

Illustrating Number Sequences

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Abstract

While critically important mathematically, number sequences can also be foundational artistically. This work presents several techniques for illustrating number sequences using geometric shapes and image processing technology. The resulting images can reveal characteristics of the sequences that are of interest to both the mathematician and the visual artist.

Introduction

Number sequences are a foundational part of mathematics, from a child's first look at arithmetic, to a number theorist's research. One of the best compendia of number sequences is the Online Encyclopedia of Integer Sequences, compiled by Neil Sloane [1]. A few of the sequences filed there and known to many others have been used in the present work to demonstrate techniques by which images can be created that are both visually pleasing and mathematically substantive.

The images herein rely extensively on image processing technology for combining several independent images (each representing a term of a sequence) into a final image. The goal was to find images that in some respect were "stationary," that is, either they include all terms of a finite sequence, or the image did not change significantly with the incorporation of additional elements. Although not necessarily fractal in nature, all the images were created using the Ultra Fractal program [2].

All Diagonals

A natural number larger than 2 can be considered to be the number of sides of a polygon for which all of the diagonals have been drawn. For example, the multiples of 5 can be considered to be polygons with 5, 10, 15, etc. sides, and their diagonals. The entirety of a sequence of natural numbers is visualized by stacking these polygons and forming the arithmetic mean of the images. This is shown in Figure 1 for the multiples of 5. The last panel shows an approximation of what I claim is in some sense the unique limiting value of that sequence.

This technique affords a great deal of flexibility to the artist, particularly with regard to the locations of the vertices of the polygons. In Figure 1, each polygon is regular and its vertices lie on a circle. Other bounding shapes can be used, such as rectangles, regular polygons, and spirals. In addition, the orientations of the elemental polygons can be varied. In Figure 1, each polygon has a vertex at the top of the circle. Instead, the polygons can be rotated, so what appear as straight dark lines in the composite image become curved. Figure 2 shows two completed works by the author, along with schematics of two of the generating polygons. In each case, the composite images were created by overlaying polygons whose numbers of sides were multiples of 3, from 6 to 63. The vertices of the polygons were spaced at equal angular increments along an equilateral triangle. In the image "Master" (second panel), the polygons were aligned such that they all had vertices at the corners of the bounding triangle. An example dodecagon is shown in the first panel. For the image "Student" (fourth panel), the polygons were rotated

such that their vertices straddled the corners of the bounding triangle. These effects caused the triangle vertices to be highlighted in “Master” and in the shadows in “Student.”

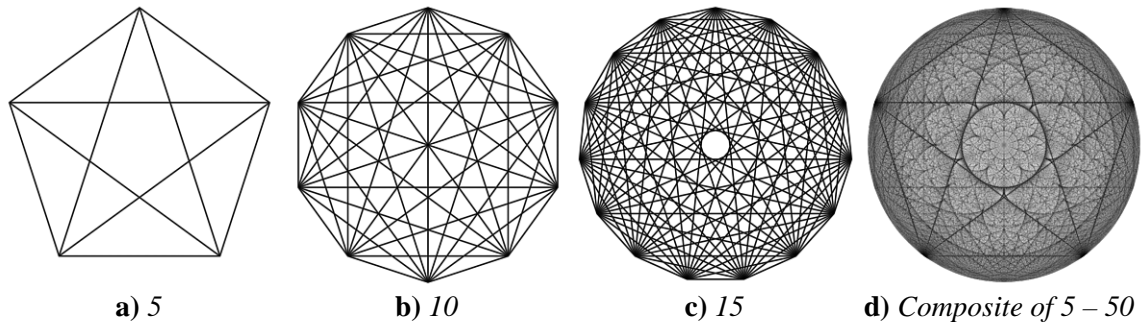


Figure 1: Polygons with multiples of 5 sides, with all diagonals

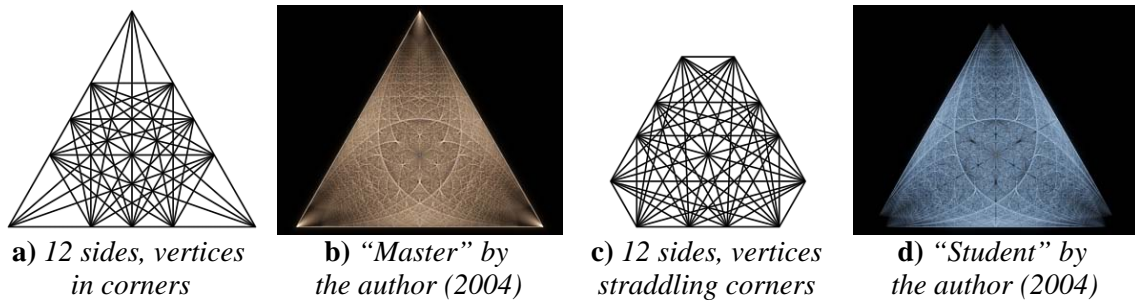


Figure 2: Polygons with multiples of 3 sides and vertices along the sides of an equilateral triangle

Circle Cutting

The “pancake” numbers are those numbers that represent the maximum number of sections into which a circle can be cut with a given number of lines. Four of the smaller pancake numbers are shown in Figure 3. In the first and second panels, the circle and the cutting lines are shown. In the third and fourth panels, the regions are alternately dark and light and the cutting lines are the boundaries.

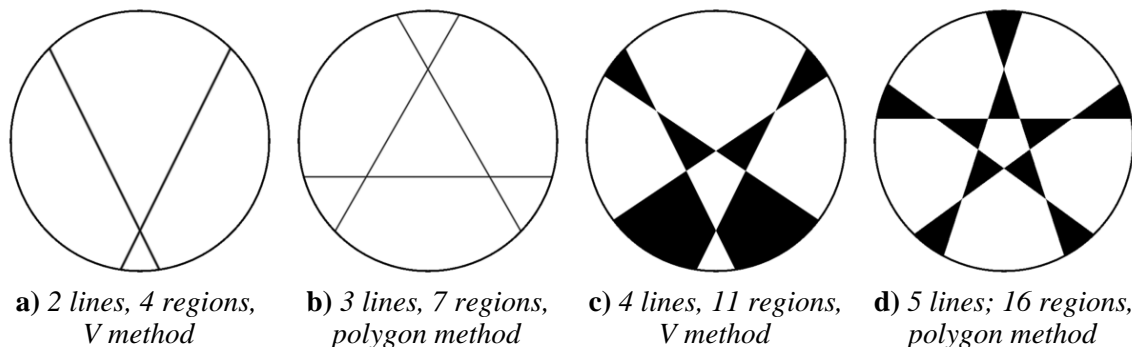


Figure 3: Circles cut into the maximum number of regions by 2 – 5 lines

The four panels illustrate two different techniques for drawing such figures. For an odd number of lines, a regular polygon with that number of sides can be drawn concurrent with the circle and its sides extended until they have all intersected inside the circle. This was done with the 3-line and 5-line cases above. For any number of lines, a V can be used as the first two (shown in Figure 2a). Each subsequent

line is then drawn from a point on one leg of the V to a point on the other leg, in a manner reminiscent of string art [3]. Figure 3c shows the V with two additional lines, which cross at the center of the circle.

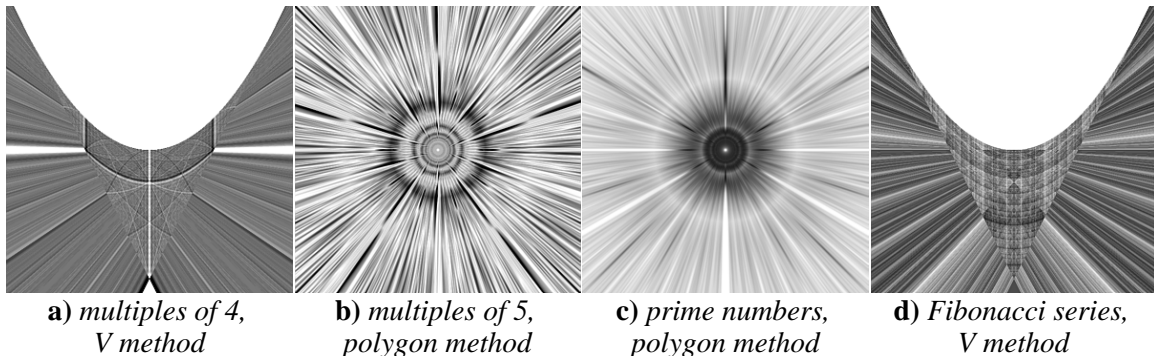


Figure 4: Sequences illustrated with circle cutting

These methods can be extended to sequences of numbers by combining the images for individual terms, as before. In Figure 4, four sequence images are shown (without the circles). The first is the multiples of 4, rendered using the V method and showing the regions as dark or light, as in Figure 3c. Note the white areas of the image, which indicate that the images for each term share some of the same white areas. In Figure 4b, the multiples of 5 are illustrated in the manner of Figure 3b: using a central regular polygon and only drawing the lines. The fivefold symmetry reflects the use of polygons with numbers of sides that are multiples of 5. In contrast, the next panel shows the same method used with the prime numbers. While there is still some structure apparent, the only symmetry is due to the fact that all the polygons had vertical lines of symmetry. The last panel of Figure 4 shows the Fibonacci series, using the same method as in the first panel. Again, the only symmetry is along the vertical axis, but there is extensive structure in the image, particularly inside the V.

Polylines

A polyline is a curve consisting of line segments, and can be thought of as “connecting the dots” with a sequence of discrete points. This technique is ideally suited for sequences that have two components. One candidate is the signature sequence for an irrational number [4]. For a particular number x , form the terms $y = i + jx$, where i and j are nonnegative integers. If x is irrational, then every y will be different and the set of y values can be ordered from least to greatest. Within this ordering, the set of i integers is the signature sequence for that x . For the golden ratio F (~ 1.618034), the signature sequence begins 0, 1, 0, 2, 1, 3, 0, 2 and is a fractal sequence in the manner of Kimberling [5, 6].

To draw a figure with this sequence, both the i and the j values were used, as the x - and y -coordinates of the vertices of the polyline. Mathematically, the sequence extends to infinity, but operationally, i and j were both limited to some maximum value. In Figure 5, the data for the first few terms of the sequence are shown, along with polylines for maximal i and j values of 4, 8, and 16. The zigzag line extends continuously from (0,0) to (*maximum*, *maximum*). For larger maximum values, the figures begin to show gradations of shading and suggest rectangles of varying dimensions. This is how the image, “Signature of Phi,” [7] was created by the author, with i and j both ranging from 0 to 64.

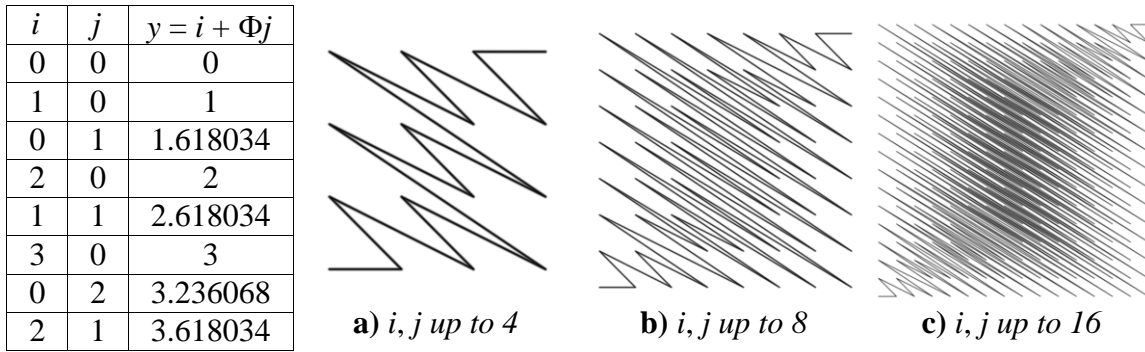


Figure 5: Signature sequence of Φ , drawn on a square grid

Alternatively, only the i or j sequence can be used. Imagine starting at the point $(1, 0)$. Subsequent points lie on the unit circle, at angle increments determined by the sequence. For example, if the maximum value of i and j were 8, then the angle increments could be $2\pi i/8$ radians, or one-eighth of the circle, times i . This idea was incorporated into the image “Eleven,” created by the author and shown in the last panel of Figure 6. The base of the signature sequence was 11; an integer was chosen so that the segments would naturally cluster around the circle. In the first three panels of Figure 6, various maximal values were used, and the voids left by the clustering are apparent. The maximal value used in “Eleven” was chosen so that there would be eleven voids (the smallest two are too small to be seen in at this scale).

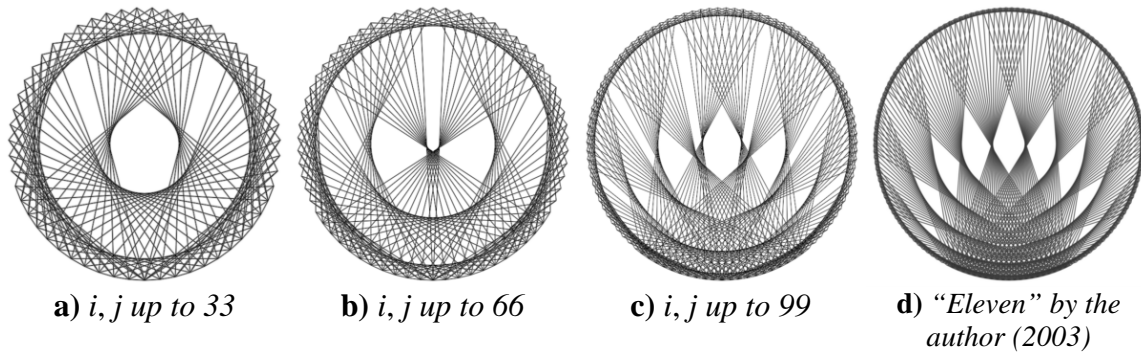


Figure 6: Signature sequence of 11, drawn on a circle

Another idea is to draw the trajectory of a billiard ball rolling on a perfect table without friction. [8] The ball begins at one corner and bounces off the rails, stopping only if it reaches a corner. For a rectangular table, this problem is ruled by fractions. Specifically, if the table has a rational ratio of length to width, and the tangent of the initial angle of the ball is a rational number, then the ball will eventually find a pocket. Several examples are shown in Figure 7, where the trajectories are shown on a table with proportions of 2:1. In each case, the ball begins in the lower left corner. Its trajectory has an angle (with respect to the bottom rail) whose tangent is a fraction. In the first case, the angle’s tangent is $1/1$ (45°) and the ball bounces off of the top rail before stopping in the bottom right pocket. In the second case, two trajectories are shown, for angles whose tangents are $1/2$ and $2/1$ (approximately 26.6° and 63.4° , respectively). The third case shows the trajectories for the five fractions $1/5$, $2/5$, $3/5$, $4/5$, and $5/5$.

In Figure 7c, a pattern is beginning to develop, with regards to the points where the trajectories intersect. This pattern is shown more clearly in Figure 8a, where the 16 trajectories with tangents $1/17$, $2/17$, etc., up to $16/17$ are shown. The image is imbued with artifacts of 17, such as the 8.5 points on either end of the table and the 17 diamond voids in the center. If the table is viewed as a number line with 0 at the left end and 1 at the right end, then at fractions m/n (in lowest terms), the number of voids is $34/n$, and the sizes of the voids decrease as n increases. In Figure 8b, an example is shown with initial trajectory angles larger than 45° . Here, 8 separate images are overlaid, with tangents equal to powers of 2

from 1 to 256. The composite process results in an image that suggests a fractal structure of elements with gray tops, bright middles, and dark bottoms.

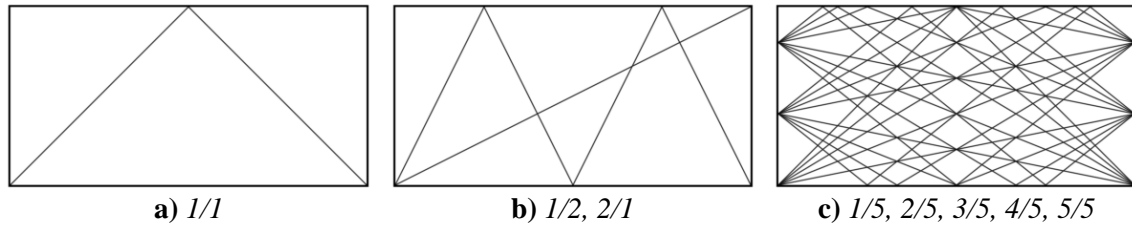
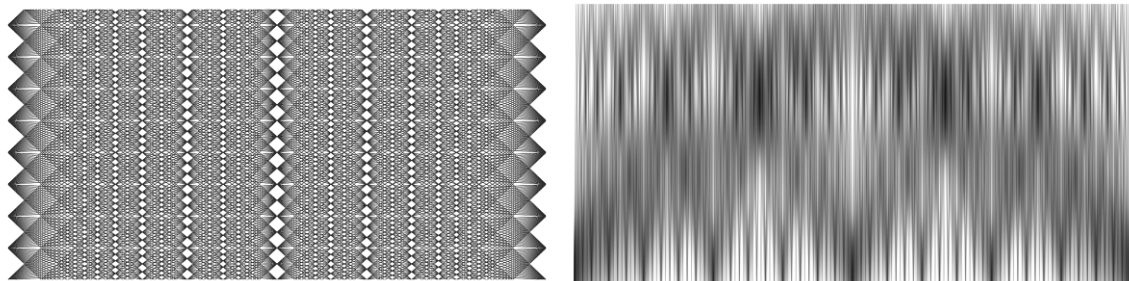


Figure 7: Billiards on a 2:1 table with initial angles whose tangents are fractions



a) tangent = $1/17, 2/17, \text{ etc., up to } 16/17$ **b)** tangent = $1, 2, 4, 8, 16, 32, 64, 128, \text{ and } 256$

Figure 8: Billiards with special initial trajectory angles

Roulettes

A roulette is a curve formed by one shape rolling along another [9]. One of the more common types of roulettes is the spirograph, wherein a circle rolls along a fixed circle. In that case, one of the relevant parameters is the ratio of the radius of the rolling circle to that of the fixed circle. If this ratio is rational, then the curve closes. Two examples are shown in Figure 9a and 9b, where the ratios of the rotating to fixed radii are $3/8$ and $8/3$, respectively. There are many ways that roulettes can be used to illustrate numbers; one way is to illustrate a series of fractions. In the last two panels of Figure 9, the fractions $1/11, 2/11, 3/11, \text{ etc., up through } 10/11$ are shown, all 10 overlaid in each image. The two versions come from the two definitions of the spirograph formula.

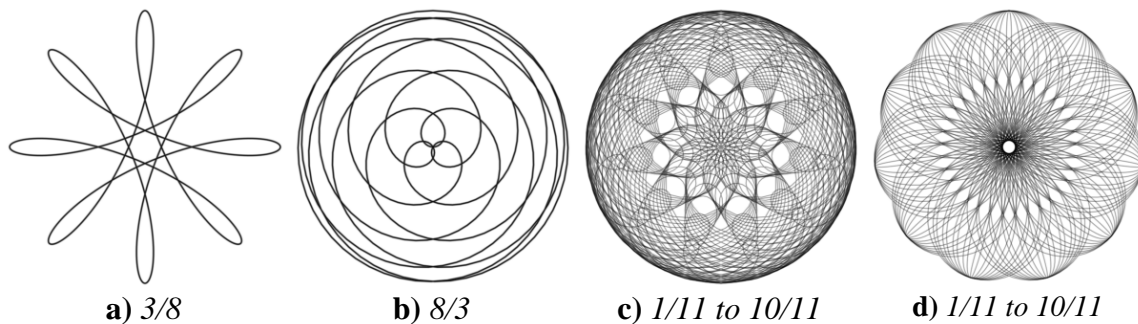


Figure 9: Spirographs for various rational ratios of rotating to fixed radii

Conclusions

Number sequences are an alive and integral part of mathematics, and can be the foundation for interesting images. By exploiting simple and complex geometric shapes and the power of image processing technology, images can be created that appeal both to the mind of the mathematician and to the eye of the

artist. Many, many techniques are possible beyond those few presented here; this will hopefully serve as an inspiration to others to expand on this work.

References

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