Aesthetic Aspects of Venn Diagrams

Barry Cipra
305 Oxford St.
Northfield, MN 55057
bcipra@rconnect.com

Peter Hamburger
Department of Mathematical Sciences
Indiana-Purdue University
Fort Wayne, IN 46805
hamburge@ipfw.edu

Edit Hepp
Windswept Drive Studio
Fort Wayne, IN 46815
hamburger1@comcast.net

Introduction

Venn diagrams are familiar to anyone who has taken high school algebra. The standard two- and three-circle Venn diagrams have a pleasing look. This stems partly from their symmetries. As customarily drawn, each has a horizontal mirror symmetry. The two-circle diagram also has a second, vertical mirror symmetry. In addition to these, each diagram also has rotational symmetry: The two-circle diagram is invariant under rotation by 180 degrees around a point inside the lens-shaped intersection of the circles, while the three-circle diagram is invariant under rotation by 120 degrees around a point inside its triple-intersection region.

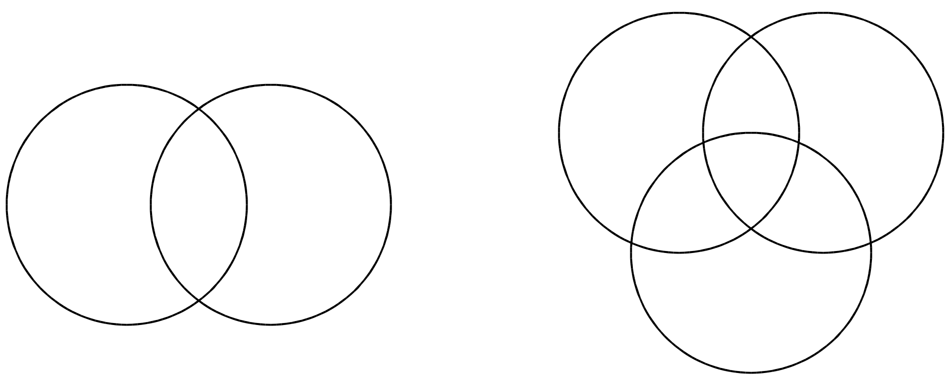

Figure 1: *The familiar 2- and 3-set Venn diagrams.*

In 1963, David Henderson at Swarthmore College raised, and partly answered, the question whether Venn diagrams with more sets could have analogous rotational symmetry [8]. To be precise, is it possible to draw a simple closed curve, rotate it by multiples of 360/*n* around some point in its interior to produce *n* congruent copies of the curve so that the result is a Venn diagram for *n* sets (each set being the interior of one of the curves)?

Henderson showed that this cannot be done if *n* is not a prime number. The basic reason is illustrated by the simplest case, *n*=4. If a Venn diagram were symmetric under rotation by 90 degrees, then each region corresponding to a pairwise intersection of sets would belong to a group of 4 such regions. This means that the number of pairwise intersections must be a multiple of 4. But with four sets, there are 6 pairwise intersections, which is not a multiple of 4. In general, n must divide each binomial coefficient $C(n,k) = n!/k!(n-k)!$ for k equal 1 to n-1, but a classic theorem, attributed to Leibniz, says this only happens if n is prime.

This negative result leaves open the question of the existence of symmetric Venn diagrams when n is prime. Henderson gave two examples of symmetric diagrams for n=5. One uses (irregular) pentagons, the other uses quadrilaterals. Branko Grünbaum at the University of Washington later gave a lovely construction with equilateral triangles [3]. He also produced a striking example with five ellipses [2].

Seven, Eleven, Etc.

It took nearly 30 years from the publication of Henderson's paper to the discovery of rotationally symmetric Venn diagrams for n=7. Grünbaum published the first in 1992 [4]. Anthony Edwards at Cambridge University subsequently found a total of six different examples. (Two Venn diagrams are "different" if it is not possible to continuously deform each to match the other or its mirror image.) Images of various symmetric Venn diagrams for n=5 and 7 are available online in a dynamic survey of Venn diagrams by Frank Ruskey at the University of Victoria [9].

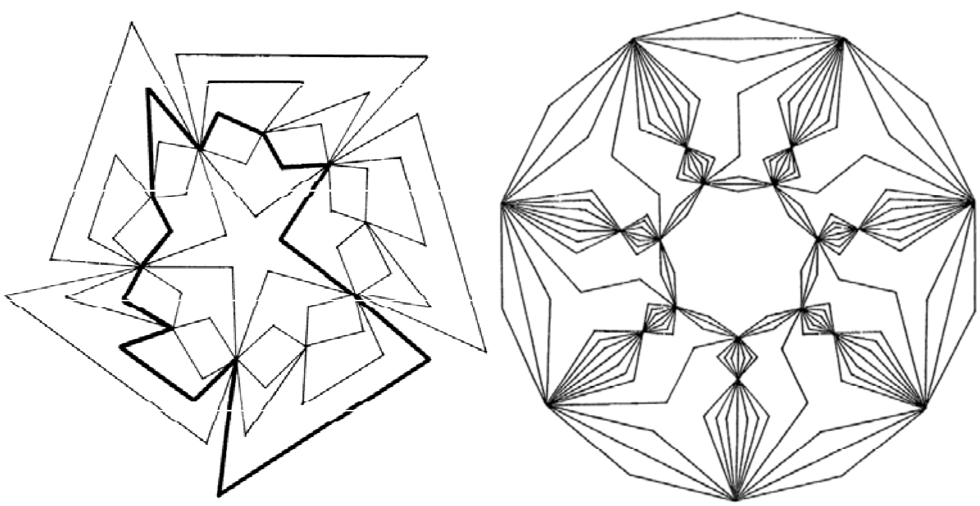

Figure 2: *Symmetric Venn diagrams with 5 and 7 sets. These diagrams have the minimum number of vertices (10 and 21, respectively) for their size. The (polygonal) curve whose rotations create the 5-set diagram is highlighted. For details, especially on the 7-set diagram, see [6] and [7].*

The answer for n=11 took nearly another decade. Peter Hamburger at Indiana-Purdue University at Fort Wayne found an approach that produced a symmetric Venn diagram with 11 sets [5]. He titled his paper "Doodles and Doilies," because the approach involves doodle-like drawings that lead to Venn diagrams with a lacy, doilie-like appearance, as least as Hamburger draws them.

Hamburger's wife, Edit Hepp, has turned his 11-set "doilies" into works of art, using color to highlight regions corresponding to different types of intersections. Hepp's originals are large (approximately 32 inches in diameter). She creates the Venn diagram by hand drawing one "wedge" of it, pasting together xerox copies of the wedge and xeroxing the result. (She also drew the diagrams in Figure 2, above.) Hepp uses colored pencils, she says, "to obtain the richest possible textures."

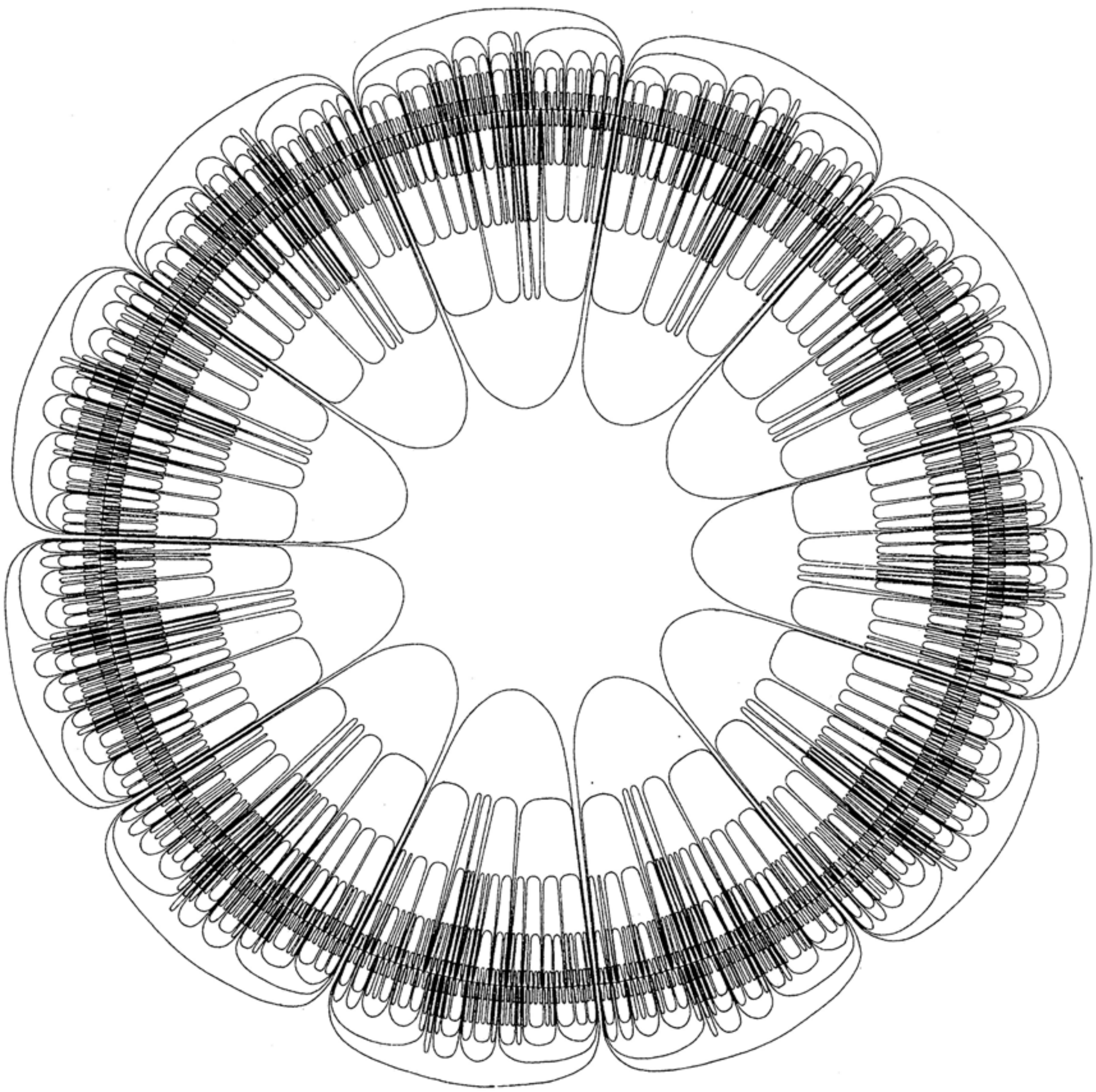

Figure 3: *Peter Hamburger's rotationally symmetric 11-set Venn diagram "doilie" [5].*

Rotationally symmetric Venn diagrams allow for innumerable artistic choices. Within each wedge, for example, the arcs can circular, sinusoidal, or even polygonal. It is also possible to disrupt the symmetry by emphasizing just one of the diagram's curves. (With the exception of Grünbaum's ellipses and equilateral triangles for *n*=5, all of the rotationally symmetric Venn diagrams with more than three sets are based on curves with little if any symmetry.) As Hepp puts it, "Symmetry creates beauty, but the most stunning images are those where the symmetries are intentionally destroyed by the artist in order to dive deep into a mathematical principle." In one picture, Hepp deletes everything outside of one curve, which removes half of the regions of the Venn diagram, and then colors the remaining regions. The result is wonderfully abstract.

What about rotational symmetry for *n*=13 or other primes? The approach Hamburger took for n=11 was expanded on by Jerry Griggs at the University of South Carolina and Charles Killian (an undergraduate) and Carla Savage at North Carolina State University, who proved that rotational symmetry can be achieved for all primes (thereby cutting short a potentially infinite sequence of papers) [1]. So far there have been no pictures–a good opportunity for an artistically oriented triskaidekaphilic.

References

[1] Griggs, J., Savage, C., and Killian, C., Venn diagrams and symmetric chain decompositions in the Boolean Lattice, *Electronic Journal of Combinatorics* 11 (2004), Research Paper 2 (available online at www.combinatorics.org).

[2] Grünbaum, B., Venn diagrams and Independent Families of Sets, *Mathematics Magazine*, 48 (Jan-Feb 1975) 12-23.

[3] Grünbaum, B., The Construction of Venn Diagrams, *College Mathematics Journal*, 15 (1984) 238-247.

[4] Grünbaum, B., Venn Diagrams II, *Geombinatorics*, Volume II, Issue 2, (1992) 25-32.

[5] Hamburger, P., Doodles and doilies, non-simple symmetric Venn diagrams, *Discrete Mathematics*, 257 (2002), 423-439

[6] Hamburger, P., Pretty drawings. More doodles and doilies, symmetric Venn diagrams, *Utilitas Mathematica*, to appear.

[7] Hamburger, P., and Pippert, R. A symmetrical beauty. A non-simple 7-Venn diagram with a minimum vertex set, *Ars Combinatoria*, 66 (2003) 129-137.

[8], Henderson, D., Venn diagrams for more than four classes, *American Mathematical Monthly*, 70 (1963) 424-426.

[9] Ruskey, F., A Survey of Venn Diagrams, *Electronic Journal of Combinatorics*, Dynamic Survey DS5, 1997, www.combinatorics.org/Surveys/ds5/VennEJC.html. (Note: The dynamic survey was first posted in 1997. It has been updated to include more recent work.)