

A Computerized Environment for Learning and Teaching 3-D Geometry

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Introduction

This is a presentation of a project in progress at Technion – Israel Institute of Technology. The ultimate goal of this project is a computerized environment integrated in a program for learning and teaching 3D Geometry in high schools, focusing on the morphology of solids. 3D Geometry is an area of mathematics that most country's curricula almost totally neglect, yet it is of utmost importance due to its applicability to other mathematical areas, to many professions, to the arts and to daily life.

The computerized environment we have started to develop will combine, in an interactive mode, the teaching and learning of spatial geometry issues, and the development of spatial capabilities. The environment will allow users to analyze, to investigate, and to animate solids, to carry out geometric constructions, and to perform upon them operations such as dissection and truncation, using a friendly graphical user interface. The interface will facilitate presenting solids and space relations in various representations simultaneously: a verbal representation, a graphical dynamic representation, and a vector representation.

This paper presents Platonic III, a computer program developed by the project team that allows the visualization and examination of the splendid features of a special class of five geometric solids – the so-called *Platonic solids*.

This computer program demonstrates many capabilities that will become a part of the computerized environment envisioned by the project's team.

1.	Tetrahedron	
2.	Cube	
3.	Octahedron	
4.	Dodecahedron	
5.	Icosahedron	

Polyhedra and the Arts

Symmetric figures of solid geometry fascinated people since ancient times. Their aesthetic value kept attracting artists' attention during the Renaissance, the Enlightenment, and up to modern time. For some artists, Polyhedra simply provided challenging models to demonstrate their mastery of perspective. For other artists, Polyhedra provided inspiration and an inventory of forms with various symmetries, while for some Polyhedra were symbolic of deep religious or philosophical truths (Hart, 1998). For example, Plato in the *Timaeus*, associated Platonic solids with the 4 elements.

The mere observation of the regular Polyhedra, with their perfect symmetries and relationships, makes one perceive the implicit information and identify patterns on their complexity. This process integrates mathematical visualization with the beauty of art. Oftentimes, human brain finds this combination very fulfilling. As Coxeter put it: "The chief reason for studying regular Polyhedra is still the same as in the time of the Pythagoreans, namely, that their symmetrical shapes appeal to one's artistic sense." H.S.M. Coxeter.

Platonic III

Platonic III provides the user with a program that allows two main options: (i) The performance of actions in 3D upon the five Platonic solids, and (ii) the examination of the results of these actions. The user interface of the program was designed to be as intuitive as possible in order to serve as an educational tool that helps young users get an insight into the special features, of the five *Platonic solids*, their inter-relationship, and their relationship with some other solids

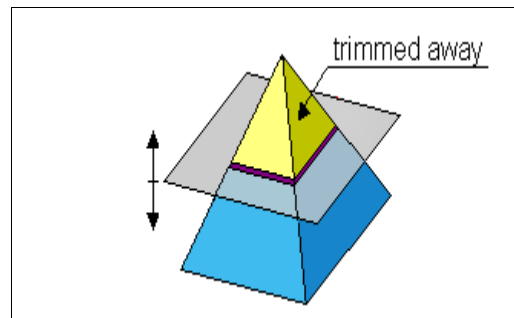
Platonic III provides users with five modes of action:



1. **Truncating vertices or edges mode**

User can perform vertex truncation and/or edge truncation by one truncating plane per vertex and one truncating plane per edge. The vertex truncation plane is perpendicular to the rotation axis through the center of the solid and that vertex. (Clearly, all five Platonic solids have a well-defined center-point.) The edge truncation plane is perpendicular to the rotation axis through the center of the solid and the midpoint of that edge.

Each truncation plane divides the whole space into two half-spaces – one half-space contains the center of the Platonic polyhedron and the other half-space contains the vertex (or edge) to be truncated. The part of the solid located in the latter half-space is trimmed away (see Figure 1a).



These truncation operations allow the user to produce new polyhedra as derivatives of the Platonic solids (See screen-shot in Figure 1b). For instance, by applying the edge / vertex truncation on the Platonic solids one can generate the so-called *Archimedean solids*. (Archimedean solids like Platonic, are symmetrical geometric polyhedra with regular polygonal faces. Unlike Platonic solids, the regular polygonal faces of Archimedean solids, do not all necessarily have the same number of edges. For example, Archimedean solids can be constructed from equilateral triangles and squares or from regular pentagons and regular hexagons, and so on.). Any newly created solid can be added to the inventory of solids and operated upon using the various modes of action the software makes available.

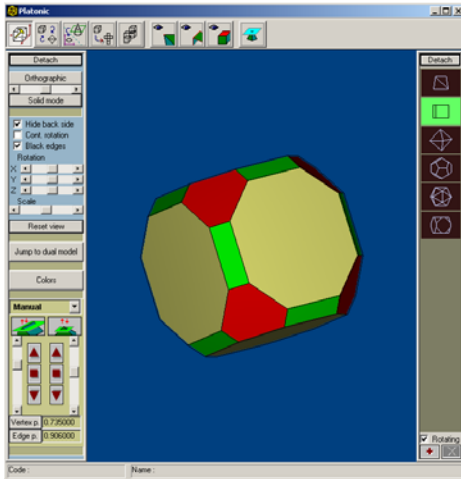
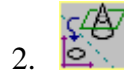


Figure 1b - A solid obtained by truncating the vertices and the edges of a cube.



2. Planar intersection mode

In this mode the user can examine planar dissection of a Platonic solid as well as of any solid obtained by truncation as described above. The intersection is highlighted on the 3D. In addition it is shown in a separate window as a 2D polygon (Figure 2).

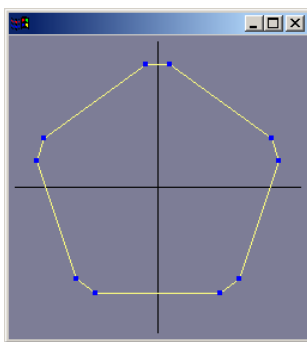
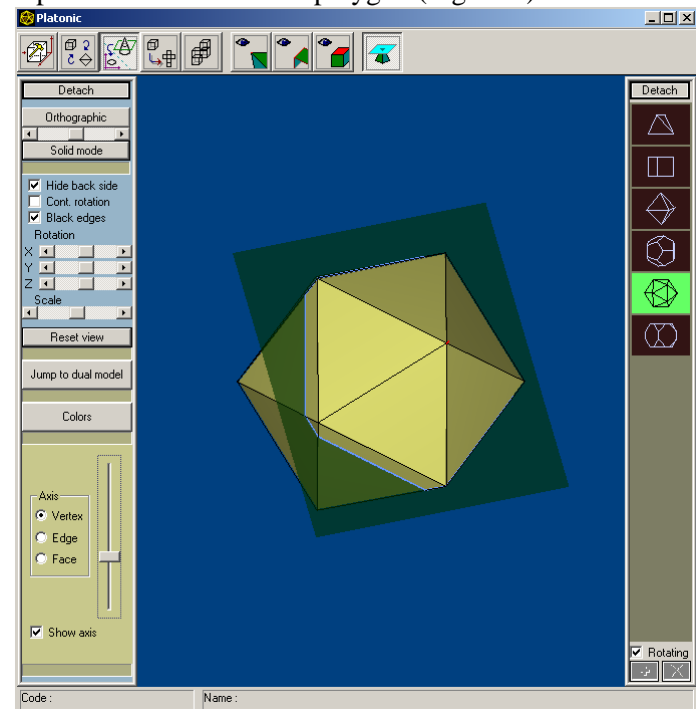


Figure 2 - Planar intersection



The cut is an intersection of the solid with a cutting plane that the user can move. The cutting plane has three possible directions. Each direction coincides with an n -fold rotational axis (which is a normal to this plane) and passes through the center of the polyhedron. The three options are:

- The axis passes also through a vertex.
- The axis passes also through the center of an edge.
- The axis passes also through the center of a face.

The user selects the desired direction by clicking on an appropriate radio button.



3. Net layout mode

In this mode, the user can "unfold" a Platonic solid (and of some of the derived solids as well), to obtain a planar net of its faces. User can then "close" the solid observing how the faces fit together. Users can use a print-out on cardboard, in order to construct a physical model of the polyhedron.

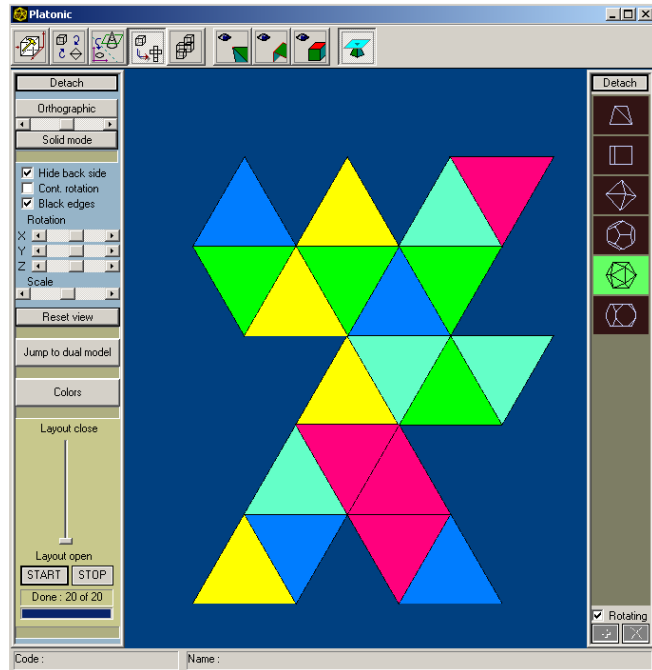



Figure 3 – Net layout

4.  **Duality visualization mode**

One of the remarkable properties of the five Platonic solids is their pairwise dual relationship: the cube and the octahedron, the dodecahedron and the Icosahedron, and the self-duality of the tetrahedron to itself. In this

mode, the software allows the user to observe and manipulate two dual Platonic solids with coinciding centers. The user can slide the control button, making one of the two, gradually more transparent, while the other becomes less transparent and consequently more visible. In the process a vertex of one solid replaces a face of the other and vice-versa.

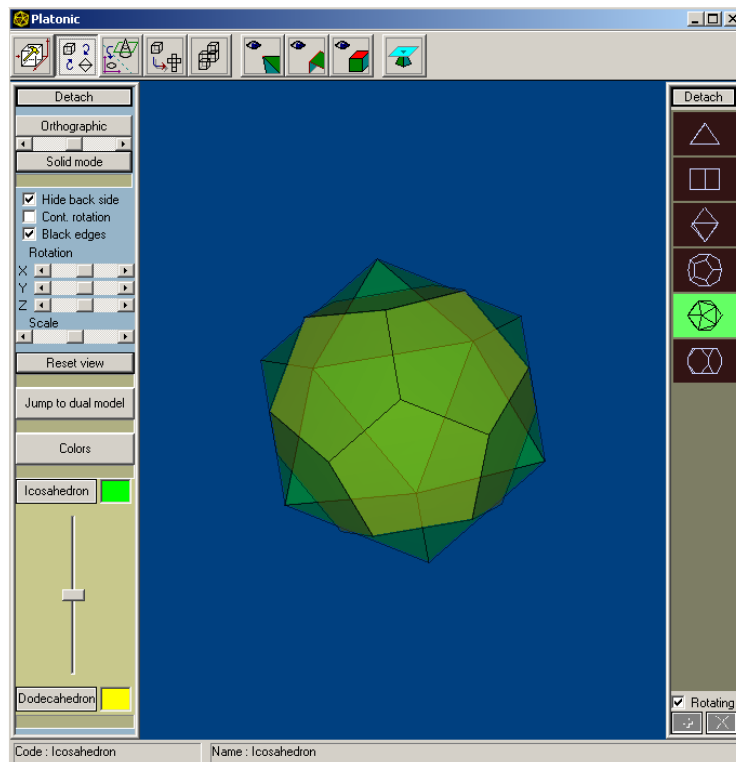
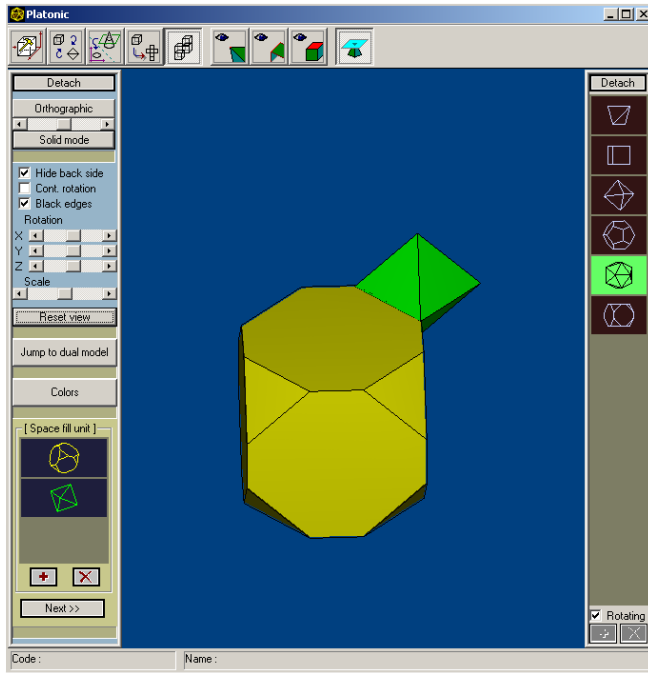


Figure 4 – Duality

5.  **Space filling mode**

The space filling mode allows the user to examine the possibility of filling the space with a single solid (Platonic polyhedra or a truncated one), or using a combination of solids. The whole process is divided into three steps:



- a. In the first step user constructs a space-filling unit, to be used in attempting to fill the space. (See Figure 5a) In order to attach two polyhedra to form one space-filling unit, user needs to manipulate one of the two polyhedra so that congruent polygonal faces of the two polyhedra touch one another. When they do, the automatic snapping is activated and the two polyhedra "stick" together.

Figure 5 a – Creation of space filling unit

- b. The second step is creating a space fill pattern according to which the space will be filled using the space-filling unit created in step a. The pattern defines the way space-filling units will be arranged to fill the space. Patterns are created automatically and (at the moment) cannot be changed manually (See a screen shot in Figure 5b).

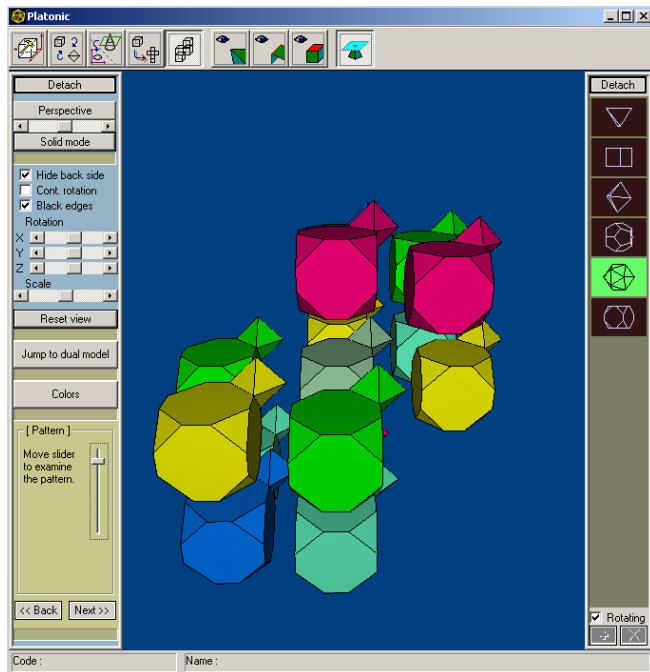


Figure 5 b: Space-filling pattern

- c. The final step is an animated space filling. Here user can observe space fill units “flying” to the unit in the center of the screen to fill the space (See a screen shot in Figure 5c.).

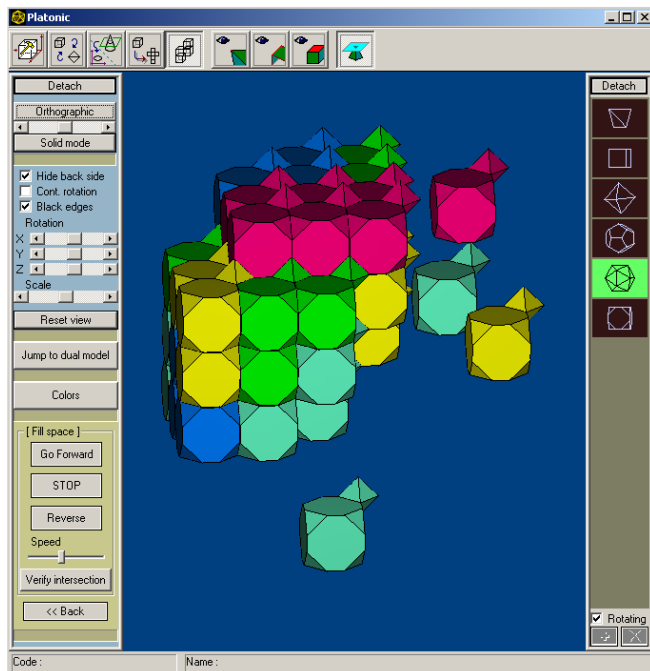


Figure 5 c – Space-filling animation.

A Few Words about the System in General

The system can process any predefined geometry and not just the regular polyhedra and the polyhedra derived from them. The system constructs the true geometry of the (semi) regular and other polyhedra, which enables further and future arbitrary processing and analysis of the models. For example, the volume of the model is now quite easy to compute. The system can also be used

as a WEB oriented environment (having an Active X control for Explorer), providing the 3D graphics support for remote teaching via the Internet.

Platonic Vs. Alternative Software

Platonic is a non-commercial software development project, as a part of a whole curriculum development in morphological 3-D geometry. In this framework special attention is given to improving visualization. Consequently it focuses on interactive processes such as solid truncation gradually yielding new solids, interactive cutting of a solid moving the cutting plane in certain directions, gradual transition between a solid opening into its net and folding back, and packing the space by identical solids, or by a combination of solids. These interactive processes seem to be unique to this program, as far as we were able to examine other available 3-D software e.g. 'Stella'¹. This interactivity feature was chosen initially in order to make the software of value for pedagogical use, assisting in building up students' understanding of abstract processes, as well as improving students' ability to associate and connect properties and forms. Empirical implementation of this curriculum in classrooms is characterized by self-exploration performed by students in small groups or individually. Teacher's role is more of a mentor than a source of knowledge.

Final Remarks

The project we have been working on is highly demanding. It combines mathematical tools, pedagogical tools and computing tools in order to provide educational systems with an integrated environment enabling the introduction of an area in mathematics that is known to be difficult to grasp for a wide range of students. The technology is available to make this happen, our team is committed to this project however the resources needed are scarce. We hope to be able to cope with the challenge we set forth, and complete our program in the near future. Future work plan includes adding of quantitative information, and presenting it adjacent to each solid or its components, improving the duality representation, and making the cuts more flexible in terms of choice of direction by the user, and manual control of the process.

Reference

Hart, G. W. (1998). Virtual Polyhedra. <http://www.georgehart.com/virtual-polyhedra/copyright.html>

¹ Stella: Polyhedron Navigator, <http://www.software3d.com>.