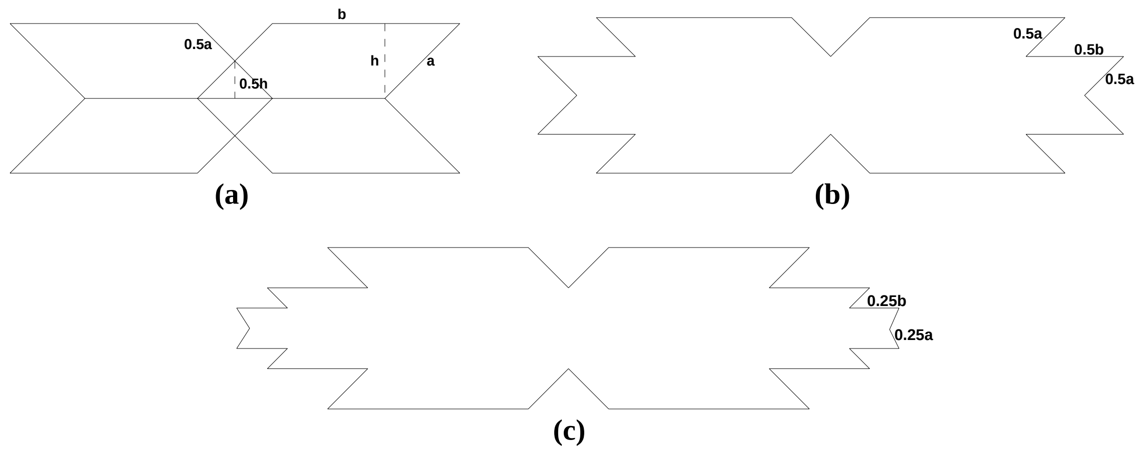


**Figure 7**

The total now is $A+bh+\frac{1}{4}bh$. On the next iteration the total area will be $A+bh+\frac{1}{4}bh+\frac{1}{16}bh$.

Continuing to infinity we will get

$$\text{The total area} = A + bh + \frac{1}{4}bh + \frac{1}{16}bh + \frac{1}{64}bh + \ldots = A + \frac{4}{3}bh.$$

Substituting for A, the final expressions for the area of the figure is = $4bh + \frac{4}{3}bh - \frac{1}{2}h\sqrt{a^2 - h^2}$

$$= \frac{16}{3}bh - \frac{1}{2}h\sqrt{a^2 - h^2}$$

## 6. Conclusion

The Algebra of Symmetries course uses the student's intuition about geometric objects and motions to introduce group theory as a topic in a less abstract setting than it is usually encountered. This has several advantages. First, group theory is seen as a useful tool and not as "abstract non-sense". Secondly, it is very natural to finish the course with continuing applications of group theory to geometry and also to art and architecture. The Design Science course finds mathematics in many things involving art and architecture. This gives in-service teachers a stock of applications, which they may use to introduce many mathematical topics in their own courses. Once again, it uses the principle that students learn best from studying concrete objects (visual or otherwise) and abstracting mathematics from them. Finally, it is our hope that the graduate students will use the pedagogy espoused in these courses in their own classes.

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[6] A. Gray and R. Sarhangi, *Study and Application of African Patterns for Use in Secondary Education*, to be submitted to the MCTM journal.