Circular Distributions and Spectra Variations in Music

How Even Is Even?

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Dedication

This paper is dedicated to the memory of John Clough (1928-2004). Without his participation and encouragement much of this work and the work referenced herein might never have come to fruition. The field of mathematical music theory owes a great debt to John Clough. The authors are privileged to have known and worked with him.

1 Introduction

In this section we review some results on circular distributions and the spectra associated with these fascinating distributions.

1.1 Circular Distributions: Consider the distribution of 8 dots, represented by small circles, spread out equally on a large circle as shown in Figure 1. Previously; Krantz, Douthett, and Clough [1] considered the case of distributing 3 of the dots, represented by small filled circles, as evenly as possible around the large circle. They showed that the most even distribution, the "maximally even distribution," of 3-out-of-8 occurred when the average chord length between either the open or filled circles was a maximum. The maximally even distribution is shown in Figure 2.

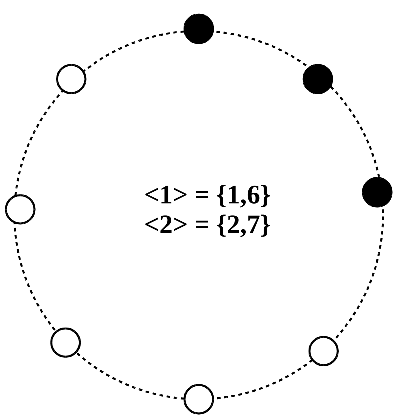


Fig. 1

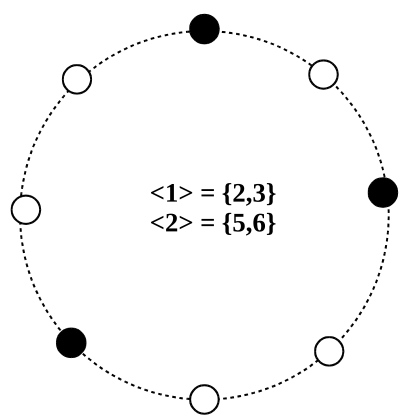


Fig. 2

1.2 Spectra: Superimposed on Figures 1 and 2 are the so-called spectra [1,2,3] associated with each distribution. The distances between dots, filled or open, is called a *specific length*. The distance between filled dots (or open dots), only, is called a *generic length*. For example, <1> represents the set of *specific lengths* between near-neighbor, specific length <1>, filled points. As seen in Figure 1 there are two such distances {1,6}. <2> represents the set of specific distances between next-nearest neighbor filled dots. As shown in Figure 1, there are two such specific lengths {2,7}. For the maximally even distribution, shown in Figure 2, the corresponding spectra are <1> = {2,3} and <2> = {5,6}.

1.3 Another Example: Consider the circular distributions shown in Figures 3 and 4. It is clear from these figures that the maximally even distribution is given in Figure 4.

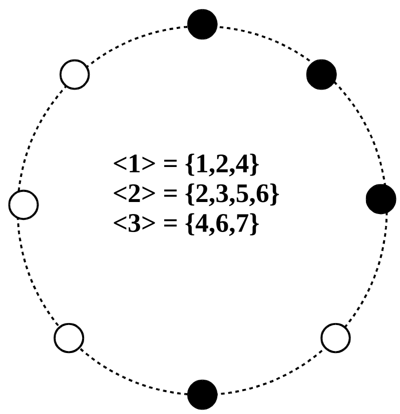


Fig. 3

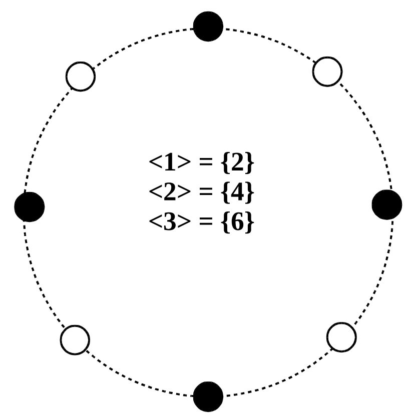


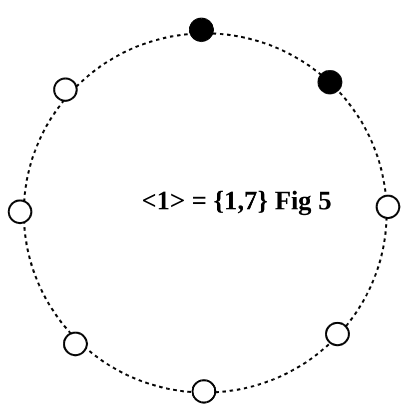
Fig. 4

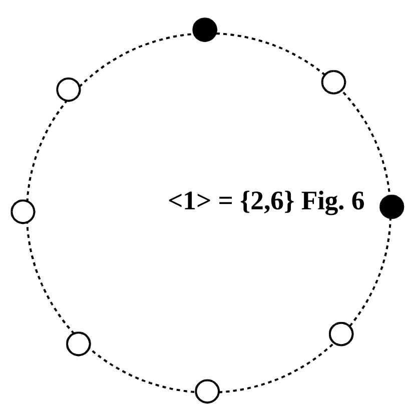
Notice that the spectra for the maximally even distribution all contain one specific length for each generic length. This is an example of a very general result. It has been shown [4] that a maximally even set is a set in which every spectrum consists of one, as in Figure 4, or two consecutive integers, as in Figure 2.

2 Spectra Variations

A question arises in light of the examples shown in section 1: "How can one compare the evenness of circular distributions that contain different numbers of filled circles?" That is, can we compare the evenness of the distributions shown in Figures 1-4 with each other or with the other, not shown, 3-out-of-8 circular distributions and all possible 4-out-of-8 distributions? More generally, can we compare the evenness of all n-out-of-8 distributions?

2.1 Simpler Example: Shown in Figures 5 through 8 are the 4 distinct ways of choosing 2-out-of-8 dots around the circle. Superimposed on each figure is the spectrum for each.





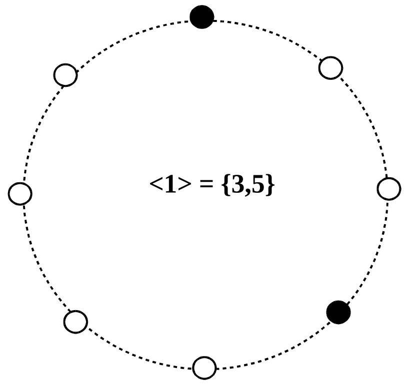


Fig. 7

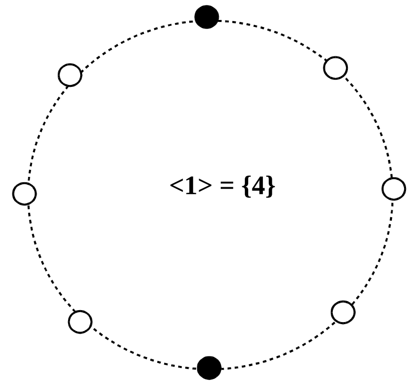


Fig. 8

2.2 Spectra Width: We define the width of the spectrum of a **d**-out-of-**c** circular distribution, written as $\Delta^d_c(I)$, as the difference between the largest and smallest member of the spectrum I; that is:

$$\Delta^d_c(I) = \max \langle I \rangle - \min \langle I \rangle \qquad (1)$$

The spectra widths for Figures 5-8 are: 6, 4, 2, and 0 respectively.

If we consider the spectrum widths of our 3-out-of-8 example, the possibilities up to rotation and inversion are shown in Figures 9 through 13:

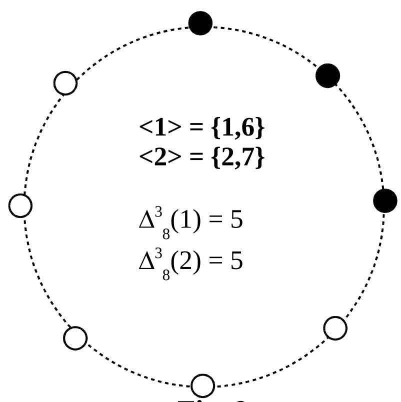


Fig. 9

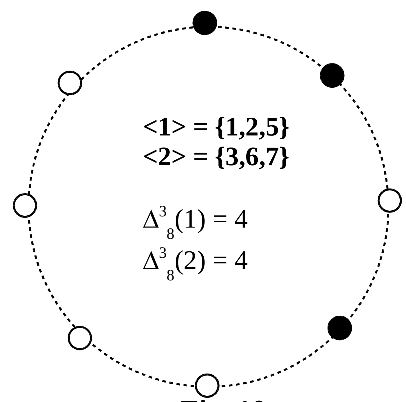


Fig. 10

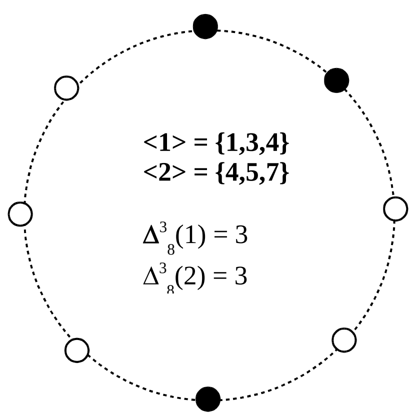


Fig. 11

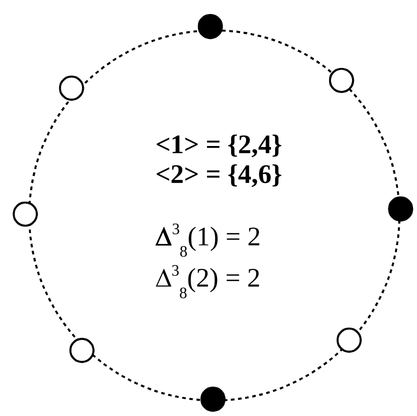


Fig. 12

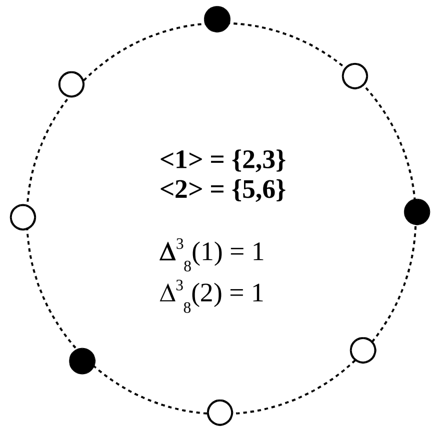


Fig. 13

Superimposed on these figures are the spectrum widths associated with each distribution.

2.3 Spectra Variation: We define the ***spectra variation*** for each distribution as the average of the spectra widths with respect to the number of filled dots:

$$V_c^d = \frac{1}{d}\sum_I \Delta_c^d(I) \qquad (2)$$

The spectra variations for Figures 9-13 are shown in Table 1.

Figure	**<1>**	**<2>**	$\boldsymbol{\Delta^3_8(1)}$	$\boldsymbol{\Delta^3_8(2)}$	$\boldsymbol{V^3_8}$
9	**{1,6}**	**{2,7}**	**5**	**5**	**3.33 (10/3)**
10	**{1,2,5}**	**{3,6,7}**	**4**	**4**	**2.33 (8/3)**
11	**{1,3,4}**	**{4,5,7}**	**3**	**3**	**2.00 (6/3)**
12	**{2,4}**	**{4,6}**	**2**	**2**	**1.33 (4/3)**
13	**{2,3}**	**{5,6}**	**1**	**1**	**0.667 (2/3)**

Table 1.

We see that the larger the spectra variation the less even the distribution. Moreover, the most even distribution, the maximally even distribution, has the smallest spectra variation and is less than one.

If we consider any exactly equal distribution, such as the one shown in Figure 4 or Figure 8, the spectra variation is exactly zero.

2.4 Another Example: The spectra variation allows us to compare the evenness of circular distributions with different numbers of filled dots. For example, consider the distributions shown in Figures 14 through 19. For brevity we consider only maximally even sets as these have variations less than one compared to all other distributions.

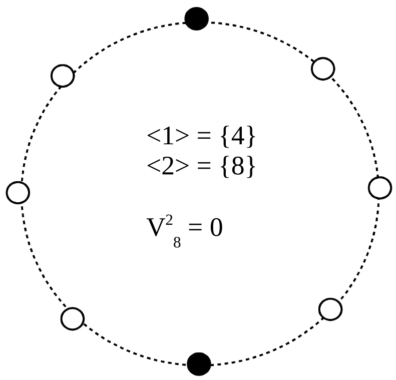


Fig. 14

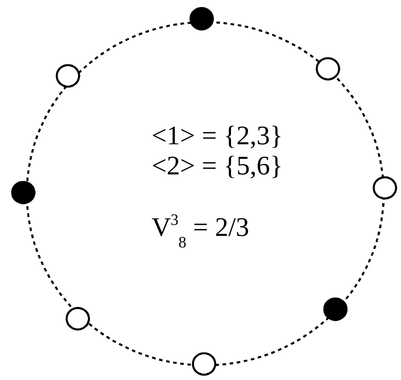


Fig. 15

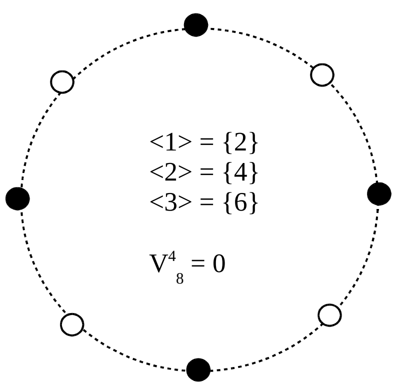


Fig. 16

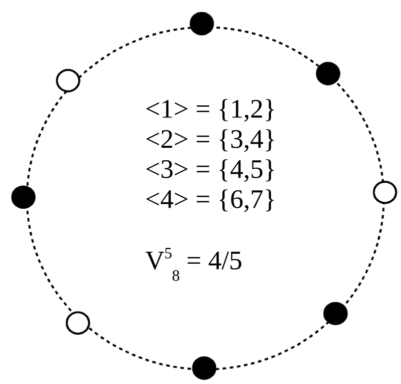


Fig. 17

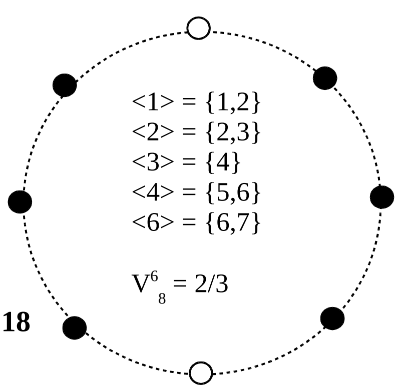


Fig. 18

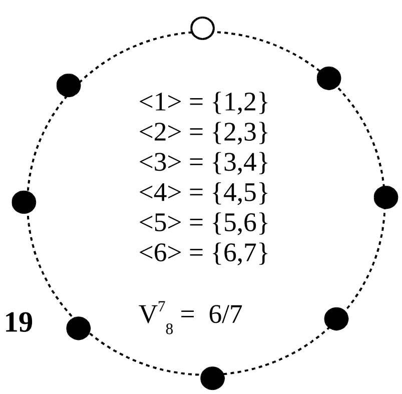


Fig. 19

Superimposed on each of these figures are the interval spectra and associated spectra variation. Shown in Table 2 are the distributions shown in Figures 14 through 19 ranked according to their spectra variations.

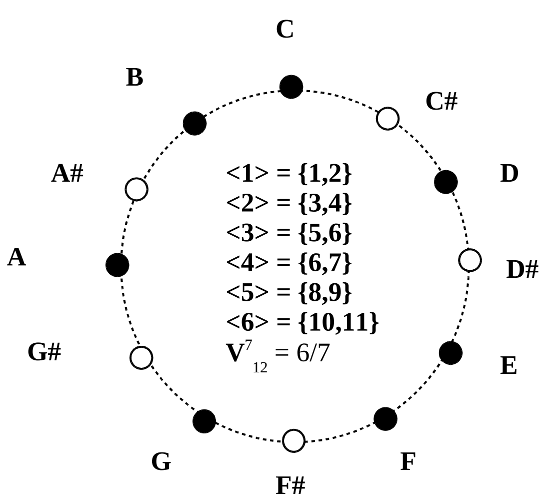
2-out-of-8	4-out-of-8	3-out-of-8	6-out-of-8	5-out-of-8	7-out-of-8
0	**0**	**0.667 (2/3)**	**0.667 (2/3)**	**0.800 (4/5)**	**0.857 (6/7)**

Table 2.

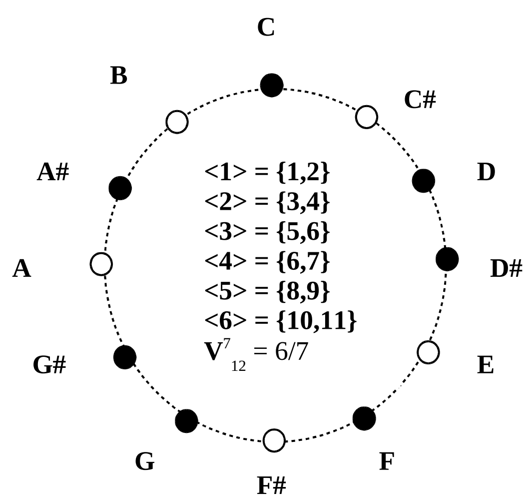
3 A Musical Example

We turn, now, to a musical example. Most people are familiar with the scale of the white keys on the piano, the so-called diatonic, or major, scale. Shown in Figure 20 is the diatonic scale as a circular distribution. The diatonic set is a maximally even set of 7-out-of-12 with a spectra variation of 6/7. Superimposed on Figure 20 are the spectra widths, spectra variation, and the usual notes of the diatonic scale.

Two other common scales are the natural minor and the descending melodic minor both of which are rotations of the diatonic scale. Each is a rotation of diatonic scale three half-steps clockwise, as shown in Figure 21.



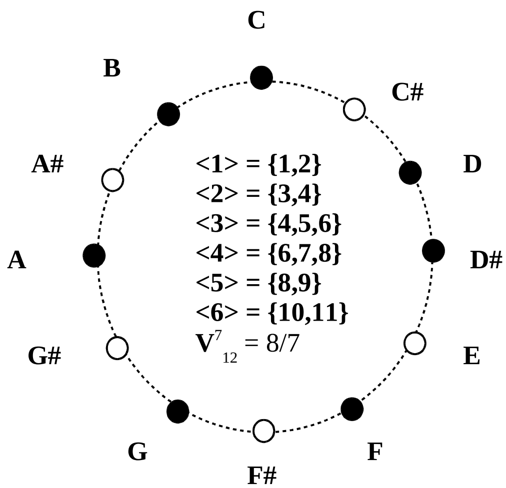
Diatonic Scale
Fig. 20



Natural Minor Scale (Descending)
Fig. 21

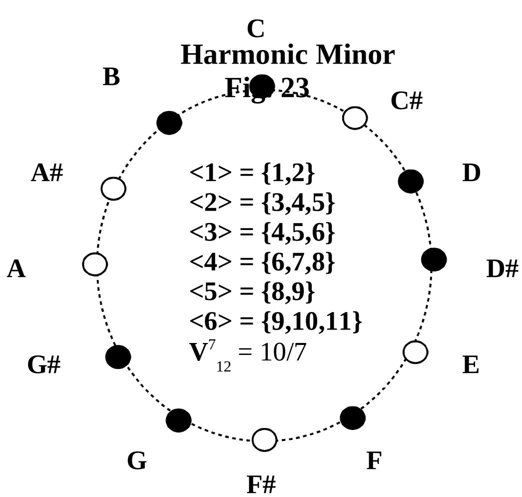
As each of these scales is a maximally even set, they all have spectra variations less than one and are the most even 7-out-of-12 scales.

Shown in Figure 22 is the next most even 7-out-of-12 scale, the ascending melodic minor with a spectra variation of 8/7. It is only one half-step away from being maximally even (D# => E).

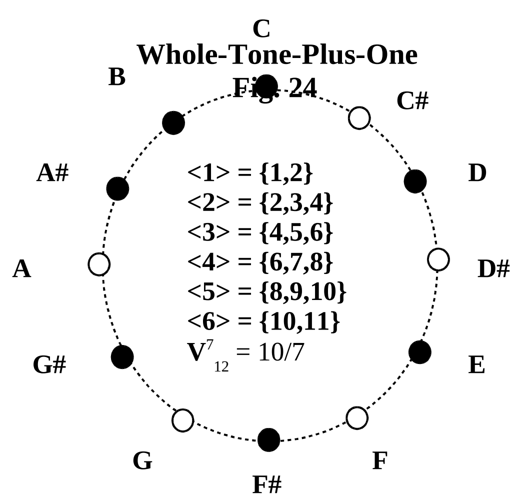


Ascending Melodic Minor
Fig. 22

The two next most even 7-out-of-12 scales are the harmonic minor and whole-tone-plus-one scale, shown in Figures 23 and 24 respectively. Each has a spectra variation of 10/7.



Harmonic Minor
Fig. 23



Whole-Tone-Plus-One
Fig. 24

Comparisons of the six most familiar 7-note scales are those that are most even as measured by the spectra variation. It is left to the reader to show that all other 7-out-of-12 circular distributions have larger spectra variations than those shown above, some 2 or 3 times larger.

4 Summary

We have developed the spectra variation method based on the, well-known, spectra of circular distributions, which allows comparison of the evenness of different circular distributions. The method is developed using a simple n-out-of-8 example and then applied to the 7-out-of-12 distribution common in Western music. We show that the smaller the spectra variation the more even the distribution. Ranking the 7-out-of-12 scales according to the smallest spectra variations shows that the most common 7-note scales in use are also those that are the most even.

5 Discussion

Clough and Myerson's work [2,3] dealt with the mathematical formalization of several musical properties. One of these properties is known as Myhill Property (MP): A set has MP if every interval spectrum is a doubleton (consists of two numbers). In particular, they focused on sets in which the members of the spectra were integers. Although their investigation was not related to evenness, they established the ground work for Clough and Douthett's [5] investigation on maximally even sets.

One class of maximally even sets consists of sets with MP in which each spectrum consists of two consecutive integers Musical scales in this class include the (black key) pentatonic and diatonic scales. In addition, Clough and Douthett's definition of maximally even sets allowed for sets in which some or all spectra are singletons (consists of a single integer). Musical sets with single integer spectra are the augmented triad, the fully-diminished seventh chord, and the whole-tone scale. The octatonic scale (also known as the diminished scale) is an example of a set in which some spectra are singletons and others consist of two consecutive integers. While the musical examples given above are in the usual chromatic universe of cardinality 12, Clough and Douthett's work extended these properties to musical universes of any size. Measures that justify the term "maximally even" can be found in Block and Douthett [4], Krantz, Douthett, and Clough [1]; and Douthett [6].

For any given chromatic and set cardinalities, Clough and Douthett identified a class of sets that were maximally even. They did not, however, discuss the comparative evenness of maximally even sets with differing cardinalities. For example, while both the diatonic scale and whole-tone scales are maximally even, intuition suggests that the whole-tone scale is "more even" that the diatonic scale. Indeed, the pitches in a whole-tone scale are distributed totally evenly around the octave (an equal-tempered system). We build on Clough's work with both Myerson and Douthett and construct a measure that compares the evenness of maximally even sets with differing cardinalities. For maximally sets, our measure varies between 0 and 1, including 0 but not 1. The value of this measure for equal-tempered sets is 0, while maximally even sets with MP measure closer to 1. Sets which are not maximally even always measure greater than or equal to 1.

References

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[6] J. Douthett, Ph.D. dissertation (University of New Mexico, Albuquerque, NM; May 1999).