Tessellations from Group Actions and the Mystery of Escher's Solid

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Abstract

In the paper [1], Joyce Frost and Peg Cagle show how to construct a tessellation with squares of the plane from another one of smaller squares, and how this process can be generalized in three dimensions to construct a tessellation of the space with rhombic dodecahedra from a tessellation with cubes. The authors then proceed and explain how to construct a stellated rhombic dodecahedron (Escher's solid), and why this solid is space-filling as well. Interestingly, the procedure of constructing a tessellation from a given one by conveniently cutting some of the tiles can be iterated in all these cases, and the tessellations obtained can be associated to group action in the plane, or three-dimensional space, respectively.

1. Plane Tesselations and Group Actions

Let's start with an easy tessellation: consider the plane covered by unit squares. If we consider the group $G_1 = \langle f, g \rangle$ of plane transformations generated by the transformations $f(\mathbf{x}) = \mathbf{x} + \mathbf{i}$ and $g(\mathbf{x}) = \mathbf{x} + \mathbf{j}$, where **x** denotes a point in the plane, and **i**, and **j** are the unit vector on the axes, then all the squares of the tessellation are equivalent under the group action of G_1.

Now imagine the squares colored in a checkerboard pattern, mark the centers of the white squares, and join them with the sides (figure 1). This procedure will divide each white square into four triangles. Add the triangles to the adjacent black squares to which they share a full edge. The plane now becomes tiled by squares with vertical and horizontal diagonals of length 2 (figure 2). This tessellation can be viewed as the result of the action of the group $G_2 = \langle h, k \rangle$ on the plane, where $h(\mathbf{x}) = \mathbf{x} + \mathbf{i} + \mathbf{j}$ and $k(\mathbf{x}) = \mathbf{x} + \mathbf{i} - \mathbf{j}$. It is not too hard to show directly that G_2 is a subgroup of G_1, and that $[G_1 : G_2]$=2. However, this is even easier if we repeat the procedure of coloring the squares and attaching the resulting triangles one more time. By doing this, we obtain a new square tessellation, in which the tiles have the edges parallel to the axes again, and the length of the edges is 2. So a group whose action generates this

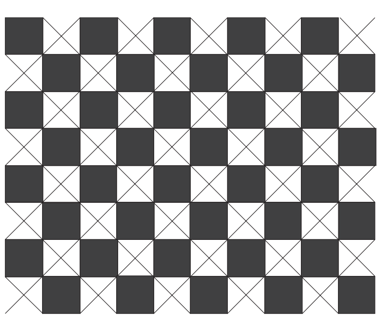

Figure 1.

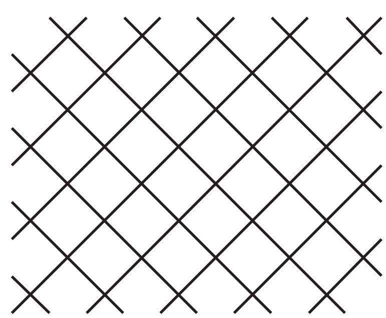

Figure 2.

tessellation is $G_3 = \langle l, m \rangle$, with $l(\mathbf{x}) = \mathbf{x} + 2\mathbf{i}$ and $m(\mathbf{x}) = \mathbf{x} + 2\mathbf{j}$. It is now obvious that $G_3 \le G_2 \le G_1$, and since $[G_1 : G_3] = [G_1 : G_2][G_2 : G_3] = 4$ and all subgroups are proper, then $[G_1 : G_2] = [G_2 : G_3] = 2$. Of course, this can be continued indefinitely.

2. Space-Filling Polyhedra

Now let's apply the same procedure in three dimensions, starting with a tessellation of the space into cubes. Imagine the cubes colored alternating in black and white. Cut the white cubes into six pyramids each, by joining the centers to the vertices. Attach the pyramids to the adjacent cubes, thus obtaining a tessellation of the space with rhombic dodecahedra. The cubic tessellation is obtained by considering the action of the group $G_1 = \langle f, g, h \rangle$, where $f(\mathbf{x}) = \mathbf{x} + \mathbf{i}$, $g(\mathbf{x}) = \mathbf{x} + \mathbf{j}$, $h(\mathbf{x}) = \mathbf{x} + \mathbf{k}$, while the rhombic dodecahedron tessellation is generated by $G_2 = \langle \mathbf{x} + \mathbf{i} + \mathbf{j}, \mathbf{x} + \mathbf{j} + \mathbf{k}, \mathbf{x} + \mathbf{k} + \mathbf{i} \rangle$.

What happens if we continue? The next step is to dissect some of the rhombic dodecahedra. Since each has 12 faces, this produces 12 pyramids with rhombic bases. Attaching these pyramids to the adjacent dodecahedra produces three-dimensional tiles, which are stellated rhombic docecahedra. The figures below show the rhombic dodecahedron and the stellated rhombic dodecahedron. The latter is also known as Escher's solid, because it was represented by Escher in his woodcut "Waterfall". The space tessellation with Escher solids, accompanied by a good illustration of how the solids fit together can be found in [2].

What is interesting is that repeating the dissecting and attaching procedure one more time gets us back to a cubic tessellation. This is easier to see with real models and it is due to the fact that each stellated rhombic dodecahedron breaks into 48 tetrahedra, out of each groups of 6 get attached together to the adjacent solid. Finally, let's denote by G_3 the group of space transformations generating the tessellation by Escher solids, and G_4 the group generating the subsequent cubic tessellation. The cubes have sides of length 2, and hence $G_4 = \langle \mathbf{x} + 2\mathbf{i}, \mathbf{x} + 2\mathbf{j}, \mathbf{x} + 2\mathbf{k} \rangle$. Thus $[G_1 : G_4] = [G_1 : G_2][G_2 : G_3][G_3 : G_4] = 8$, and each group has index 2 in the next one.

As a final observation, the fact that in this type of construction the index of each subgroup in the next group is equal to 2 is in agreement with the fact that the area or volume of the tiles used in the tessellation doubles at each iteration.

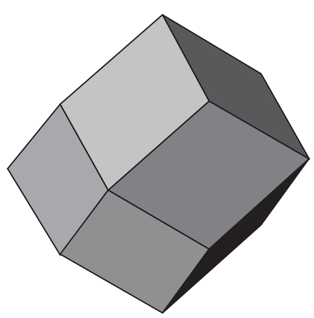

Figure 3.

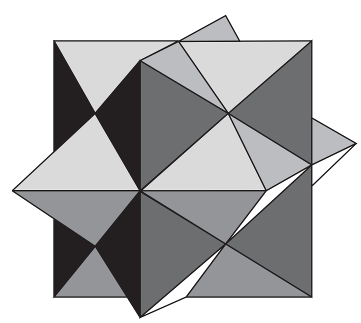

Figure 4.

References

[1] Joyce Frost, Peg Cagle, *An Amazing, Space-Filling, Non-regular Tetrahedron*, http://mathforum.org/pcmi/hstp/resources/dodeca/paper.html

[2] Alan Holden, Shapes, Space and Symmetry, Columbia University Press, NY, 1971.