ζ-Irrationality Search: After a Golden Section Approach, Another Esthetic but Vain Attempt

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Abstract

The proof of the irrationality of ζ(5) is a long standing open problem. The present paper abandons a golden section inspiration (as many artists may have done in their field), and suggests a different approach. Yet, it appears as vain as the first one, though it does offer an opportunity to resuscitate interest in the topic, while an extra esthetic zeta-formula is encountered concurrently.

1. ζ(2), ζ(3) and the golden section.

Although a previous paper was at first sight but a summary of existing proofs for the irrationality of π, ln2, ζ(2) = $1+\frac{1}{2^2}+\frac{1}{3^2}\ldots$ and ζ(3) = $1+\frac{1}{2^3}+\frac{1}{3^3}\ldots$ (see [2]), it was given the "Lester Ford Award 2002" by the Mathematical Association of America, while some found an inspiration in it for a query about still other famous mathematical constants, such as e and Euler's constant (see [4]), and others continued their computer search for similar constants (see [3]). To F. Beukers (see [1]), the reason for these reactions was the lack of progress in this field at the time, and thus any sensible new impulse is meaningful.
Furthermore, there was a link to mathematical notions used more often in artistic circles, though not so well known to pure number specialists: the golden section, noted φ, τ, g or σ_{Au}, and the silver and bronze sections, σ_{Ag} and σ_{Br}. They are the positive roots of $x^2-nx-1=0$, for n=1, 2, 3…, and they emerged as follows in the explained proofs.

For ζ(2), 0 < $\left|\int_0^1\int_0^1\frac{(x(1-x)y(1-y))^n}{(1-xy)^{n+1}}dxdy\right|$ = $\left|\frac{R_{n+1}+S_{n+1}\zeta(2)}{T_{n+1}}\right|$ for any n∈N, while M_2= $\left|\max(\frac{x(1-x)y(1-y)}{1-xy})\right|$ and $(M_2)^n.T_{n+1}\le(M_2)^n.(3^{n+1})^2\le 1$. Thus, the rationality of ζ(2) would lead to a contradiction as R_n and S_n (and T_n) are integers and $\left|R_{n+1}+S_{n+1}\zeta(2)\right|\to 0$.

For ζ(3), 0 < $\left|\int_0^1\int_0^1\int_0^1\frac{(x(1-x)y(1-y)z(1-z))^n}{(1-(1-xy)z)^{n+1}}dxdydz\right|$ = $\left|\frac{R_{n+1}+S_{n+1}\zeta(3)}{T_{n+1}}\right|$ for any n∈N, while M_3= $\left|\max(\frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z})\right|$ and $(M_3)^n.T_{n+1}\le(M_3)^n.(3^{n+1})^3\le 1$. Thus, the rationality of ζ(3) would lead to a contradiction as $\left|R_{n+1}+S_{n+1}\zeta(3)\right|\to 0$.

For ζ(4), it was expected the following expression had potential for attempting a proof (and its extension, eventually, for ζ(5)):

$$\left|\int_0^1\int_0^1\int_0^1\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)w(1-w))^n.(1-xy))^n}{((1-(1-xy)z)(1-(1-xy)w))^{n+1}}dxdydzdw\right|.$$

Indeed, the maximum $M_2=|(1/\sigma_{Au})^5|$ is attained for $x=y=-1/\sigma_{Au}$, and $M_3=|(1/\sigma_{Ag})^4|$ for $x=y=-1/\sigma_{Ag}$, and now the M_4-maximum is obtained for $x=y=-1/\sigma_{Br}$. However, the same paper also pointed out this option failed since the integral is not of the form $(R_{n+1}+S_{n+1}\zeta(4))/T_{n+1}$. Thus, the golden-silver-bronze section connection was misleading (partially - but this happened in art too: see references given in [2]).

2. Another approach for a ζ-irrationality proof.

An esthetic expression, based on the logic in the form of the integrand in the given proofs, seemed promising to overcome some surprising difficulties of ζ-irrationality proof attempts:

(E) $$\zeta(m) = \frac{1}{m-1}\int_0^1\int_0^1\int_0^1 \ldots\int_0^1 \frac{1}{(1-xy)(1-xyz)\ldots(1-xyz\ldots w)}dxdydz\ldots dw.$$

Now, the proof could go by checking the following conjectures:

(I) $$0<\left|\int_0^1\int_0^1\int_0^1 \ldots\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)\ldots w(1-w))^n}{((1-xy)(1-xyz)\ldots(1-xyz\ldots w))^{n+1}}dxdydz\ldots dw\right| = \left|\frac{R_n+S_n\zeta(m)}{T_n}\right|,\ R_n, S_n, T_n \in Z.$$

(II) $$M_m = \max\left|\frac{x(1-x)y(1-y)z(1-z)\ldots w(1-w)}{(1-xy)(1-xyz)\ldots(1-xyz\ldots w)}\right| \text{ with } \left|M_m.3^m\right|\le 1.$$

For ζ(2), the proposal coincides with the well-known proof, while it can be shown suitable substitutions transform the proposed ζ(3) integral into Beuker's type. For ζ(4), the (very large) algebraic expression for the maximum value M_4 has no more relation to the bronze mean but, numerically at least, condition (II) can be verified: $M_4.3^4<1$; that is a good start. Now some substitutions lead to:

$$\left|\int_0^1\ldots\int_0^1 \frac{(x(1-x)y(1-y)z(1-z)w(1-w))^n}{((1-xy)(1-xyz)(1-xyzw))^{n+1}}dx\ldots dw\right| = \left|\int_0^1\ldots\int_0^1 \frac{(x(1-x)q(1-q)r(1-r)w(1-w))^n}{(1-(1-qrw)x)^{n+1}}dx\ldots dw\right|$$

As in [1], it establishes the expression (E) for $n=0$, while the general expression now transforms into

$$\left|\int_0^1\int_0^1\int_0^1 \frac{(1-q)^n \frac{1}{n!}\frac{d^n}{dr^n}(r(1-r))^n \frac{1}{n!}\frac{d^n}{dw^n}(w(1-w))^n}{1-qrw}\log(qrw).dqdrdw\right|$$

Already for n=1, it is seen that the numerator does not only contain terms in $(qrw)^j$, j=0…n, yielding a fraction times ζ(4), but other terms as well. That is, the above calculations only show that 0 < $\left|R_{n+1}+S_{n+1}\zeta(3)+U_{n+1}\zeta(4)\right|\to 0$, for R_n, S_n, U_n ∈Z. Thus, the only thing to remember from the present paper may be the esthetic expression (E) for ζ(m), but, alas, the author did not have the nerve to check if this expression deserves a proof, in despite of J. Sondow's encouragement.

References

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[4] J. Sondow, *Criteria for irrationality of Euler's constant*, Proc. Amer. Math. Soc 131, pp. 3335-3344 2003.