

# Meanders

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## Abstract

The shape of a meandering river is approximated by a curve that has a particularly simple equation when expressed in intrinsic coordinates. It is the same shape as the form taken by a strip of spring-steel constrained by bending forces at its ends. These curves are easily generated using computer turtle graphics, as are some related forms. An extension into three dimensions produces a range of sculptural forms, some of which correspond to existing works of art.

## The Forms of Rivers

It is well known that a river flowing in a flat landscape does not take a straight path but follows a sinuous curve, e.g. Figure 1, which changes over time. The form of this curve was studied in detail about forty years ago by Luna Leopold [1]<sup>R</sup> who called it a "sine-generated" curve, since it has the equation

$$\psi = a \sin s$$

where  $a$  is a constant

$\psi$  is the angle to the mean down-valley direction

$s$  is distance measured along the curve.

This is an example of an intrinsic equation, so called because it does not relate to external coordinate axes. In this form it uses a system introduced by Whewell, but it will be more convenient to use Cesàro's system, which uses radius of curvature,  $\rho$ , rather than  $\psi$ .

$$\rho = \frac{ds}{d\psi}$$
$$1/\rho = \frac{d\psi}{ds} = a \cos s$$

The reciprocal of the radius,  $1/\rho$ , is known as the curvature,  $\kappa$ .



**Figure 1:** *The river Coquet at Swindon (near Rothbury), Northumberland.*

## The Elastic Curve

The sine-generated curve is actually an approximation to the "elastic curve", first described by James Bernoulli, which is the form taken by a flat spring when acted on by forces at its ends (see Figure 2). Eagles [2]<sup>l</sup> gives an approximate method of construction using classical methods, but there is an easier method using computer graphics.

**Figure 2:** A distorted elastic curve illustrated (Windflow



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### Turtle G

Many people will be familiar with turtle geometry. A domain version, MSW Logo, can be downloaded from [http://www.mslogo.com](#), although it can be implemented in almost any language. The commands FORWARD (move by some distance), and RIGHT and LEFT (turn by some angle) do not refer to any external coordinate system, it is purely local geometry. If, after moving forward by a small step of length  $\delta s$ , the path can be seen as an arc of a circle with radius,  $\rho$ :

$$\delta s = \rho \delta \alpha, \quad \rho = \frac{\delta s}{\delta \alpha}, \quad \kappa = 1/\rho = \frac{\delta \alpha}{\delta s}$$

Making the step size unity for simplicity, here is a Logo procedure for drawing a sine-generated curve:

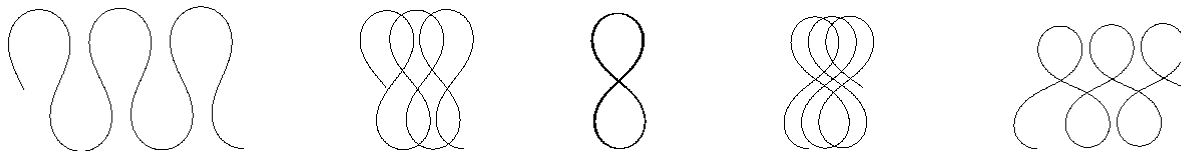
TO MEANDER :a :b

make "s 0

repeat 10000 [fd 1 rt :a \*(cos (:s/:b)) make "s :s + 1]

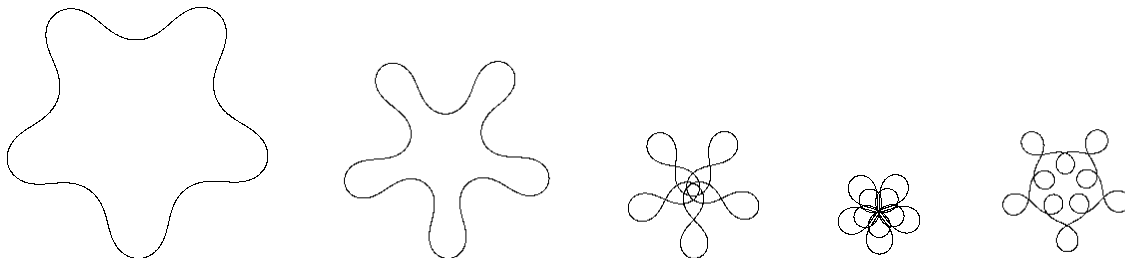
END

If the *cos* function takes a value in degrees one cycle will take 360 steps. The additional scaling parameter, *b*, is needed for other sizes, or if the argument is in radians. In this case a convenient value is  $b = 20.0535$ , which gives a cycle of  $2 \times 20.0535$ , which is approximately 126 steps. An equivalent in another language needs a stored variable (the turtle heading) that is changed when the turtle turns. (See Abelson and diSessa [3] for details on implementing turtle geometry in two and three dimensions.) As the parameter, *a*, is increased the meander curve gets more tightly bent (Figure 3).



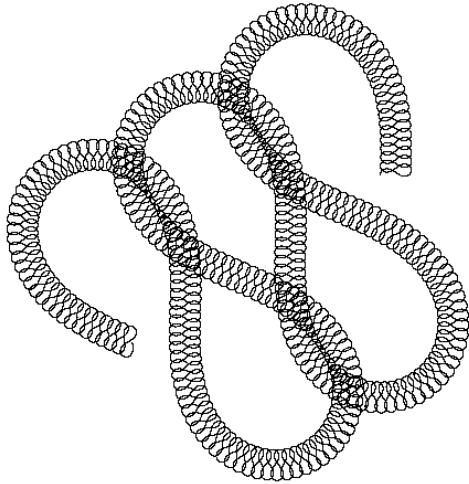
**Figure 3:** Increasing the parameter of a meander ( $a = 0.57, 0.64, 0.686, 0.715, 0.775$ ).

A curve of constant curvature is a circle, of course, and adding a constant, *c*, will bend the meander around a circle (Figure 4). If we require the curve to close after one circuit it must complete a whole number of cycles as it goes around, and the contribution to the total turn from the cosine term will be zero. The added constant, *c*, is determined by the number of steps, for example for a curve with symmetry order 5 there will be five cycles, and  $c = 0.2$ , if  $b = 1$ . If angles are in radians, with  $b = 20.0535$ , then  $c = 0.2/b = 0.09973$ , and the curve closes after  $2\pi \times 20.0535 \times 5 = 630$  steps



**Figure 4:** A circular meander of symmetry order 5 as the parameter increases

( $a = 0.3, 0.4, 0.57, 0.715, 0.86$ ).



An interesting variation occurs if the  $\cos(s)$  term is replaced by  $\cos(s + \cos(ks))$ , with  $k$  close to unity. (Again suitable scaling parameters might be needed.) The whole meander curve itself meanders, suggesting the possibility of self-similar meander curves (Figure 5).

**Figure 5:** A modified meander suggesting the possibility of self-similarity

$a = 2.2 \quad k = 0.991$

### Meanders in Three Dimensions

In three dimensions there are three types of turn: *roll* around the direction in which the turtle is heading, *yaw* to left or right, and *pitch* up or down. Since we are not concerned with gravity there is no preferred up direction, so yaw and pitch are equivalent. Most versions of Logo include the possibility of three dimensional geometry but the better display available in VRML makes it worth writing scripts (in Java or JavaScript) to generate an extrusion along a turtle path. Abelson and diSessa [3] pp. 140-144 explains how to implement movement in three dimensions.

A sine-generated curve can be incorporated into a three dimensional space-curve in many ways. One of the most interesting has a constant pitch (so constant curvature), with the roll defined by a cosine function. A ribbon-like extrusion along this curve is a ruled surface of constant curvature.

The intrinsic equations of the curve are

$$\begin{aligned} \kappa &= c \\ \tau &= a \cos s/b, \end{aligned}$$

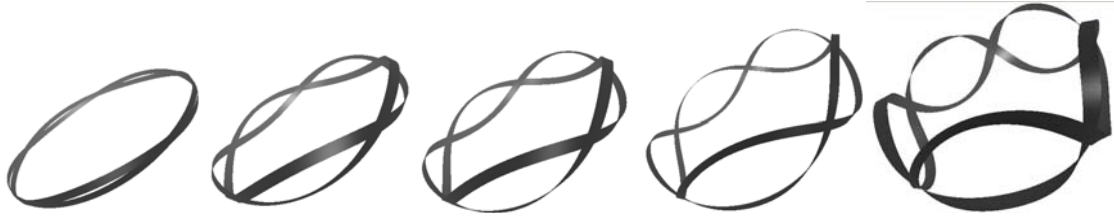
where the curvature,  $\kappa$ , gives the pitch, and the torsion,  $\tau$ , gives the roll. The constant curvature parameter,  $c$ , will be called the *bend*, and the torsion parameter,  $a$ , the *twist*. As before  $b$  is a scaling parameter.

Although things are more complicated than in two dimensions some of the same principles apply. If the curve is to close, the turtle's intrinsic axes should have an unchanged orientation when it returns to the start (although it might pass through the starting position several times before closing the curve). The roll must complete a whole number of cycles, so, as before, with  $b = 20.0535$  a curve with symmetry order 5 closes after approximately 630 steps. The total roll must be zero, so a ribbon along the curve is topologically equivalent to a cylinder.

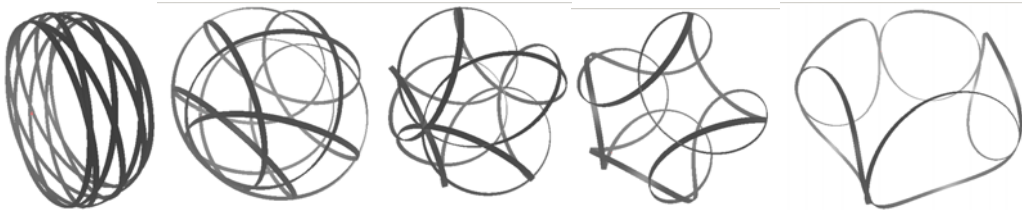
A meander curve in three dimensions is defined (to similarity) by two parameters, the bend and the twist, and there is a range of possible space-curves as they vary. Considering only those curves with the same symmetry, say order 5, is sufficient to understand the full range of curves, since the same principles apply to curves of any symmetry, and any that does not close is arbitrarily close to one that does.

Begin by thinking about curves with twist = 0. They consist of some number of loops around a circle. What happens if the twist is increased? (See Figure 6.) The loops begin to separate, and in Figure 6 begin

to curl inwards. The bend parameter needs to change to take account of this, but eventually the concave parts of the curve cancel the convex parts and any further increase in the twist parameter leads to a curve that cannot close after 630 steps, whatever the bend parameter.



**Figure 6:** Increasing the twist of a 3-D meander that starts with a 2 loop.

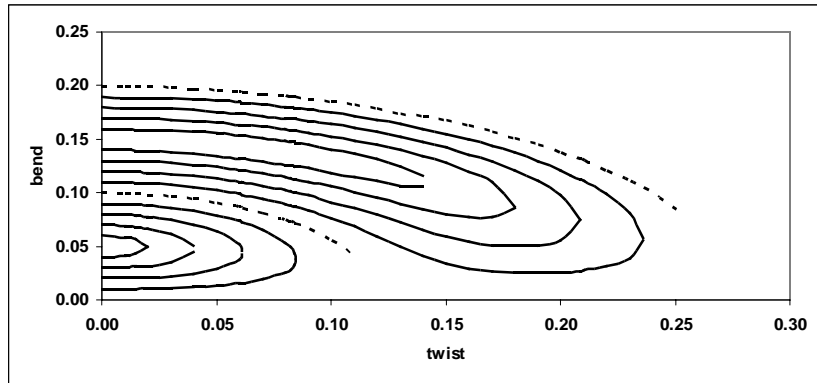


**Figure 7:** The 3-D meander from an 8 loop moves towards the same limit as twist increases.

The same limiting case can arise starting with a different number of loops (Figure 7), allowing a smooth trajectory between the two starting points.

Figure 8 shows the values of bend and twist that give curves with 5-symmetry. Notice that a circle with five loops is equivalent to a single circuit, since each loop is identical, and it is a limiting case. It does not close for any non-zero twist. The dotted lines correspond to space-curves where the concave and convex parts exactly cancel, so they go to infinity (Figure 9). Curves on opposite sides of these lines have centres of symmetry on opposite sides

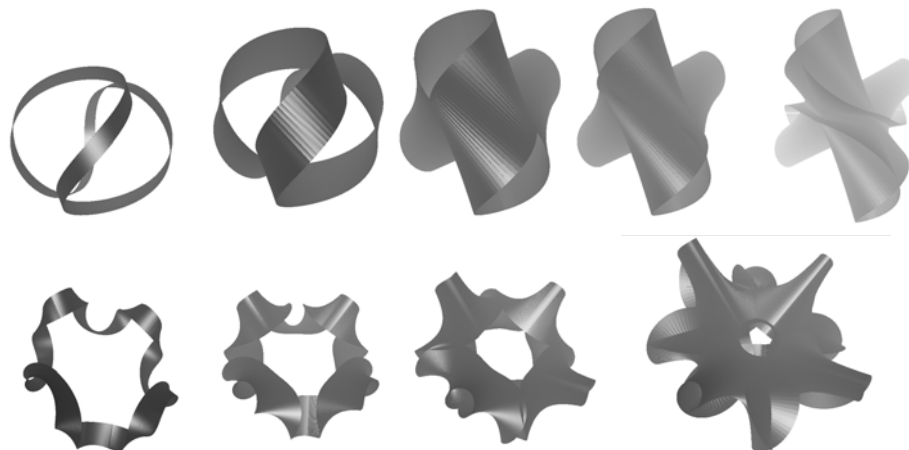
**Figure 8:** Values of bend and twist that produce 3-D meanders with 5-symmetry.





**Figure 9:** *Infinite 3-D meanders.*

These looping curves have an obvious aesthetic appeal, and many variations are possible. Figure 10 illustrates how changing the width of the ribbon can significantly alter their appearance. It is not surprising that some sculptors have produced works with similar features.



**Figure 10:** *Changing the ribbon width of two 3-D meanders (not to scale).*



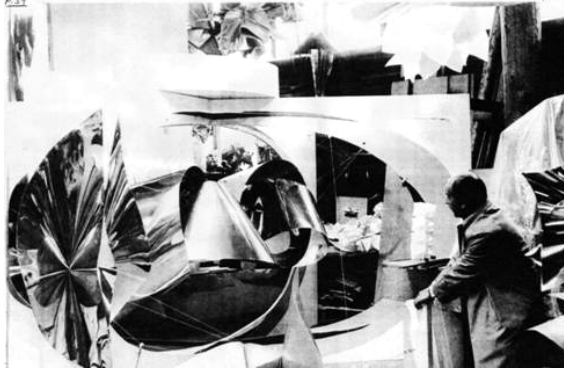
Many pieces by Brent Collins [4] have the overall shape of 3-D meanders, but usually the twist maintains the same sign rather than oscillating, so they are not topological cylinders. His ribbon sculptures are exceptions [5], and Figure 11, *Pax Mundi* (Figure 22 in [5]), is a particularly obvious example, although it does not appear to be a ruled surface.

**Figure 11:** *Pax Mundi, by Brent Collins.*

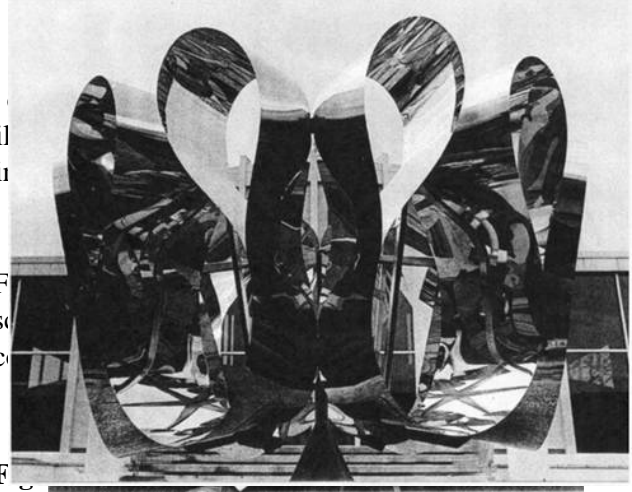
Jan Zach's work [6] with stainless steel in the late 1960s inevitably expresses elastic curves, since the steel automatically takes on these shapes, although other forces are also at work, in particular gravity. *Windflower No.1* (Figure 12) is a good example.

**Figure 12:** *Windflower No.1* (1967).

In *Flower of Freedom No.6* (Figure 13) the ribbon is obviously constrained near the centre. This illusionary curves provide believable images of ribbons, but they spring back to a circular cylinder.



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## References

- [1] Leopold, L. B. and Langbein, W.B., "River Meanders", *Scientific American*", **214**, (Jun 1966), pp.60-70.
- [2] Eagles, T.H., *Constructive Geometry of Plane Curves*, Macmillan, 1885, pp.348-354.
- [4] Collins, B., Visualization: From Biology to Culture, *Bridges Proceedings 2000*, pp.309-315.
- [6] Hall, R., *Intersections, The Life and Art of Jan Zach*, Hallie Ford Museum of Art, Willamette University, 2002.