An Introduction to the Golden Tangram and its Tiling Properties

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Abstract

The Author of this paper has developed a Tangram based upon the Golden Ratio. This introduction to the tiling properties of the Golden Tangram focuses upon the property called Preciousness and a Matrix called the Precious Matrix. It includes a discussion of some of the convex polygons that can be created and their non periodic tessellating patterns. It continues with examples of the unique way in which they can produce an infinite number of tiling patterns. It explains the iterative nature of the process as applied to designs for Mosaics and Quilts.

1. Introduction

This work follows on from previous work on Precious Triangles and Polygons [1],[2],[4], [5]. The idea has been expanded to include a set of polygons related to the Golden Ratio. The set of polygons introduced here have edges of only two dimensions. The ratio of the larger to the smaller being the Golden Ratio. Together, the tiles can be formed into a number of convex polygons, one of which is the regular Pentagon, see figure 1. Most people will be familiar with the standard Tangram puzzle [3] from which pictures and designs can be created. The normal Tangram rules are that all the tiles should be used and they should touch but not overlap another tile.

2. What are Precious Polygons?

Precious Polygons are sets of different polygons which can be used to form other sets of similar polygons. The necessary conditions for preciousness are that a larger version of each polygon can be produced using only the original polygons, secondly, the enlargement factor in each case must be the same and finally, all the elements of Nth power of the Precious Matrix must be non zero, where N is the number of different tiles [1]. This process is similar to that of Solomon Golomb's Rep-tiles [7],[8] but involving sets of polygons rather than a single polygon. Self similarity was also a feature of 14th and 15th Century Islamic Geometry [9].

3. Precious Polygons from the Golden Tangram.

There are six different polygons that form the Golden Tangram in figure 1, two triangles, one trapezium, two rhombi, and a pentagon. It also shows a scheme whereby each of the original polygons can combine together to form a similar but larger version of the original set. It can be quite easily shown that each large version is larger than the original by a factor known as the Golden Ratio or φ [4]. This enlargement factor must be a constant for the set to be Precious and is known as the Precious Ratio. Now that we have a Precious set then we can take any design made from the original Tangram shapes and produce a larger version using the scheme in figure 1. Since we end up with a design using only the original shapes, we can repeat the process ad infinitum. Each successive design is larger than the previous one by a factor of φ. Figure 2 shows the development of the series of designs based upon a cat design.

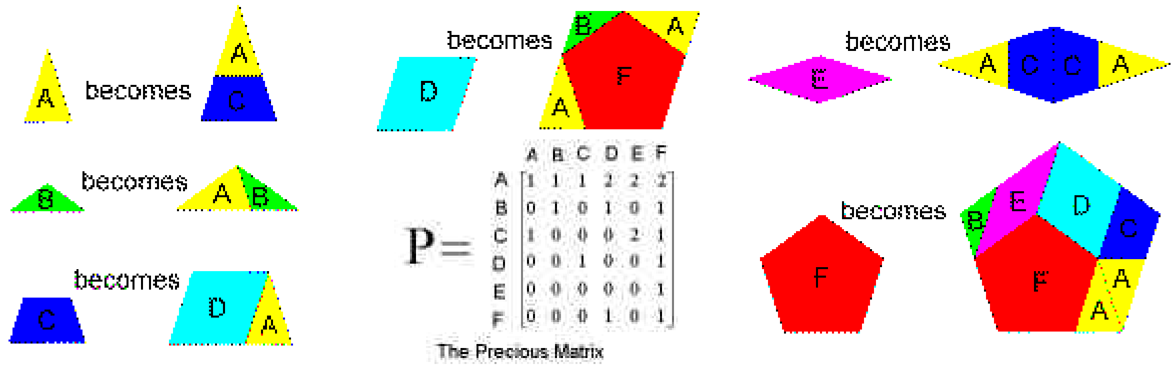


Figure 1: *The six Precious Polygons forming the Golden Tangram.*

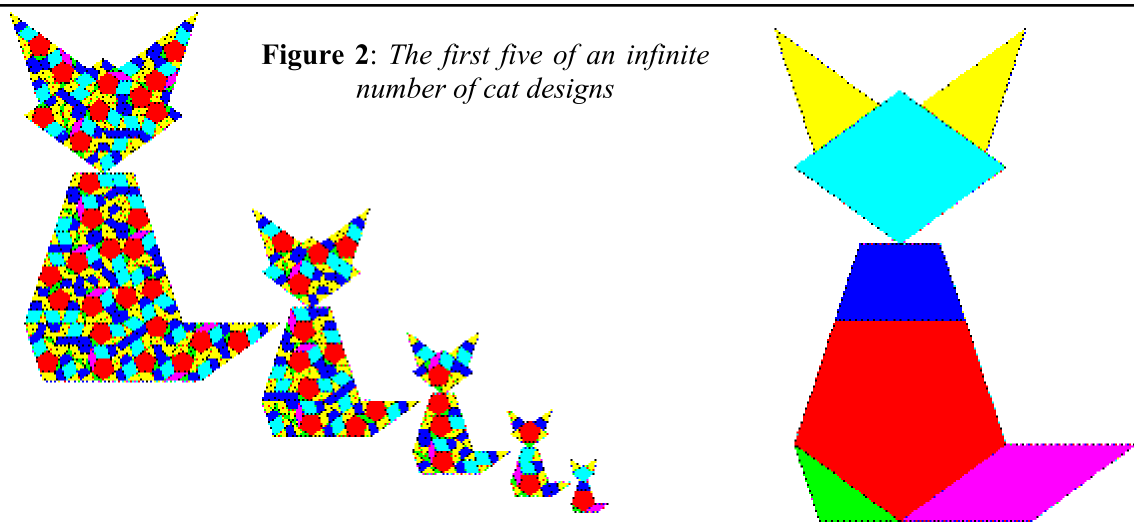


Figure 2: *The first five of an infinite number of cat designs*

4 Some interesting ratios.

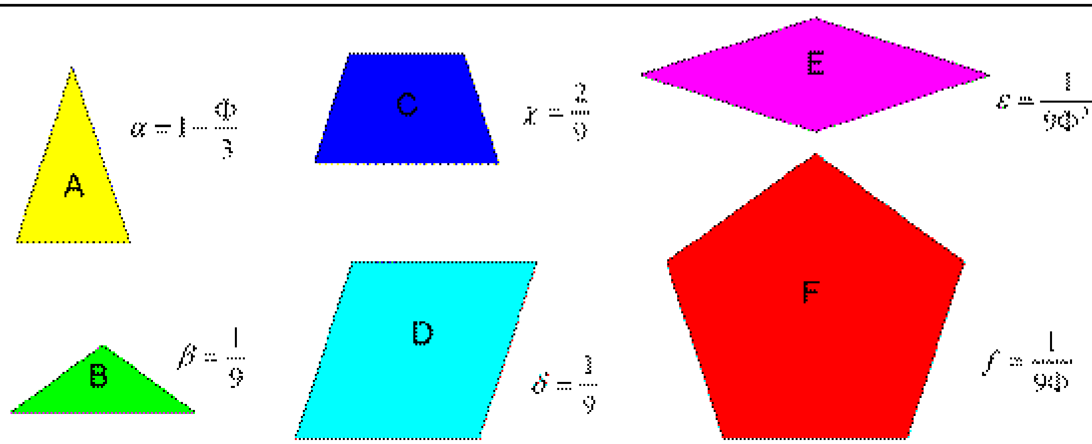


Figure 3: *The proportion of each tile in any design approaches a constant value which is independent of the original design. For example the proportion of the B triangle approaches* $\beta = \frac{1}{9}$ *Even if there are no B triangles in the original design*

The proportion of each tile in the final design seems to be independent of the number of tiles in the original design. This has been the case with other sets of precious shapes. Whether it is a design using all the tiles (figure 2) or a single tile, after 10 or so generations, the proportion of each tile approaches the constants shown in figure 3. Using a simulation exercise these conjectures are true to 5 or 6 decimal places after the tenth generation of design. There are proofs [1],[4] for other sets of shapes. The proof for the Golden Tangram are the subject of further research.

5 The Precious Matrix

The relationship between the original set of shapes and the enlarged set can be described with a Matrix. If you refer to figure 1 then it can be seen that each triangle A is replaced with a triangle A and a Trapezium C. B is replaced with A and B, C with A and D, D with 2 As, a B and an F and so on. Sometimes, one of the shapes will disappear after a few generations. A discussion of this related to the normal Tangram can be found at reference [1]. For a continuous scheme to infinity it is necessary for all the elements of the Nth power of P to be greater than zero. (Where N is the number of different tiles). It can be seen from figure 5 that all the elements of the 6th power of the Precious matrix are greater than zero, so the final condition for Preciousness is satisfied. It is, perhaps, worth pointing out that the Precious Matrix for a Reptile has a dimension of 1 by 1.

$$P = \begin{array}{c|cccccc} & A & B & C & D & E & F \\ \hline A & 1 & 1 & 1 & 2 & 2 & 2 \\ B & 0 & 1 & 0 & 1 & 0 & 1 \\ C & 1 & 0 & 0 & 0 & 2 & 1 \\ D & 0 & 0 & 1 & 0 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 & 1 \\ F & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$$

The Precious Matrix

$$P^6 = \begin{bmatrix} 87 & 54 & 141 & 282 & 174 & 510 \\ 21 & 13 & 34 & 68 & 42 & 123 \\ 42 & 26 & 68 & 136 & 84 & 246 \\ 21 & 13 & 34 & 68 & 42 & 123 \\ 5 & 3 & 8 & 16 & 10 & 29 \\ 13 & 8 & 21 & 42 & 26 & 76 \end{bmatrix}$$

Figure 4: *The Precious Matrix P and its Nth power. N = 6 because there are 6 different*

6 Using the Precious Matrix to calculate the total number of tiles in a design.

The number of tiles in the original design can be represented by a Matrix. For instance the Matrix N for the cat in can be seen in figure 5. In the initial design there are 2 A tiles, 1 B tile and so on... The matrix $P^N N$ gives the number of each tile in the Nth generation of the picture.

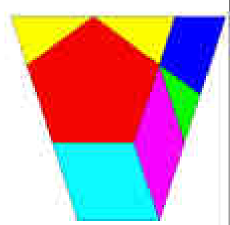
$$N = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad P^2 N = \begin{bmatrix} 28 \\ 7 \\ 14 \\ 7 \\ 2 \\ 4 \end{bmatrix} \quad P^2 = \begin{bmatrix} 2 & 2 & 3 & 5 & 4 & 10 \\ 0 & 1 & 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 3 & 2 & 5 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

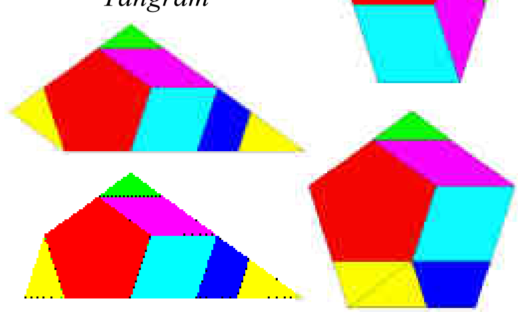
Figure 5: *The matrix showing the number of each tile in generation 0 and 2 for the cat in figure 2.*

7 Convex polygons

The seven tiles from the normal Tangram based upon a square can be formed into only thirteen convex polygons. This was proved by Fu Traing Wang and Chuan-Chi Hsuing in 1942 [6]. A similar proof for this Golden Tangram is the subject of on going research. It is possible to show that there are no triangles, and that the maximum extent of the polygons is the decagon, even this is unlikely. A number of quadrilaterals, pentagons (including the regular pentagon), hexagons and heptagons have been identified. Not all of them will be discussed here. The quadrilaterals identified so far will tessellate in a regular fashion, as is normal for any quadrilateral. What is unusual is that they also tessellate in an non periodic fashion. Figure 6 shows a few of the convex polygons identified so far.

Figure 6: *A few of the Convex polygons so far identified for the Golden Tangram*





8 Tessellating Convex polygons

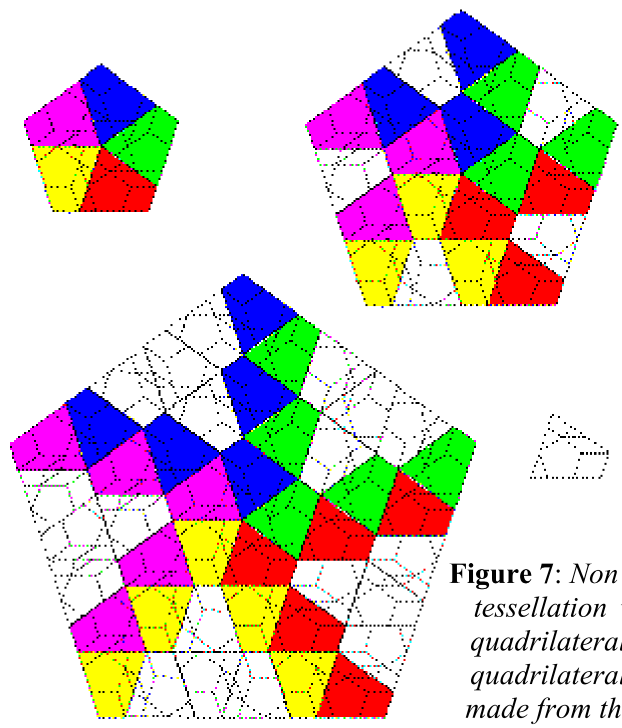


Figure 7: *Non periodic tessellation with a quadrilateral. Each quadrilateral being made from the seven Golden Tangram tiles.*

Tangrams that tessellate can often lead to a series of interesting patterns. An account of an approach using the normal Tangram can be found at [1]. It is easy to show that quadrilaterals will always tessellate in a periodic fashion. The convex quadrilaterals so far identified using the tiles from the Golden Tangram display some interesting non periodic tessellations. The one shown in figure 7 will cover the plain as a series of concentric pentagons. Five of the tiles form a pentagon . A larger pentagon can be made by copying the tiles in pairs and positioning them at the five corners of the original pentagon. The remaining gaps can be filled with the original tile. This process can be repeated ad infinitum producing a larger pentagon at each stage. Each of the quadrilaterals identified so far can cover a plane in a similar fashion. The design can be further expanded using the precious properties of the underlying tiles.

9 An interesting Precious Couplet.

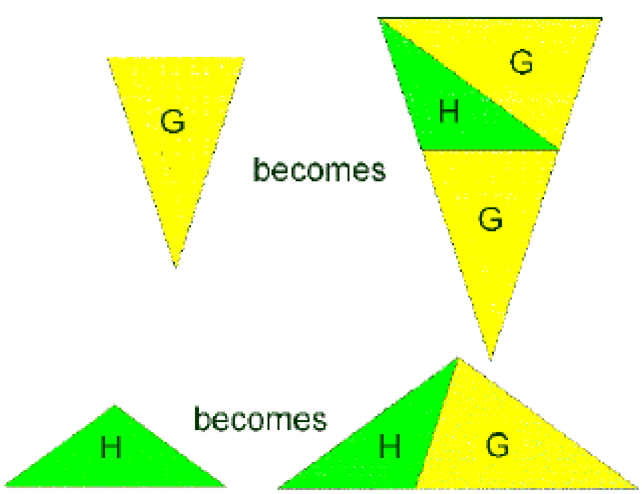


Figure 9: *A Precious couplet using the two triangles originally called A and B*

$$P=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} P^2=\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

Figure 8: *The Precious Matrix and its square for triangles G and H*

The two triangles A and B form a Precious set in their own right. They are shapes that are not similar, they can be used to create larger but similar shapes, each with the same enlargement and, because there are only two tiles, the second power of P should contain all non zero elements. As before the enlargement factor is φ, the golden ratio. They are referred to in this paper as G and H because their relationship is different to that between triangles A and B. These triangles are the only triangles whose angles are multiples of 36 degrees. They are the basis of a range of interesting designs and are worthy of a paper in their own right. Section 11 shows an application of these designs in the area of patchwork quilting.

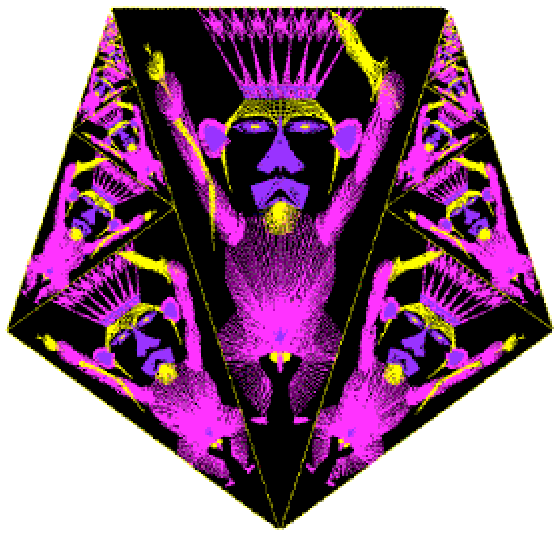


Figure 10: *Bes, my favourite Egyptian God*

Figures 10 and 11 show further examples of designs using the precious couplet

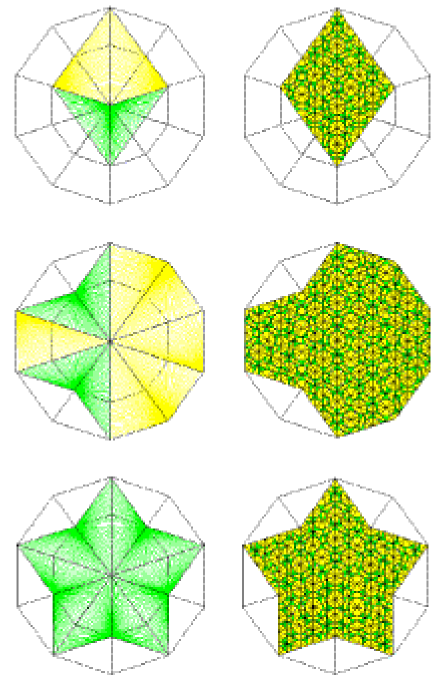


Figure 11: *Examples of generations 0 and 4 of some tiling designs around a point using the two triangles G and H.*

10 The creation of Mosaics from the Golden Tangram

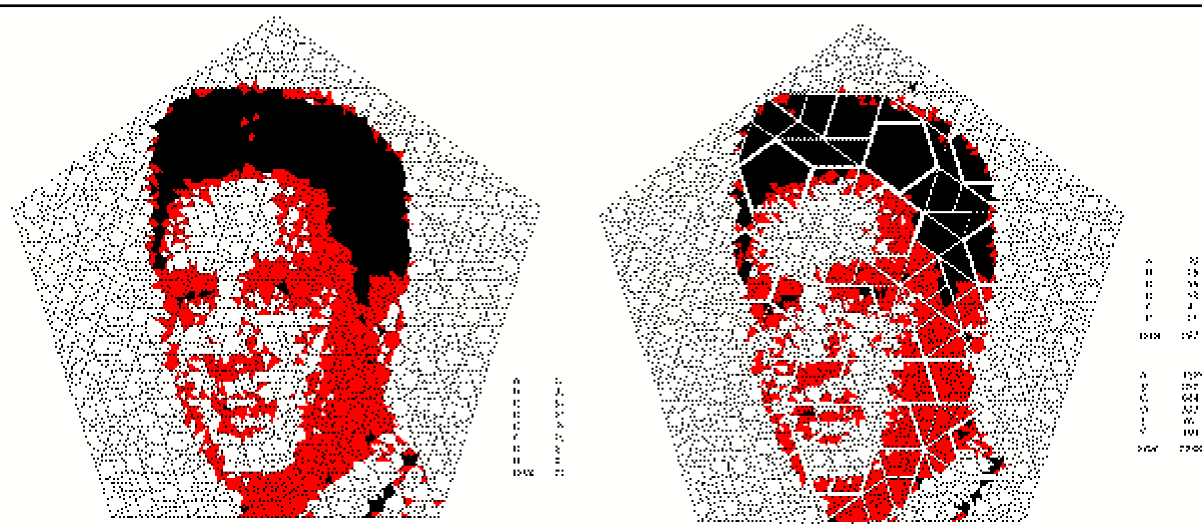


Figure 12: *The details of a ceramic Mosaic using different generations of tile.*

Conventional mosaics use small tesserae to form a picture. The picture at the left of figure 12 is made from several thousand tesserae, each being one of the six Precious Shapes. The picture on the right shows how the picture can be created from 162 higher order tiles. Each one being one of the same original 6 shapes, but larger. To maintain the detail the large tiles would need to be embossed with the original smaller tiles. The outlines of the 6 moulds can be seen on the right hand picture. The software, written by the Author, used to create the pictures in this paper also produces a disc that controls a CNC milling machine that makes the moulds from which the high order ceramic tiles can be accurately produced,

11 Application of the Couplet to the creation of Patchwork.

Conventional quilts often use blocks which form squares, rectangles or any shape that tessellates periodically. The designs being developed here use non periodic tessellation. Many of the systems of Precious polygons are quiltable. That is, the pieces can be sewn together without sewing around corners. For instance, the pentagonal design here is made from the two triangles introduced in section 9. The first stage is to sew the triangles as shown in figure 13. Sufficient will be needed for stage 1 in figure 15. The resulting triangles can then be sewn to form stage 2. This process can be repeated through stages 3, 4 and 5 to produce the final pentagon.

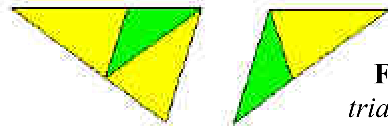


Figure 13: *Sewing the triangles to produce Stage 1*

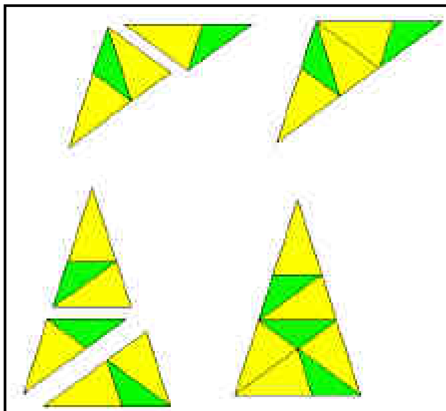


Figure 14: *Sewing scheme to produce Stage 2*

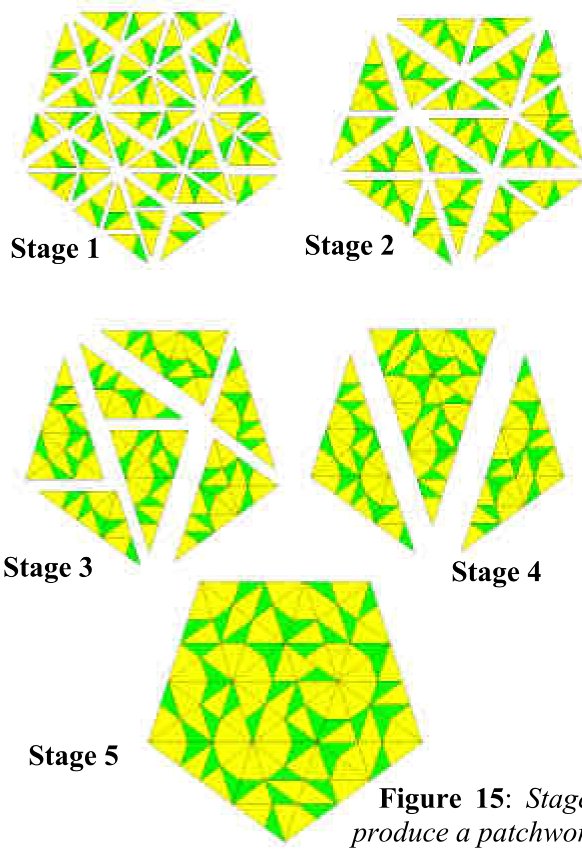


Figure 15: *Stages to produce a patchwork from precious triangles*

[1] Spencer, Stanley J, *The Tangram Route to Infinity* ISBN 141202917-1
[2] *Mathematical Connections in Art, Music and Science*. Bridges Conference 2004 ISBN 0-9665201-5-7.
[3] *Tangrams,* (Accessed 6.12.2003)<http://tangrams.ca>
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[6] Fu Traing Wang and Chuan-Chih Hsiung *A Theorem on the Tangram*. American Mathematical Monthly, vol. 49, 1942
[7] Steven Dutch *Rep-Tiles* (Accessed 1.4.2004) http://www.uwgb.edu/dutchs/symmetry/reptile1.htm
[8] Solomon W Golomb, *Polynomials Puzzles, Patterns, Problems, and Packings,* pg. 8, Appendix C, Pg 148, Princeton University Press, 2nd edition, 1994.
[9] Jay Bonner, *Three Traditions of Self Similarity in 14th and 15th Century Islamic Geometric Ornament,* Meeting Alhambra, ISAMA Bridges Conference Proceedings 2003, ISBN 84-930669-1-5.