On Parsimonious Sequences as Scales in Western Music

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Abstract

Musicians have narrowed the continuous range of frequency into discrete sequences of frequencies, interpreted as pitches, of various types called scales from the earliest writings on music onwards. This study provides some answers as to scope and relationships between modes and scales in areas where, surprisingly, little systematic study has been done. The approach reveals that the Fibonacci sequence provides the key to unlocking the question of scope. Techniques from post-tonal and modal music theory provide answers to relational questions.

1. Preliminaries

Musicians have narrowed the continuous range of frequency into discrete sequences of frequencies, interpreted as pitches, of various types called scales from the earliest writings about music onwards.[1] We pass over other topics deeply intertwined with scales such as tuning and temperament as well as referential pitch level in order to focus on an enumeration technique for stepwise sequences. These partition the space between terminal pitches formed by boundary intervals of less than or equal to the octave. Each of these sequences may be thought of as a scale with a distinct structure selecting the pitches for an instance of that scale type.

Thinking of a scale as a sequence of intervals rather than of pitches has the natural advantage of abstraction. These sequences form equivalence-classes induced by transposition (translation in mathematical terms); the pitches change while the intervallic sequence remains invariant. Usually, the first and last notes of a scale mark a 2:1 frequency ratio between them called the octave, and this acts as a modulus that maps the octave related frequencies into the type representative within the scale. Parsimonious for this study means that the interval between any two adjacent notes of the scale must be either a semitone or a whole tone (two semitones), and given historical musical practice, this is a reasonable limit for adjacent-note intervals, parsimonious intervals.[2] For example, the C major scale or

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[1] The earliest theoretical writings on music are Asian and predate Western theories by about a thousand years. For a recent study of such thought, see Clough, Douthett, Ramanathan, and Rowell [1].

[2] While there are from a parsimonious point of view, "gapped" scales such as the harmonic minor, these play a lesser role in actual musical practice.

Richard Cohn [2a] first emphasized the idea of parsimony with respect to voice-leading as a direct outgrowth from work by Hugo Riemann. We have The further developed this idea in Hermann and

Ionian mode is represented as follows where the letter names of the pitches are listed and aligned beneath are distances given in semitones that constitute the parsimonious sequence:[3]

C D E F G A B C
<2 2 1 2 2 2 1>

As has been long well known, using this parsimonious sequence starting on any other pitch generates another member of the major scale equivalence-class induced by transposition. Here the two Cs form the boundary pitches and the boundary interval between them is the octave. Naturally, the intervals of these parsimonious sequences sum to the boundary interval, 12, and that intervallic distance is another definition for the octave.

Other well known parsimonious sequences that have the boundary interval of the octave include the octatonic, <12121212>, and whole tone scales, <222222>; however, there are many other less well known as well as the ubiquitous modern church modes that we will specifically revisit later. As we can see from the parsimonious sequences above, there is more than one cardinality of parsimonious intervals that generates a scale between the boundary intervals of an octave: the major scale has 7, the octatonic 8, and the whole tone has 6 parsimonious intervals partitioning the octave. We q be the number of intervals in the parsimonious sequence.

Musicians also find it useful to study scales with other boundary intervals. For instance in the Medieval and Renaissance eras, tetrachords, pentachords, and hexachords (four, five, and six note sequences) figured prominently in music theory; these have boundary intervals of 5, 7, and 9 semitones respectively.[4] Thus, we study parsimonious sequences for all sizes of boundary intervals between 1 and 12. The enumeration technique presented here is general for parsimonious sequences of any sized boundary interval.[5] While there are historical and current uses for parsimonious sequences where n is less than 12, the octave modulus remains in effect for all sequences in this study.

2. An Enumeration Technique for Parsimonious Sequences

An insight that some of the resulting intervallic distances in a scale sum to Fibonacci numbers led us to investigating its relevance. Taking the C major scale above, we have <2212221> where the parsimonious interval values of 1 and 2 themselves, the sum of the second two digits, and the sum of the first three digits are Fibonacci numbers.[6]

Douthett [2b], Douthett and Hermann [2c], and here.

[3] Renaissance church modes differ from the modern church modes as used by composers such as Debussy and Ravel as well as by jazz musicians to this date. Because the tuning systems of the Renaissance did not result in closed systems, very few transpositions were available, and several of the intervals had two different sizes. Today's equal temperament system is closed allowing intervallic patterns to be transposed to start from any pitch in the system and each interval is of only one size. It is ironic that the Renaissance church modes are named for locations in the ancient Hellenic world predating Christianity and that the Italian scholars of the middle ages mistranslated the ancient texts and associated the wrong place names with these scalar patterns. See David E. Cohn [3a] for more on this historical mistake and Cristle Collins Judd [3b] for information on the modes during the Renaissance. The Locrian mode is of modern invention predating World War II. It is used today in jazz pedagogy and is helpful for our purposes as it provides a name for that rotation-class member.

[4] These music theoretical works were frequently pedagogical. For a brief study on music theory pedagogy from Antiquity to the present that touches on these intervals and their role in context, see Wason [4]. intervals 5, 7, and 9 are know as the perfect fourth, perfect fifth, and major sixth in music of those eras.

[5] For more on these mappings and the operation of transposition (translation), see Morris [5].

[6] See Kramer [6a] for more on the Fibonacci series in 20th-century music and Huntley [6b] for other

In tackling this problem, there are two questions to be answered initially.

Question 1: Given a boundary interval of length *n* and parsimonious sequences with *m* whole-steps, how many distinct sequences are there?

Question 2: Given a boundary interval of length *n*, what is the total number of distinct parsimonious sequences?

To address the first question, we need the formula that determines the number of ways to choose *m* objects from a set of *n* distinct objects (*n* choose *m*):

$$\mathrm{C}(n,m)=\frac{n!}{m!(n-m)!}.$$

The numbers generated by this formula are also called *binomial coefficients*. This relates to the question in the following way: Suppose our boundary length is $n = 5$. If there are 5 intervals in the sequence, then all 5 are half-steps and there are 0 whole-steps. One can think of this sequence as having 5 distinct positions and that 0 of them will be whole-steps. Whence, there are 5 choose 0 sequences with 5 half-steps and no whole-steps:

$$\mathrm{C}(5,0)=\frac{5!}{0!5!}=1.$$

On the other hand, with 4 intervals and a boundary length of 5, 1 of the intervals must be a whole-step. Whence, of the 4 distinct positions, 1 must be chosen to be whole-step. It follows that there are 4 choose 1 sequences with 3 half-steps and 1 whole-step:

$$\mathrm{C}(4,1)=\frac{4!}{1!4!}=4.$$

The sequences are

<2111>, <1211>, <1121>, and <1112>.

Finally, with 3 intervals and a boundary interval of length 5, there are 2 whole-steps. So, there are 3 choose 2 ways of placing the whole-steps in the sequence:

$$\mathrm{C}(3,2)=\frac{3!}{2!1!}=3.$$

These sequences are

<221>, <212>, and <122>.

Any fewer than 3 intervals in our sequence with boundary length 5 and our sequence could not be parsimonious. So, we stop. For a boundary interval of length *n* and *m* whole-steps in the sequence, there are C(*n*–*m*,*m*), $0 \le m \le \lfloor n/2 \rfloor$, distinct parsimonious sequences (if $m > \lfloor n/2 \rfloor$, then the sequence cannot be parsimonious).

On the second question, we need the Fibonacci numbers: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, and so forth ($F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$). Note that the total number of parsimonious sequences with boundary length 5 in the example above is a Fibonacci number:

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applications of the series.

$$\text{Total} = C(5,0) + (4,1) + (3,2) = 8 = F_6.$$

This is not a coincidence! In general, to get the total number of parsimonious sequences for a boundary interval of length *n*, all the cases above must be added together:

$$\text{Total} = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n - \lfloor n/2 \rfloor, \lfloor n/2 \rfloor).$$

It is known in mathematics that this sum is the Fibonacci number F_{n+1}; that is,

$$F_{n+1} = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n - \lfloor n/2 \rfloor, \lfloor n/2 \rfloor).$$

For *n* = 1 through 12, Table 1 below shows the number of parsimonious sequences in which *m* whole-steps appear (Column 3) and the total number of parsimonious sequences (Column 2). For each boundary interval length *n* in Column 1, the sum of the values in Column 3 yields the corresponding Fibonacci number shown in Column 2.

n	F_{n+1}	Values of C(*n*–*m*,*m*) where *n* is the interval boundary length						
1	1	C(1,0) = 1						
2	2	C(2,0) = 1	C(1,1) = 1					
3	3	C(3,0) = 1	C(2,1) = 2					
4	5	C(4,0) = 1	C(3,1) = 3	C(2,2) = 1				
5	8	C(5,0) = 1	C(4,1) = 4	C(3,2) = 3				
6	13	C(6,0) = 1	C(5,1) = 5	C(4,2) = 6	C(3,3) = 1			
7	21	C(7,0) = 1	C(6,1) = 6	C(5,2) = 10	C(4,3) = 4			
8	34	C(8,0) = 1	C(7,1) = 7	C(6,2) = 15	C(5,3) = 10	C(4,4) = 1		
9	55	C(9,0) = 1	C(8,1) = 8	C(7,2) = 21	C(6,3) = 20	C(5,4) = 5		
10	89	C(10,0) =1	C(9,1) = 9	C(8,2) = 28	C(7,3) = 35	C(6,4) = 15	C(5,5) = 1	
11	144	C(11,0) = 1	C(10,1) = 10	C(9,2) = 36	C(8,3) = 56	C(7,4) = 35	C(6,5) = 6	
12	233	C(12,0) = 1	C(11,1) = 11	C(10,2) = 45	C(9,3) = 84	C(8,4) = 70	C(7,5) = 21	C(6,6) = 1

Table 1: *The Number of Parsimonious Sequences for n = 1 to 12.*

These sums to Fibonacci numbers can also be seen in the Pascal triangle in Table 2. Each arrow goes through the binomial coefficients that sum to the Fibonacci number at the head of the arrow. The coefficients that sum to the Fibonacci number F_{n+1} are the number of parsimonious sequences with *m* whole-steps and boundary length *n*. For example, the arrow that points to F_6 goes through the numbers 1, 4, and 3, which are the number of parsimonious sequences with boundary length 5 that contain 0, 1, and 2 whole-steps, respectively.

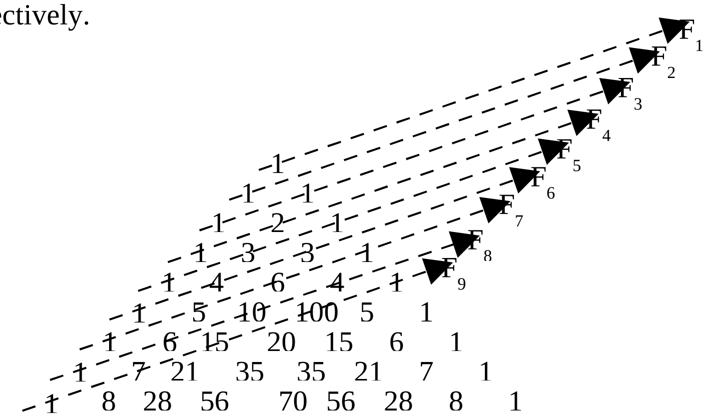

Table 2: *The Pascal Triangle with Diagonal Sums to Fibonacci Numbers.*

While it is easy to generate the specific parsimonious sequences when the value of n is low, it becomes progressively less so as the F_{n+1} values make their rapid climb. So a third question emerges.

> <u>Question 3:</u> What are efficient methods for generating all specific parsimonious sequences with a boundary interval of n?

Before we answer this question directly, it is convenient to return to the modern church modes. Recall the major scale—Ionian mode presented above. Clearly, rotating the notes so the last note becomes the first preserves its parsimonious structure. Since there are 7 distinct notes, there are 7 different rotations, and it is these rotations that define the modern church modes. Table 3 lists these rotations, and each mode's name is given to the write of the sequence. These modes form a *rotation-class* of parsimonious sequences with a boundary interval of length n = 12. The rotations of the octatonic scale, discussed above form another rotation class of parsimonious sequences with a boundary length of 12. In this case, there are only 2 sequences in this class; <12121212> and <21212121>. The whole-tone scale rotation-class has only 1 member: <222222>. Whence, every parsimonious sequence with boundary length 12 belongs to some rotation-class.[7] This observation will be useful in answering the question posed above.

C	D	E	F	G	A	B	C	Ionian
<2	2	1	2	2	2	1>		
B	C	D	E	F	G	A	B	Locrian
<1	2	2	1	2	2	2>		
A	B	C	D	E	F	G	A	Aeolian
<2	1	2	2	1	2	2>		
G	A	B	C	D	E	F	G	Mixolydian
<2	2	1	2	2	1	2>		
F	G	A	B	C	D	E	F	Lydian
<2	2	2	1	2	2	1>		
E	F	G	A	B	C	D	E	Phrygian
<1	2	2	2	1	2	2>		
D	E	F	G	A	B	C	D	Dorian
<2	1	2	2	2	1	2>		

Table 3: *The Rotation-Class of the Modern Church Modes.*

For our purposes here, we may informally describe a method for obtaining all specific parsimonious sequences. We start with the sequence of all parsimonious interval 1s summing to n and then successively adding a parsimonious interval 2 (subtracting the requisite number of 1s), which generates all of the members of each rotation-class that sum to n. As we know how many parsimonious sequences are in a rotation-class of length n, we can divide each $C(n,m)$ within some specific F_{n+1} situation by the appropriate q to remind us of how many rotation-classes are needed. The process ends when the rotation-class of all parsimonious interval 2s in the case of even values for n or all 2s plus one 1 in the case of odd values for n is recorded. Table 4 uses this method to generate all parsimonious sequences for n = 9, F_{10} = 55. The rotation-classes are separated by rows of asterisks.

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[7]See Rahn [7] for more on this operation in post-tonal (atonal) music theory, but it is not limited to parsimonious sequences.

<111111111>

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<11111112>, <21111111>, <12111111>, <11211111>, <11121111>, <11112111>, <11111211>,
<11111121>

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<1111122>, **<2111112>**, <2211111>, <1221111>, <1122111>, <1112211>, <1111221>

* * * * * * * * * * *

<1111212>, <2111121>, <1211112>, <2121111>, <1212111>, **<1121211>**, <1112121>

* * * * * * * * * * *

<1112112>, <2111211>, **<1211121>**, <1121112>, <2112111>, <1211211>, <1121121>

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<111222>, <211122>, <221112>, <222111>, <122211>, <112221>

* * * * * * * * * * *

<112122>, <211212>, <221121>, <122112>, <212211>, <121221> &

* * * * * * * * * * *

<121212>, <212121>

* * * * * * * * * * *

<121122>, <212112>, <221211>, <122121>, <112212>, <211221> &

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<12222>, <21222>, **<22122>**, <22212>, <22221>

Table 4: *The Parsimonious Sequences of* $n = 9$, F_{10}.

3. Observations on Parsimonious Sequences

Just as today's musicians may think of the modern church modes as a set of scales preserving intervals under rotation that is based on the traditional diatonic collection, any of the rotation-classes of parsimonious sequences may be thought of as a unique collection of "parsimonious modes." In the diatonic collection, the choice of the mode Ionian—major scale as the type representative of this rotation-class is arbitrary from this point of view as it would be for any such class.[8]

Many composers from Debussy onwards have been intensely interested in the effects of mirror symmetry (reflection) in musical materials including scales as demonstrated by actual passages of music.[9] Returning to Table 3, we may see reflection within or between the modern church modes. Note that the Dorian mode's sequence of parsimonious intervals are the same whether read from left to right or the reverse. This mode's intervallic structure and any other with this feature are invariant under reflection. This operation is called inversion by musicians. Modes that are *not* their own reflection, not invariant, are paired with another that holds the same sequence of intervals when read in the opposite direction. These inversionally equivalent pairs in the modern church modes are Ionian/Phyrgian, Locrian/Lydian, and Aeolian/Mixolydian as can be confirmed in Table 3. When the value of *n* is less than or equal to 8, the inversionally equivalent pairs are found within the same rotation-class. For values of *n* greater than 8, this is usually but not always the case. Returning to Table 4, note that two of the rotation-classes have ampersands, "&," after the last member of the class. In these cases, one member of each inversionally equivalent pair is found within one ampersand marked rotation-class and the other of the pair is found in

[8] For a recent study on transformations between modes in a similar sense of the same value for n, see Santa [8].

[9] For well known examples, see Bartók [9a], Debussy [9b], and Webern [9c].

the other ampersand marked rotation-class. As *n* increases, so does the number of inversionally paired rotation-classes. As a last point made here on Table 4, those parsimonious sequences that are inversionally invariant (self reflections) are presented in boldface type.

4. Concluding Thoughts

With this enumeration technique, musicians can quickly generate all parsimonious sequences ("scales") and recognize their rotational ("modes") and inversional equivalences for any value of *n* of interest. Jazz musicians in particular may find this of interest as their pedagogy is today centered on the application of a wide variety of scales and modes.[10] While some post-tonal composers and theorists have studied pitch-class sets and their various transformational groups quite extensively, they have not given the same kind of attention to modes and scales much less to parsimonious sequences.[11] Given this intense interest in pitch structures and scales/modes during the last century or so, it is odd that we know of no previous study that provides a map of the terrain and a means for traversing it.[12] For our musician friends, here is that map and vehicle; enjoy the trip!

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[10] Slonimsky [10a], initially published in 1947, provided the first influential source on scales for jazz musicians. Later Russell [10b], circulated in manuscript in 1955, became and remains in its following editions the most respected source on scales in jazz theory. Reeves [10c] is a typical pedagogical text used in American universities for jazz studies.

[11] See Morris [4] and Morris [11] for important summaries of this work.

[12] Nonetheless, much valuable work has been done on the topic. For some recent work that may be of interest to mathematicians, see Carey and Clampitt [12a], Clough, Engebretsen, and Kochavi [12b] and Vieru [12c].

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