The Complexity of the Musical Vocabulary of the Nzakara Harpists

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Abstract

We present a new method for evaluating the complexity of music. We explore this method as it applies to the Nzakara harpists of Central Africa. In particular, the movement of the harpist's hands is measured for short, repeated patterns. Longer harp patterns are then compared to shorter patterns to determine a measure of the difficulty level of the piece based upon the amount the player's hands must move. Following explanation of the structure of the patterns, the method of measuring complexity is described. We then give a table of measurements for various harp patterns, some of which fit the grammatical rules of the Nzakara and some of which do not. We give an evaluation of some of these results and discuss the difficulty of achieving a whole picture of the difficulty or complexity of a musical structure.

Introduction

In contrast to our modern language full of symbols and definitions to describe mathematics, cultures have often relied upon the arts to express mathematical ideas and principles. Following a series of researchers including Simha Arom, Eric de Dampierre, Marc Chemillier, and Klaus-Peter Brenner, we examine the mathematical principles at work in the traditional harp music of the Nzakara people of Central Africa. In [3], Chemillier presents an in-depth review of his experiences with the practitioners of this art. Though there are few practitioners left, Chemillier has found unique mathematical structure in the harpists' language. In 2004, Brenner and Gregory each examined Chemillier's work. In [1], Brenner gave an ethnomuscological review of Chemillier's approach to harp pattern analysis. In [5] and [6], Gregory counted the number of patterns that fit the observed grammatical structure of the tunes.

The following section discusses the patterns Chemillier observed among the Nzakara harp tunes and develops the vocabulary which refers to various aspects of the tunes. The third section presents the definition of our complexity measure as it applies to the harp patterns. Next we give examples of one type of complexity of several harp tunes, some following Chemillier's observed patterns and some not. Finally, we conclude with a discussion of the difficulty of gaining a complete picture of the complexity of music because of the multidimensionality inherent in musical structure.

The language of the harps

The traditional harps have 5 strings. The strings are played in pairs as dyads, which Chemillier terms *bichords* and which can be viewed as the letters in the alphabet of the Nzakara harpists. There are five allowable bichords: those consisting of neighboring strings are disallowed, as is the pair of the lowest and highest strings. Each harp tune is a succession of bichords repeated over and over again to accompany chant and dance. The smallest repeated set of bichords is the *core* of the harp tune. Within each core, the same letter never appears twice in succession, and the core does not begin and end with the same letter (to avoid repeating the letter upon repeat, or concatenation). Further, the core will not "factor" into repeated smaller words since it is defined to be the smallest repeated word. Chemillier encoded the five allowable bichords into $\mathbb{Z}_5$, ordering them from lowest-pitched to highest by the lower note of the bichord and using the upper note to break ties as shown in Figure 1. He found that each allowable core is made up of a generator followed by four successive translations of the generator modulo 5. Since for any translation in $\mathbb{Z}_5$ the pattern returns to the beginning of the core at the fifth translation, each harp tune is a never-ending cyclical "ladder" created by perpetually translating a word in $\mathbb{Z}_5$. An example is shown in Figure 2. We should note that harpists often improvise their own variations on these base themes, and there may be rules governing these variances that are susceptible to mathematical or grammatical analysis. However, such variations are not recorded or documented at this time.

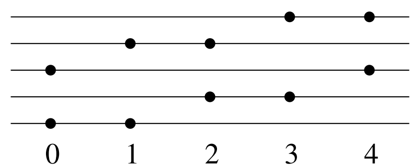

Figure 1: The five allowable bichords of the Nzakara harpists

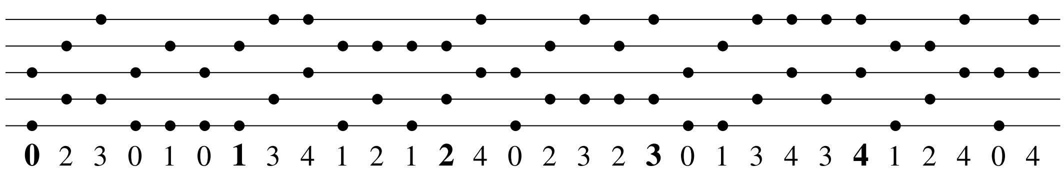

Figure 2: Example of a *limanza* canon

Measuring the complexity

One way to evaluate the complexity or richness of the vocabulary is to examine the number of cores following the grammatical rules outlined by Chemillier, as done in [5] and [6]. Here we examines another measure of the complexity, inspired by Jeffrey Pressing's method of measuring rhythmic complexity [7],[8]. It is impossible to capture the multi-dimensional nature of music in a single, simple mathematical procedure. Therefore, any measurement of the complexity of these tunes is limited to one aspect or one dimension of the tunes. The measure described below examines the difficulty of playing the harp tune due to the movement of the harpist's hands.

According to Pressing's algorithm for the cognitive complexity of rhythms, each rhythm is divided into subparts of equal length repeatedly until we reach subdivisions of the smallest prime length. Then the rhythm as a whole and each subdivision are examined for certain properties which contribute to the syncopation of any western-style rhythm. Each of these properties has a certain cognitive cost, or difficulty level assigned. For example, playing only on the upbeat, or the last pulse of a subdivision, would be more difficult or more costly than playing only on the downbeat. The cognitive complexity is calculated by summing each of the cognitive costs, weighted by the proportion of the total pattern of the subdivision.

We follow a similar model and formulate a definition of complexity for these tunes based upon the amount of movement by the hands between bichords. To begin we establish a base set of patterns which have complexity zero. We then build a catalog of complexities for cores of different length by comparing each new core with shorter cores.

Consider the five allowed bichords and their labelling in $\mathbb{Z}_5$, as shown in Figure 1. The least complex (in our sense of complex) string of notes to play is a single repeated bichord since the hands will not move among the strings. The complexity of such a string (e.g., ...000000... or ...222222...) is defined to be zero.

Notice that to move from bichord 0 to bichord 1, the player must move one finger one string. So bichords 0 and 1 are considered to have distance equal to 1. To move from bichord 4 to bichord 1, the player must move one finger down one string and the other down two strings. So we say bichords 1 and 4 have a distance of 3. Thus we define the *distance between bichords* to be equal to the total number of strings which the fingers must move to change from one bichord to the other. Because of the way the labelling has been defined, we can see that the distance between two bichords is the absolute value of the difference between the numbers which represent them in $\mathbb{Z}_5$: $\text{dist}(B_1, B_2) = |B_1 - B_2|$.

As in Pressing's definition of cognitive complexity, the complexity of each harp core is defined through comparison to more basic patterns. In this case, the more basic patterns are the shorter cores. This definition applies to both allowable harp patterns and those which are disallowed, allowing comparison between "grammatical" harp patterns and those which do not follow the observed rules. We define three measures of complexity, denoted C_1, C_2, and C_A. C_1 and C_2 are variations on the same theme of comparing the core in question with a single "closest" core of a smaller size. C_A will use an "average" over the smaller cores.

Each complexity measure relies upon a definition of the distance between two words. We define a distance closely related to the Hamming distance or $\overline{d}$-distance of information theory and to the swap distance described in [4]. Let l and d be positive integers. Let $u = u_1u_2...u_l$ and $v = v_1v_2...v_l$ be two finite strings of elements of $\mathbb{Z}_d$, each of length l. If $\text{dist}(u_i, v_i) = |u_i - v_i|$ is the distance between u_i and v_i as defined above, then the distance between u and v is defined to be

$$\text{dist}(u, v) = \frac{1}{l}\sum_{i=1}^{l} \text{dist}(u_i, v_i). \tag{1}$$

In short, the distance between two finite words of the same length is the average distance between their corresponding letters.

Since the cores used in the harp patterns are repeated over and over in a cyclical pattern, there is a straightforward definition of the distance between two finite cores of different lengths. Consider u and v of positive lengths s and t, respectively. Define u' to be u concatenated upon itself $\text{lcm}(s,t)/s$ times to create a word of length $\text{lcm}(s,t)$, and define v' to be v concatenated to itself $\text{lcm}(s,t)/t$ times to create another word of length $\text{lcm}(s,t)$. Then the distance between u and v is defined to be the distance between u' and v'.

We define the *nearest-neighbor complexity* of a core of length 2 as the minimum of the distances to cores of length 1. Both the C_1 and C_2 complexities of cores of length 2 are defined this way. For example, the complexity of the core 01 is 0.5, since its distance from 0 and 1 is equal to 0.5: $C_1(01) = C_2(01) = 0.5$. Similarly, $C_1(03) = C_2(03) = 1.5$, demonstrating that the player must move both fingers between the 0 bichord and the 3 bichord further than between the nearby 0 and 1.

The nearest-neighbor complexity of a core of length longer than 2 takes into account all shorter cores which are "close to" the core under consideration as well as their complexities. There are many reasonable variations on this theme, two of which are discussed here. In the first version, C_1, finding a nearby neighbor is considered of more importance than the complexity of that nearby neighbor: the set of words of minimum distance is found and then complexity is minimized within that set. In the second version, C_2, complexity and distance are minimized together by finding the minimum of their product.

More precisely, given a word u of length $k > 2$, let V_i be the set of words of length i which have the minimum distance to u for all $i < k$:

$$V_i = \{v | v \text{ is of length } i,\ \text{dist}(u,v) \leq \text{dist}(u,w) \text{ for all words } w \text{ of length } i\}.$$

If $i = 1$, then let $s_1(u) = \text{dist}(u,v)$ for any v in V_1. (Note that $s_1(u)$ is well-defined since every element of V_i has the same distance to u.) For $i > 1$ we define

$$s_i(u) = \min\{\text{dist}(u,v) \cdot C_1(v) | v \in V_i\}.$$

Then the C_1 complexity of u is defined to be

$$C_1(u) = \sum_{i=1}^{k-1} \frac{s_i(u)}{\omega_i} \ ,$$

where ω_i is a weighting term.

Some possible weighting variables ω_i are i, 5^i, 5^{i-1}, or $i!$. We have chosen to use $\omega_i = 5^{i-1}$. Since we are working with an alphabet of size 5, the number of possible words of length i is given by 5^i. Thus with this choice of ω_i, the weight on each sum term approximates the number of choices among which we minimized. Furthermore, because of computing resource limitations, the summations resulting in Table 1 are truncated at 5. (Note that the distance function is bounded by 4, so as i increases, the weight variable we chose will quickly dominate and the contribution from the i'th term becomes inconsequential.)

Our second version of the nearest-neighbor complexity of a core u of length k minimizes both the distance and complexity of the comparison words with equal importance:

$$s_1(u) = \min\{\text{dist}(u,v) | v \text{ is a word of length } 1\}$$

$$\text{for } i > 1,\ s_i(u) = \min\{\text{dist}(u,v) \cdot C_2(v) | v \text{ is a word of length } i\}$$

Then C_2 is the weighted sum of the s_i's, as before:

$$C_2(u) = \sum_{i=1}^{k-1} \frac{s_i(u)}{\omega_i} \ .$$

The calculations in Table 1 use the same weighting variable as the first version and truncate the summation term in the same location.

The final complexity measure in the table, the *complexity over averages*, is calculated using the average of distance times complexity over all the smaller v. In other words,

$$C_A(u) = \sum_{i=1}^{k-1} \text{mean} \frac{(\text{dist}(u,v_i) \cdot C_A(v_i))}{\omega_i}, \tag{2}$$

where the arithmetic means are taken over all V_i with length i.

The effectiveness of the measure

As we see in Table 1, the C_2 complexity measure appears to do the best job of finding the distance a player's hands must travel to play any pattern of bichords. For example, a harpist's hands can never move

Table 1: The computed complexities of selected bichord patterns

Bichord pattern	C_1	C_2	C_A
...01010101010101...	0.5	0.5	1.7
...02020202020202...	1	1	1.6
...03030303030303...	1.5	1.5	1.7
...04040404040404...	2	2	2
...010010010010010...	0.38	0.38	2.38
...012012012012012...	0.75	0.75	2.03
...013013013013013...	1.12	1.12	2.12
...024024024024024...	1.48	1.48	2.29
...000010000100001...	0.26	0.26	2.67
...000020000200002...	0.48	0.48	2.62
...012340123401234...*	1.35	1.35	2.29
...024130241302413...*	1.35	1.35	2.29
...031420314203142...*	1.35	1.35	2.29
...021340213402134...**	1.35	1.35	2.29
...113231132311323...**	0.91	0.91	1.95
...034140341403414...**	1.58	1.58	2.51
...00000000010000000001...	0.54	0.15	2.74
...01341240230134124023...*	3.00	1.35	2.06
...02132430410213243041...*	3.05	1.35	2.06
...01403423120140342312...*	3.00	1.35	2.06
...01340243140134024314...**	3.27	1.55	2.07
...04241414420424141442...**	3.25	1.53	1.85
...01120440210112044021...**	2.70	1.24	2.27

* These patterns follow the grammatical rules of the harp tunes.

** These patterns were generated randomly for comparison.

more than 4 strings with each succeeding note, yielding a "tune" alternating between bichords 0 and 4, or $\ldots 040404040\ldots$. This tune does indeed have the largest C_2 complexity in the table, with a measure of 2. However, it has among the lowest complexities with averages. In contrast, a harpist playing the "tune" $\ldots 00000000010000000001\ldots$ moves his hands very little, and this pattern has the lowest nearest-neighbor complexity in the table (0.15), while it has the highest complexities with averages (2.74).

C_A seems to be a poor measure for hand movement, as demonstrated by a simple example. The hands of a harpist playing a sequence of bichord 0 alternating with bichord 1 should move the same amount between each bichord no matter the number of total strings on the harp (i.e., the size of the alphabet). Consequently, the complexity of 01 should remain constant across alphabet sizes. However, as the size of the harp grows, $C_A(01)$ will also grow as it is compared to words farther and farther away. In general, cores far away from the word in question with high complexity have at least as much influence as close cores with near complexity. We might try to counteract this problem by dividing by the distance instead of multiplying so that cores further away would have less influence over the complexity of the core in question.

The two nearest-neighbor complexities give similar results, and we begin to see differences for cores of length 10 or more. The values given by C_1 are naturally higher, since the minimum is being taken over a smaller set. Within each set of period lengths, C_1 and C_2 demonstrate similar ratios among the patterns. However, we can see a telling difference between the two when we examine 00001 and 0000000001. We see that $C_1(00001)$ is actually smaller than $C_1(0000000001)$, indicating that 00001 is less complex than the string with more zeros. It may indeed be mentally less complex in that the musician has to keep track of a smaller number of 0's before he plays a 1. However, in terms of the complexity we are trying to measure, namely the movement of the hands, C_2 gives a more satisfying conclusion with $C_2(0000000001) < C_2(00001)$.

Conclusion

Our goal was to find a complexity measure to evaluate the total amount of movement of the harpists' hands. Other factors that impact the difficulty of learning and playing a piece might include the speed at which the piece is played, the positioning of the hands (for example, whether the piece is played with one hand or two – a choice which varies by harpist – or whether fingers or hands must cross over each other), the length of the period of the piece (i.e., the memory required), the care with which the player has to count rhythms, finger fatigue, and many other factors. The particular dimension of hand movement seems to be measured well by the two nearest-neighbor complexities. Other measures may satisfactorily evaluate these other aspects. With enough such complexity measures and an understanding of what each one means, we might be able to combine them to gain a reasonable profile to understand the overall complexity of a piece of music.

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